

# Optimization under fuzzy rule constraints

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## Abstract

Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part is a linear combination of the crisp values of the decision variables. We suggest the use of Takagi and Sugeno fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem.

**Keywords:** fuzzy number, constrained optimization, fuzzy reasoning

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# 1 Introduction

Fuzzy optimization problems can be stated and solved in many different ways [2, 3, 14, 21, 22]. Usually the authors consider optimization problems of the form

$$\max/\min f(x); \text{ subject to } x \in X,$$

where  $f$  or/and  $X$  are defined by fuzzy terms. Then they are searching for a crisp  $x^*$  which (in a certain) sense maximizes  $f$  on  $X$ . For example, fuzzy linear programming (FLP) problems can be stated as [9, 13, 15, 17]

$$\max/\min f(x) = \tilde{c}x; \text{ subject to } \tilde{A}x \lesssim \tilde{b}, \quad (1)$$

where the fuzzy terms are denoted by tilde. Fullér and Zimmermann [12] interpreted FLP problems (1) with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes, where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem. Their solution process consists of two steps: first, for every decision variable  $x \in \mathbb{R}^n$ , compute the (fuzzy) value of the objective function,  $\text{MAX}(x)$ , via sup-min convolution of the antecedents/constraints and the fact/objective. Then an (optimal) solution to the FLP problem is any point which produces a maximal element of the set  $\{\text{MAX}(x) \mid x \in \mathbb{R}^n\}$ .

Unlike in (1) the fuzzy value of the objective function  $f(x)$  may not be known for any  $x \in \mathbb{R}^n$ . More often than not we are only able to describe the causal link between  $x$  and  $f(x)$  linguistically using fuzzy if-then rules.

In [8] we have considered constrained fuzzy optimization problems of the form

$$\max/\min f(x); \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x) \mid x \in X \subset \mathbb{R}^n\}, \quad (2)$$

with

$$\mathfrak{R}_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

where  $A_{ij}$  and  $C_i$  are fuzzy numbers; and we have suggested the use of Tsukamoto's fuzzy reasoning method [19] to determine the crisp values of  $f$ .

In this paper we suppose that our knowledge base contains fuzzy if-then rules of the form

$$\mathfrak{R}_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) = a_{i1}x_1 + \dots + a_{in}x_n + b_i \quad (3)$$

where  $A_{ij}$  is a fuzzy number, and  $a_{ij}$  and  $b_i$  are real numbers. Then we determine the crisp value of  $f$  at  $u \in \mathbb{R}^n$  by the Takagi and Sugeno fuzzy reasoning method, and obtain an optimal solution to (2) by solving the resulting (usually nonlinear) optimization problem

$$\max/\min f(u), \text{ subject to } u \in X.$$

We illustrate the proposed method by several examples.

## 2 Constrained Optimization under Fuzzy If-then Rules

A linguistic variable [20] can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. A fuzzy set  $A$  in  $X$  is called a fuzzy point if there exists a  $u \in X$  such that  $A(t) = 1$  if  $t = u$  and  $A(t) = 0$  otherwise. We will use the notation  $A = \bar{u}$ . Fuzzy points are used to represent crisp values of linguistic variables. If  $x$  is a linguistic variable in the universe of discourse  $X$  and  $u \in X$  then we simply write " $x = u$ " or " $x$  is  $\bar{u}$ " to indicate that  $u$  is a crisp value of  $x$ .

To find a fair solution to the fuzzy optimization problem

$$\max/\min f(x); \text{ subject to } \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x) \mid x \in X\}, \quad (4)$$

with fuzzy if-then rules of form (3) we first determine the crisp value of the objective function  $f$  at  $u \in \mathbb{R}^n$ , denoted also by  $f(u)$ , by the compositional rule of inference

$$f(u) := (x \text{ is } \bar{u}) \circ \{\mathfrak{R}_1(x), \dots, \mathfrak{R}_m(x)\}$$

using the Takagi and Sugeno fuzzy reasoning method as

$$f(u) := \frac{\alpha_1 z_1(u) + \dots + \alpha_m z_m(u)}{\alpha_1 + \dots + \alpha_m}.$$

where the firing levels of the rules are computed by

$$\alpha_i = \prod_{j=1}^n A_{ij}(u_j), \quad (5)$$

and the individual rule outputs, denoted by  $z_i$ , are derived from the relationships

$$z_i(u) = \sum_{j=1}^n a_{ij} u_j + b_i.$$

To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function). In this manner our constrained optimization problem (4) turns into the following crisp (usually nonlinear) mathematical programming problem

$$\max/\min f(u); \text{ subject to } u \in X.$$

If  $X$  is a fuzzy set with membership function  $\mu_X$  (e.g. given by soft constraints as in [21]) then following Bellman and Zadeh [1] we define the fuzzy solution to problem (4) as

$$D = \mu_X \cap \mu_f, \quad (6)$$

where  $\mu_f$  is an appropriate transformation of the values (computed by the Takagi and Sugeno reasoning method) of  $f$  to the unit interval [10], and an optimal solution to (4) is defined to be as any maximizing element of  $D$ .

**Example 1.** Consider the optimization problem

$$\max f(x); \text{ subject to } \{\mathfrak{R}_1(x), \mathfrak{R}_2(x) \mid x \in X = [0, 1]\}, \quad (7)$$

where

$$\begin{aligned} \mathfrak{R}_1(x) &: \text{ if } x \text{ is } \textit{small} \text{ then } f(x) = x \\ \mathfrak{R}_2(x) &: \text{ if } x \text{ is } \textit{big} \text{ then } f(x) = 1 - x. \end{aligned}$$

If  $\text{small}(x) = 1 - x$  and  $\text{big}(x) = x$ , and  $u$  is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = 1 - u, \alpha_2 = u.$$

Then we get

$$f(u) = (1 - u)u + u(1 - u) = 2(1 - u)u.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max 2u(1 - u); \text{ subject to } u \in [0, 1].$$

which has the optimal solution  $u^* = 1/2$ .

If the membership functions in the rules are defined by

$$\text{small}(x) = \frac{1}{1 + e^{-(1/2-x)}}, \quad \text{big}(x) = \frac{1}{1 + e^{(1/2-x)}}$$

and  $u$  is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = \frac{1}{1 + e^{-(1/2-u)}}, \quad \alpha_2 = \frac{1}{1 + e^{(1/2-u)}}.$$

Then we get

$$f(u) = \frac{u}{1 + e^{-(1/2-u)}} + \frac{1 - u}{1 + e^{(1/2-u)}}.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max \left( \frac{u}{1 + e^{-(1/2-u)}} + \frac{1 - u}{1 + e^{(1/2-u)}} \right); \text{ subject to } u \in [0, 1].$$

which has the optimal solution  $u^* = 1/2$ .

If we use nonsymmetric membership functions in the rules, for example

$$\text{small}(x) = \frac{1}{1 + e^{-10(3/4-x)}}, \quad \text{big}(x) = \frac{1}{1 + e^{0.1(1/3-x)}}$$

and  $u$  is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = \frac{1}{1 + e^{-10(3/4-u)}}, \quad \alpha_2 = \frac{1}{1 + e^{0.1(1/3-u)}}.$$

Then we get

$$f(u) = \frac{\frac{u}{1 + e^{-10(3/4-u)}} + \frac{1-u}{1 + e^{0.1(1/3-u)}}}{\frac{1}{1 + e^{-10(3/4-u)}} + \frac{1}{1 + e^{0.1(1/3-u)}}}.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max \left( \frac{\frac{u}{1 + e^{-10(3/4-u)}} + \frac{1-u}{1 + e^{0.1(1/3-u)}}}{\frac{1}{1 + e^{-10(3/4-u)}} + \frac{1}{1 + e^{0.1(1/3-u)}}} \right); \text{ subject to } u \in [0, 1].$$

which has the optimal solution  $u^* = 0.65$  and  $f(u^*) = 0.52..$

Even though Example 1 is probably the simplest one, it clearly shows the complexity of the problem of optimization under fuzzy if-then rules. Namely, the only way to increase  $f(u)$  is to decrease the feasibility of  $u$ .

**Example 2.** Consider the optimization problem

$$\min f(x); \text{ subject to } \{x_1 + x_2 = 1/2, 0 \leq x_1, x_2 \leq 1\}, \quad (8)$$

where

$$\begin{aligned} \mathfrak{R}_1(x) &: \text{ if } x_1 \text{ is } \textit{small} \text{ and } x_2 \text{ is } \textit{small} \text{ then } f(x) = x_1 + x_2, \\ \mathfrak{R}_2(x) &: \text{ if } x_1 \text{ is } \textit{small} \text{ and } x_2 \text{ is } \textit{big} \text{ then } f(x) = -x_1 + x_2. \end{aligned}$$

Let  $u = (u_1, u_2)$  be an input to the fuzzy system. Then the firing levels of the rules are

$$\alpha_1 = (1 - u_1)(1 - u_2), \quad \alpha_2 = (1 - u_1)u_2,$$

It is clear that if  $u_1 = 1$  then no rule applies because  $\alpha_1 = \alpha_2 = 0$ . So we can exclude the value  $u_1 = 1$  from the set of feasible solutions. The individual rule outputs are computed by

$$z_1 = u_1 + u_2, \quad z_2 = -u_1 + u_2.$$

and, therefore, the overall system output, interpreted as the crisp value of  $f$  at  $u$  is

$$f(u) = \frac{(1 - u_1)(1 - u_2)(u_1 + u_2) + (1 - u_1)u_2(-u_1 + u_2)}{(1 - u_1)(1 - u_2) + (1 - u_1)u_2} = u_1 + u_2 - 2u_1u_2.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\min (u_1 + u_2 - 2u_1u_2); \text{ subject to } \{u_1 + u_2 = 1/2, 0 \leq u_1 < 1, 0 \leq u_2 \leq 1\}.$$

which has the optimal solution  $u_1^* = u_2^* = 1/4$  and its optimal value is  $f(u^*) = 3/8$ .

Even though the individual rule outputs are linear functions of  $u_1$  and  $u_2$ , the computed input/output function  $f(u) = u_1 + u_2 - 2u_1u_2$  is a nonlinear one.

**Example 3.** Consider the problem

$$\max_X f \tag{9}$$

where  $X$  is a fuzzy subset of the unit interval with membership function

$$\mu_X(u) = 1 - (1/2 - u)^2$$

for  $u \in [0, 1]$  and the fuzzy rules are

$$\begin{aligned} \mathfrak{R}_1(x) : & \text{ if } x \text{ is } \textit{small} \text{ then } f(x) = 1 - x, \\ \mathfrak{R}_2(x) : & \text{ if } x \text{ is } \textit{big} \text{ then } f(x) = x. \end{aligned}$$

Let  $u \in [0, 1]$  be an input to the fuzzy system  $\{\mathfrak{R}_1(x), \mathfrak{R}_2(x)\}$ . Then the firing levels of the rules are  $\alpha_1 = 1 - u$ ,  $\alpha_2 = u$ . The individual rule outputs are  $z_1 = (1 - u)(1 - u)$ ,  $z_2 = u^2$  and, therefore, the overall system output is

$$f(u) = (1 - u)^2 + u^2 = 2u^2 + 2u + 1.$$

Then according to (6) our original fuzzy problem (9) turns into the following crisp biobjective mathematical programming problem

$$\max \min\{2u^2 + 2u + 1, 1 - (1/2 - u)^2\}; \text{ subject to } u \in [0, 1],$$

which has the optimal value of 0.8333 and two optimal solutions  $\{0.09, 0.91\}$ .

The rules represent our knowledge-base for the fuzzy optimization problem. The fuzzy partitions for linguistic variables will not usually satisfy  $\varepsilon$ -completeness, normality and convexity. In many cases we have only a few (and contradictory) rules. Therefore, we can not make any preselection procedure to remove the rules which *do not play any role* in the optimization problem. All rules should be considered when we derive the crisp values of the objective function. We have chosen the Takagi and Sugeno fuzzy reasoning scheme, because the individual rule outputs are crisp functions, and therefore, the functional relationship between the input vector  $u$  and the system output  $f(u)$  can be easily identified.

### 3 Summary

We have addressed mathematical programming problems in which the functional relationship between the decision variables and the objective function is known linguistically. We suggested the use of the Takagi and Sugeno fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem. We can refine the fuzzy rule base by introducing new linguistic variables modeling the linguistic dependencies between the variables and the objectives [4, 5, 6, 11]. These will be the subjects of our future research.

## References

- [1] R. E. Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sciences*, Ser. B 17(1970) 141-164.
- [2] E. Canestrelli and S. Giove, Optimizing a quadratic function with fuzzy linear coefficients, *Control and Cybernetics*, 20(1991) 25-36.
- [3] E. Canestrelli and S. Giove, Bidimensional approach to fuzzy linear goal programming, in: M. Delgado, J. Kacprzyk, J.L. Verdegay and M.A. Vila eds., *Fuzzy Optimization* (Physical Verlag, Heidelberg, 1994) 234-245.
- [4] C. Carlsson and R. Fullér, Interdependence in fuzzy multiple objective programming, *Fuzzy Sets and Systems*, 65(1994) 19-29.
- [5] C. Carlsson and R. Fullér, Fuzzy if-then rules for modeling interdependencies in FMOP problems, in: *Proceedings of EUFIT'94 Conference*, September 20-23, 1994 Aachen, Germany, Verlag der Augustinus Buchhandlung, Aachen, 1994 1504-1508.
- [6] C. Carlsson and R. Fullér, Multiple Criteria Decision Making: The Case for Interdependence, *Computers & Operations Research*, 22(1995) 251-260.
- [7] C. Carlsson and R. Fullér, Optimization with linguistic values, *TUCS Technical Reports*, Turku Centre for Computer Science, No. 157/1998. [ISBN 952-12-0138-X, ISSN 1239-1891].
- [8] C. Carlsson and R. Fullér, Optimization under fuzzy if-then rules, *Fuzzy Sets and Systems*, 2000, (to appear).
- [9] M.Delgado,J.L.Verdegay and M.A.Vila, A possibilistic approach for multiobjective programming problems. Efficiency of solutions, in: R.Słowiński and J.Teghem eds., *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Kluwer Academic Publisher, Dordrecht, 1990 229-248.
- [10] D. Dubois and H. Prade, Decision making under fuzzy constraints and fuzzy criteria – mathematical programming vs. rule-based system approach, in: M.Delgado, J.Kacprzyk, J.L.Verdegay and M.A.Vila eds., *Fuzzy Optimization: Recent Advances*, Studies Fuzziness, Vol. 2, Physica-Verlag, Heidelberg, 1994 21-32.
- [11] R.Felix, Relationships between goals in multiple attribute decision making, *Fuzzy sets and Systems*, 67(1994) 47-52.
- [12] R.Fullér and H.-J.Zimmermann, Fuzzy reasoning for solving fuzzy mathematical programming problems, *Fuzzy Sets and Systems* 60(1993) 121-133.

- [13] F. Herrera, M. Kovács, and J. L. Verdegay, Fuzzy linear programming problems with homogeneous linear fuzzy functions, in: *Proc. of IPMU'92*, Universitat de les Illes Balears, 1992 361-364.
- [14] M.Inuiguchi, H.Ichihashi and H. Tanaka, Fuzzy Programming: A Survey of Recent Developments, in: Slowinski and Teghem eds., *Stochastic versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Kluwer Academic Publishers, Dordrecht 1990, pp 45-68
- [15] M. Kovács, Linear programming with centered fuzzy numbers, *Annales Univ. Sci. Budapest, Sectio Computatorica*, 12(1991) 159-165.
- [16] A. Kusiak and J. Wang, Dependency analysis in constraint negotiation, *IEEE Transactions on Systems, Man, and Cybernetics*, 25(1995) 1301- 1313.
- [17] H. Rommelfanger, Fuzzy linear programming and applications, *European Journal of Operational Research*, 92(1996) 512-527.
- [18] T.Takagi and M.Sugeno, Fuzzy identification of systems and its applications to modeling and control, *IEEE Trans. Syst. Man Cybernet.*, 1985, 116-132.
- [19] Y. Tsukamoto, An approach to fuzzy reasoning method, in: M.M. Gupta, R.K. Ragade and R.R. Yager eds., *Advances in Fuzzy Set Theory and Applications* (North-Holland, New-York, 1979).
- [20] L.A.Zadeh, The concept of linguistic variable and its applications to approximate reasoning, Parts I,II,III, *Information Sciences*, 8(1975) 199-251; 8(1975) 301-357; 9(1975) 43-80.
- [21] H.-J. Zimmermann, Description and optimization of fuzzy systems, *Internat. J. General Systems* 2(1975) 209-215.
- [22] H.-J.Zimmermann, Fuzzy Mathematical Programming, in: Stuart C. Shapiro ed., *Encyclopedia of Artificial Intelligence*, John Wiley & Sons, Inc., Vol. 1, 1992 521-528.