

Qualitative properties of nonlinear parabolic operators

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Abstract

It is a natural expectation that the mathematical models of real-life phenomena have to possess some characteristic qualitative properties of the original process. For parabolic problems the main known qualitative properties are the maximum-minimum principles, nonnegativity-nonpositivity preservation and maximum norm contractivity. These properties have a fundamental relevance concerning the validity of the mathematical or numerical model: without them, the model might produce unphysical quantities that contradict reality. For linear problems with Dirichlet boundary conditions, these properties have been thoroughly investigated and their relations have been characterized. In the present paper, we extend the linear results to nonlinear problems with general boundary conditions. Firstly, we characterize various implications between the qualitative properties. Some of them are given in general, and in certain cases we restrict our study to operators with gradient-dependent principal part or to operators with heat conduction coefficient. Secondly, we give general sufficient conditions to ensure these qualitative properties, both separately and all of them together. The relations are illustrated with several examples.

Keywords: nonlinear parabolic problems, qualitative properties, maximum principle, numerical solution

1. Introduction

A large number of time-dependent real-life phenomena can be modelled mathematically by parabolic partial differential equations, such as heat conduction, diffusion, air pollution, option pricing, disease propagation [1, 3, 18, 21, 22, 24, 27], to name a few. The qualitative theory of partial differential

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equations came into being as an independent research field in the mid-fifties, e.g. [15, 16]. At that time, researchers turned from the investigation of existence and uniqueness problems to the branch of the theory dealing with other questions, namely with the properties of the solutions. These questions addressed for example different lower and upper bounds for the solutions and the growth and the regularity of the solutions. A comprehensive survey of the qualitative properties of second order linear partial differential equations can be found e.g. in [14]. In the last decades we have seen stunning applications in many other fields (e.g. environmental sciences and population dynamics) of the study of qualitative properties of nonlinear partial differential equations. Some good examples here are designing absorbing boundary conditions for systems of PDEs in fluid dynamics (Burgers' equation, Euler equations for compressible flow, Navier–Stokes equations for incompressible flow), environmental sciences (oceanography and meteorology), medicine (simulation of blood flows in human vascular system), reaction-diffusion systems, see e.g. [2, 7, 8, 11, 20].

In practice, parabolic equations are solved numerically using certain approximation techniques [21, 22, 24, 27]. It is a natural recommendation that both the exact solution and the numerical solution have to reproduce the basic qualitative properties of the original phenomenon. In this context the main properties, which arise generally in many typical situations of the mentioned models, are the *maximum–minimum principles*, the *nonnegativity and nonpositivity preservation* properties and the *maximum norm contractivity* property. Moreover, these qualitative properties have a fundamental relevance concerning the validity of the mathematical or numerical model. Without these properties, the model might produce unphysical quantities that are in conflict with the reality, such as negative concentration or temperature in certain subdomains, or the modelled heat would flow from cold to hot, contradicting the 2nd law of thermodynamics.

For the linear parabolic case the topic of qualitative properties has been widely studied in the authors' papers [4, 5, 6, 12, 13, 27]. In [5], a characterization was given on the relation of qualitative properties on both continuous and discrete level for general linear parabolic operators. The maximum–minimum principle and its variants, the nonnegativity and nonpositivity preservation properties and the maximum norm contractivity property were investigated for problems with Dirichlet boundary conditions. It was shown that for the linear operator, defined below in (4), the maximum principle implies the nonpositivity preservation, and the nonpositivity preservation together with the condition $\mathcal{L}1 = h(x, t) \geq 0$ imply the maximum principle and the maximum norm contractivity property. (The exact definitions of these concepts will be given in the next section). The relations were summarized as shown in Figure 1. Qualitative properties have also been studied for nonlinear operators, and some particular results have been proved. Comparison theorems have been established for a general class of operators in [17] for one space dimension and in [23] for several space dimensions. An Alexandrov maximum principle has been shown in [10], whereas the failure of a strong form of maximum principle was given for the porous medium equation in [26].

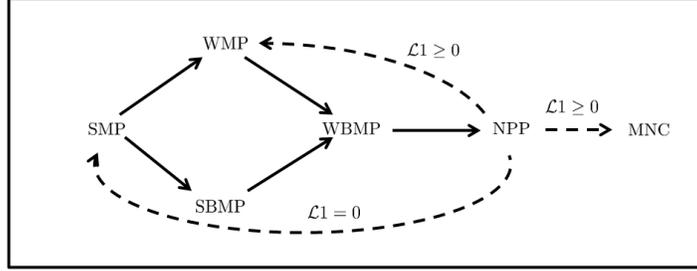


Figure 1: *Connections between the qualitative properties of the linear operator (4).*

In turn, in this paper, our goal is to give a characterization of the connections between the main qualitative properties for a class of nonlinear parabolic problems as general as possible. That is, we wish to extend the above mentioned results of [5], shown in Figure 1, from the linear to the nonlinear case and to general boundary conditions. The structure of the paper is as follows. In Section 2 we define the studied qualitative properties. In Section 3 we present the characterization of connections in three steps: elementary properties are given for the widest class of problems, then deeper connections are shown for two more special classes of operators: one with gradient-dependent principal part and one with heat conduction type coefficient. We also give sufficient conditions that guarantee the qualitative properties separately and all of them together, examples and counterexamples are also given.

2. The investigated qualitative properties

Let Ω be a bounded domain in \mathbb{R}^d and $T > 0$. We consider parabolic operators in the cylinder

$$Q_T := \Omega \times (0, T).$$

Let us decompose the boundary as $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ such that $\partial\Omega_D$ is a relatively closed subset of $\partial\Omega$ (thus $\partial\Omega_N$ is a relatively open subset of $\partial\Omega$) consisting of finite number of components. For the different boundaries we introduce the notations

$$\Gamma_D = \partial\Omega_D \times (0, T), \quad \tilde{\Gamma}_D = \partial\Omega_D \times [0, T], \quad \Gamma_N = \partial\Omega_N \times (0, T), \quad \tilde{\Gamma}_N = \partial\Omega_N \times [0, T],$$

$$\Omega_0 = \Omega \times \{0\}, \quad \Gamma_{par} = \tilde{\Gamma}_D \cup \bar{\Omega}_0,$$

where the last set is called *parabolic boundary*. We will mainly study two different special types of operators, but we can formulate the definitions of the studied properties for the general nonlinear operator

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \operatorname{div}(D(x, t, u, \nabla u)) + q(x, t, u) \quad \text{in } Q_T, \quad (1)$$

where $D : Q_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $q : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ are given sufficiently smooth coefficient functions. We define the domain of the operator \mathcal{N} as

$$\text{dom}(\mathcal{N}) = C^1(\overline{Q}_T) \cap C^{2,1}(Q_T),$$

i.e. as the set of those functions $u : \overline{Q}_T \rightarrow \mathbb{R}$ for which the first time and space derivatives exist and are continuous in \overline{Q}_T and the second space derivatives exist and are continuous in Q_T . (The requirement C^1 up to the boundary is made in order that we may readily include Neumann or mixed boundary conditions.)

Our results will mainly involve two special cases of operator (1): either operators with gradient-dependent principal part:

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \text{div}(K(x, t, \nabla u)) + q(x, t, u), \quad (2)$$

or operators with "heat conduction" coefficient (i.e. depending on u but not on ∇u):

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \text{div}(p(x, t, u)\nabla u) + q(x, t, u). \quad (3)$$

We note that the linear parabolic operator, defined as

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \text{div}(A(x, t)\nabla u) + h(x, t)u \quad \text{in } Q_T \quad (4)$$

(with given coefficients $A \in C(Q_T, \mathbb{R}^{d \times d})$ and $h \in C(Q_T)$), is a special case of the above operators. The domain of definition of this operator is defined as in the previous case.

Definition 1. The nonlinear operator (1) satisfies

(a) the *nonnegativity preservation (NNP)* property if:

$$\mathcal{N}[u] \geq 0, \quad u|_{\Gamma_{par}} \geq 0, \quad (D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0 \quad \Rightarrow \quad u \geq 0 \text{ in } \overline{Q}_T$$

(where ν denotes the outward normal unit vector on the boundary $\partial\Omega_N$);

(b) the *nonpositivity preservation (NPP)* property if:

$$\mathcal{N}[u] \leq 0, \quad u|_{\Gamma_{par}} \leq 0, \quad (D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0 \quad \Rightarrow \quad u \leq 0 \text{ in } \overline{Q}_T;$$

(c) the *weak boundary maximum principle (WBMP)* and the *strong boundary maximum principle (SBMP)*, respectively, if for all $(x, t) \in \overline{Q}_T$ and $u \in \text{dom}(\mathcal{N})$ with $\mathcal{N}[u] \leq 0$:

$$u(x, t) \leq \begin{cases} \max\{0, \max u|_{\Gamma_{par}}\} & \text{(WBMP)}, \\ \max u|_{\Gamma_{par}} & \text{(SBMP)} \end{cases}$$

(in other words, SBMP means that u attains its maximum on the parabolic boundary, and WBMP means the same but only for a nonnegative maximum);

- (d) the *weak boundary minimum principle (WBmP)* and the *strong boundary minimum principle (SBmP)*, respectively, if for all $(x, t) \in \overline{Q}_T$ and $u \in \text{dom}(\mathcal{N})$ with $\mathcal{N}[u] \geq 0$:

$$u(x, t) \geq \begin{cases} \min\{0, \min u|_{\Gamma_{par}}\} & \text{(WBmP)}, \\ \min u|_{\Gamma_{par}} & \text{(SBmP)} \end{cases}$$

(in other words, SBmP means that u attains its minimum on the parabolic boundary, and WBmP means the same but only for a nonpositive minimum);

- (e) the *weak maximum principle (WMP)* and the *strong maximum principle (SMP)*, respectively, if for all $(x, t) \in \overline{Q}_T$ and $u \in \text{dom}(\mathcal{N})$:

$$u(x, t) \leq \begin{cases} t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\} + \max\{0, \max u|_{\Gamma_{par}}\} & \text{(WMP)}, \\ t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\} + \max u|_{\Gamma_{par}} & \text{(SMP)}; \end{cases}$$

(in other words, WMP and SMP complete the bound in WBmP and SBmP, respectively, with a term including $\mathcal{N}[u]$ when the latter has no prescribed sign);

- (f) the *weak minimum principle (WmP)* and the *strong minimum principle (SmP)*, respectively, if for all $(x, t) \in \overline{Q}_T$ and $u \in \text{dom}(\mathcal{N})$:

$$u(x, t) \geq \begin{cases} t \cdot \min\{0, \inf_{Q_T} \mathcal{N}[u]\} + \min\{0, \min u|_{\Gamma_{par}}\} & \text{(WmP)}, \\ t \cdot \min\{0, \inf_{Q_T} \mathcal{N}[u]\} + \min u|_{\Gamma_{par}} & \text{(SmP)}; \end{cases}$$

(in other words, WmP and SmP complete the bound in WBmP and SBmP, respectively, with a term including $\mathcal{N}[u]$ when the latter has no prescribed sign);

- (g) the *maximum norm contractivity (MNC)* property if:

$$\begin{aligned} \mathcal{N}[u] = \mathcal{N}[v] \text{ in } Q_T, \quad u|_{\overline{\Gamma}_D} = v|_{\overline{\Gamma}_D}, \quad (D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} \\ = (D(x, t, u, \nabla v) \cdot \nu)|_{\Gamma_N} \end{aligned}$$

imply

$$\max_{x \in \overline{\Omega}} |u(x, t) - v(x, t)| \leq \max_{x \in \overline{\Omega}} |u(x, 0) - v(x, 0)| \quad \forall t \in [0, T].$$

Remark 1. We have formulated the qualitative properties for operators and not for initial-boundary value problems. The reason for this is that for the linear case in [5] the properties were also formulated for operators, moreover, certain definitions can thus be given in a shorter form. The analogous properties for equations can be formulated in an obvious way, for example, in the case of

the NNP, we say that the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div} (D(x, t, u, \nabla u)) + q(x, t, u) &= f(x, t) \quad \text{in } Q_T, \\ u|_{\Gamma_D} &= g, \\ (D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} &= \gamma, \\ u|_{\Omega_0} &= u_0 \end{aligned}$$

possesses the NNP property if

$$f \geq 0, u_0 \geq 0, g \geq 0, \gamma \geq 0 \quad \Rightarrow \quad u \geq 0.$$

In the sequel we will sometimes consider families of linear operators. Then the following concept will be useful.

Definition 2. Let us consider a family of linear operators of the type (4) with coefficients depending continuously on a parameter $0 \leq s \leq 1$:

$$\mathcal{L}_s u \equiv \frac{\partial u}{\partial t} - \operatorname{div} (A(x, t; s) \nabla u) + h(x, t; s) u \quad \text{in } Q_T, \quad (5)$$

Let \mathcal{P} be a given property. The family of linear operators (5) is called *closed under averaging for property \mathcal{P}* when the following holds: if for all $0 \leq s \leq 1$ the linear operator with coefficients $A(\cdot, \cdot; s)$ and $h(\cdot, \cdot; s)$ possesses property \mathcal{P} , then also the linear operator with coefficients

$$A(x, t) := \int_0^1 A(x, t; s) ds \quad \text{and} \quad h(x, t) := \int_0^1 h(x, t; s) ds, \quad (6)$$

where $\{A(x, t)\}_{kl} := \int_0^1 A_{kl}(x, t; s) ds$, $k, l = 1, \dots, d$, possesses property \mathcal{P} .

3. Relations between the qualitative properties of nonlinear parabolic operators

In this section, we give some important relations between the qualitative properties of parabolic operators. Some relations between the different types of maximum-minimum principles can be formulated for the general operator (1), but in order to obtain further relations, we need to restrict the form of the operator to the previously defined special types (2) and (3).

3.1. General relations

In this section we list the relations that can be formulated for the general operator (1).

Proposition 1. *For the nonlinear operator (1), the strong maximum principles SMP and SBMP imply, respectively, the weak maximum principles WMP, WBMP. The maximum principles SMP and WMP imply, respectively, the boundary maximum principles SBMP and WBMP. Similar statement is true for the minimum principles.*

PROOF. The first statement follows from the trivial relation

$$\max u|_{\Gamma_{par}} \leq \max\{0, \max u|_{\Gamma_{par}}\}.$$

The second statement is also trivial because if $\mathcal{N}[u] \leq 0$ then

$$t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\} = 0.$$

The relations for the minimum principles follow similarly. \square

Proposition 2. *If operator (1) satisfies one of the maximum principles then it also preserves the nonpositivity (NPP). If operator (1) satisfies one of the minimum principles then it also preserves the nonnegativity (NNP).*

PROOF. Each of SMP, WMP and SBMP implies the WBMP property (Theorem 1), and the WBMP property trivially implies the NPP. Similarly, each of SmP, WmP and SBmP implies the WBmP property, and the WBmP property trivially implies the NNP. \square

For linear operators the maximum principles are equivalent with the minimum principles and the nonnegativity preservation property is equivalent with the nonpositivity preservation property. For nonlinear operators this is generally not true, as will be illustrated later in Remark 4. The next proposition presents a sufficient condition for such an equivalence.

Proposition 3. *If the relations*

$$\begin{aligned} D(x, t, -u, -\nabla u) &= -D(x, t, u, \nabla u) \\ q(x, t, -u) &= -q(x, t, u) \end{aligned}$$

are true for the operator (1) for all $u \in \text{dom}(\mathcal{N})$, then the maximum principles are equivalent with the minimum principles, and the NNP property is equivalent with the NPP property.

PROOF. We prove the first statement for the WMP property. The other cases can be proven similarly. Let us suppose that (1) satisfies the WMP, we need to show that it also satisfies the WmP. Let u be an arbitrary function from $\text{dom}(\mathcal{N})$. Then $-u \in \text{dom}(\mathcal{N})$ and the assumptions imply that $\mathcal{N}[-u] = -\mathcal{N}[u]$. Based on the WMP we have

$$\begin{aligned} -u(x, t) &\leq t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[-u]\} + \max\{0, \max(-u|_{\Gamma_{par}})\} \\ &= -t \cdot \min\{0, \inf_{Q_T} \mathcal{N}[u]\} - \min\{0, \min u|_{\Gamma_{par}}\}. \end{aligned}$$

Multiplying by (-1) we obtain that the WmP is satisfied. The opposite direction can be proven similarly.

Now we prove that the NNP property implies the NPP property. Let $u \in \text{dom}(\mathcal{N})$ be a function with the properties $\mathcal{N}[u] \leq 0$, $u|_{\Gamma_{par}} \leq 0$, and

$(D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0$. Then the function $-u$ satisfies the properties $\mathcal{N}[-u] = -\mathcal{N}[u] \geq 0$, $-u|_{\Gamma_{par}} \geq 0$ and

$$(D(x, t, -u, \nabla(-u)) \cdot \nu)|_{\Gamma_N} = -(D(x, t, u, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0.$$

Based on the NNP property we obtain that $-u \geq 0$ on \overline{Q}_T , thus $u \leq 0$. This shows that the NPP property is satisfied. The opposite direction can be proven similarly. \square

3.2. Relations between the qualitative properties of nonlinear parabolic operators with gradient-dependent principal part

3.2.1. The considered type of operators

We first study in detail the operators, defined in (7) with $D(x, t, u, \nabla u) \equiv K(x, t, \nabla u)$ ($K \in C^1(Q_T \times \mathbb{R}^d)$), i.e., the nonlinearity in the principal part depends on ∇u but not directly on u . Recall that thus the operator has the form

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \operatorname{div}(K(x, t, \nabla u)) + q(x, t, u) \quad \text{in } Q_T. \quad (7)$$

The formal linearization of the nonlinear operator (7) at a given function $z = z(x, t)$ is

$$\mathcal{L}'_z u \equiv \frac{\partial u}{\partial t} - \operatorname{div}(K'_\eta(x, t, \nabla z) \nabla u) + q'_\xi(x, t, z)u \quad \text{in } Q_T \quad (8)$$

(cf. (4)). Here the derivatives with respect to η and ξ denote the derivative of the functions K and q with respect to their third arguments, respectively.

3.2.2. Connections between the maximum-minimum principles and the nonnegativity-nonpositivity preservations

Theorem 4. *Let us consider the nonlinear operator (7). If $\xi \mapsto q(x, t, \xi)$ is nondecreasing then the NPP implies the WMP for functions u with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$.)*

PROOF. Assume that operator (7) possesses the NPP. Let u be a function from $\operatorname{dom}(\mathcal{N})$ with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0$. If $\partial\Omega_N = \emptyset$ then u can be any function from $\operatorname{dom}(\mathcal{N})$. Let

$$M_1 := \max\{0, \sup_{Q_T} \mathcal{N}[u]\}, \quad M_2 := \max\{0, \max u|_{\Gamma_{par}}\}.$$

For the WMP to hold, we must prove that

$$u(x, t) \leq M_1 t + M_2 \quad \forall (x, t) \in \overline{Q}_T.$$

Let us define

$$\begin{aligned} \mathcal{T}[u] &:= -\operatorname{div}(K(x, t, \nabla u)) + q(x, t, u), \\ v(x, t) &= u(x, t) - M_1 t - M_2 \quad \forall (x, t) \in Q_T, \end{aligned}$$

where naturally $v \in \text{dom}(\mathcal{N})$. Then

$$\begin{aligned}
\mathcal{N}[v] &= \frac{\partial v}{\partial t} + \mathcal{T}[v] = \frac{\partial u}{\partial t} - M_1 - \text{div}(K(x, t, \nabla v)) + q(x, t, v) \\
&= \frac{\partial u}{\partial t} - M_1 - \text{div}(K(x, t, \nabla u)) + q(x, t, u - M_1 t - M_2) \\
&= \frac{\partial u}{\partial t} - M_1 + \mathcal{T}[u] + q(x, t, u - M_1 t - M_2) - q(x, t, u) \\
&\leq \frac{\partial u}{\partial t} - M_1 + \mathcal{T}[u] = \mathcal{N}[u] - M_1 \leq 0 \quad \text{in } Q_T,
\end{aligned} \tag{9}$$

$$v(x, t) = u(x, t) - M_1 t - M_2 \leq u(x, t) - M_2 \leq 0 \quad \text{for } (x, t) \in \Gamma_{par}, \tag{10}$$

$$K(x, t, \nabla v) \cdot \nu = K(x, t, \nabla u) \cdot \nu \leq 0 \quad \text{for } (x, t) \in \Gamma_N. \tag{11}$$

Thus the nonpositivity property (NPP) for v implies that $v \leq 0$ in \overline{Q}_T , i.e. $u(x, t) \leq M_1 t + M_2$ in \overline{Q}_T as required.

By reversing signs, we obtain in the same way:

Corollary 5. *Let us consider the nonlinear operator (7). If $\xi \mapsto q(x, t, \xi)$ is nondecreasing then the NNP implies the WmP for functions u with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$).*

Remark 2. As a special case, when \mathcal{N} is the linear operator (4) and only Dirichlet boundary condition is considered, then we get back the condition presented in [5]: WmP \Rightarrow NNP, NNP + ($h \geq 0$) \Rightarrow WmP.

Theorem 6. *Let us consider the nonlinear operator (7). If $q(x, t, \xi) \equiv 0$ then the NPP implies the SMP for functions u with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0$, and the NNP implies the SmP for functions u with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$).*

PROOF. The proof for SMP is similar to the proof of Theorem 4, such that we redefine M_2 as $M_2 := \max u|_{\Gamma_{par}}$. Since $q \equiv 0$, we do not need that part of the proof of Theorem 4 where the property $M_2 \geq 0$ was used. The case of SmP is obtained by reversing signs. \square

3.2.3. Condition for maximum norm contractivity

Theorem 7. *Let us consider the nonlinear operator (7). Assume that*

(i) $\xi \mapsto q(x, t, \xi)$ is nondecreasing,

further, the linearized operators (8)

(ii) possess the nonnegativity property (NNP),

(iii) possess the nonpositivity property (NPP), and

(iv) are closed under averaging for both properties NNP and NPP.

Then the nonlinear operator (7) has the maximum norm contractivity (MNC) property.

PROOF. Let us suppose that u and v are two arbitrary functions from $\text{dom}(\mathcal{N})$ with

$$\mathcal{N}[u] = \mathcal{N}[v] \text{ in } Q_T, \quad u|_{\bar{\Gamma}_D} = v|_{\bar{\Gamma}_D}, \quad (K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} = (K(x, t, \nabla v) \cdot \nu)|_{\Gamma_N}.$$

Let $M := \max_{x \in \bar{\Omega}} |u(x, 0) - v(x, 0)|$. Then we must prove that for all $x \in \bar{\Omega}$ and all $t \in [0, T]$, $|u(x, t) - v(x, t)| \leq M$.

We will apply the Newton-Leibniz formula in the following manner. Let us define the matrix-valued functions $K'_{[\hat{\eta}, \tilde{\eta}]} : \Omega \times (0, T) \rightarrow \mathbb{R}^{d \times d}$ as

$$\begin{aligned} K'_{[\hat{\eta}, \tilde{\eta}]}(x, t) &:= \int_0^1 K'_\eta(x, t, s\hat{\eta} + (1-s)\tilde{\eta}) ds \\ &:= \left\{ \int_0^1 \frac{\partial K_k}{\partial \eta_l}(x, t, s\hat{\eta} + (1-s)\tilde{\eta}) ds \right\}_{k, l=1, \dots, d} \end{aligned} \quad (12)$$

where K_k are the components of K , and introduce the notation

$$A(x, t) := K'_{[\nabla u(x, t), \nabla v(x, t)]}(x, t) \quad (x \in \Omega, \quad 0 < t < T). \quad (13)$$

Then, based on the Newton-Leibniz formula, we obtain

$$K(x, t, \nabla u) - K(x, t, \nabla v) = A(x, t)(\nabla u - \nabla v).$$

Similarly, defining

$$q'_{[\hat{\xi}, \tilde{\xi}]}(x, t) := \int_0^1 q'_\xi(x, t, s\hat{\xi} + (1-s)\tilde{\xi}) \quad (x \in \Omega, \quad 0 < t < T)$$

and using notation

$$h(x, t) := q'_{[u(x, t), v(x, t)]}(x, t) \quad (x \in \Omega, \quad 0 < t < T), \quad (14)$$

we have

$$q(x, t, u) - q(x, t, v) = h(x, t)(u - v).$$

Here $h \geq 0$, since $\xi \mapsto q(x, t, \xi)$ is nondecreasing. According to assumptions (ii) – (iii), the operators

$$\mathcal{L}_{su+(1-s)v} w \equiv \frac{\partial w}{\partial t} - \text{div} \left(K'_\eta(x, t, s\nabla u + (1-s)\nabla v) \nabla w \right) + q'_\xi(x, t, su + (1-s)v) w \quad (15)$$

in Q_T possess the NPP and NNP properties. Furthermore, in view of assumption (iv) and applying Definition 2 with

$$A(x, t; s) := K'_\eta(x, t, s\nabla u + (1-s)\nabla v), \quad h(x, t; s) := q'_\xi(x, t, su + (1-s)v),$$

we obtain that the operator

$$Lw \equiv \frac{\partial w}{\partial t} - \operatorname{div} (A(x, t)\nabla w) + h(x, t)w \quad \text{in } Q_T \quad (16)$$

also possess the NPP and NNP properties.

To see the contractivity, it is sufficient to show that

$$\begin{aligned} w^+(x, t) &:= u(x, t) - v(x, t) + M \geq 0 \quad \text{and} \\ w^-(x, t) &:= u(x, t) - v(x, t) - M \leq 0 \quad \forall x \in \bar{\Omega}. \end{aligned} \quad (17)$$

Let us apply operator (16) to the functions w^+ and w^- in Q_T .

$$\begin{aligned} Lw^\pm &= \frac{\partial w^\pm}{\partial t} - \operatorname{div} (A(x, t)\nabla w^\pm) + h(x, t)w^\pm \\ &= \frac{\partial(u-v)}{\partial t} - \operatorname{div} (A(x, t)\nabla(u-v)) + h(x, t)(u-v) \pm h(x, t)M \\ &= \frac{\partial(u-v)}{\partial t} - \operatorname{div} (K(x, t, \nabla u) - K(x, t, \nabla v)) + q(x, t, u) \\ &\quad - q(x, t, v) \pm h(x, t)M \\ &= \mathcal{N}[u] - \mathcal{N}[v] \pm h(x, t)M = \pm h(x, t)M. \end{aligned}$$

This is nonnegative for w^+ and nonpositive for w^- . Moreover

$$\begin{aligned} w^\pm|_{\bar{\Gamma}_D} &= (u - v \pm M)|_{\bar{\Gamma}_D} = \pm M, \\ w^\pm|_{\bar{\Omega}_0} &= (u - v \pm M)|_{\bar{\Omega}_0} = u(\cdot, 0) - v(\cdot, 0) \pm M, \\ (A(x, t)\nabla w^\pm \cdot \nu)|_{\Gamma_N} &= (A(x, t)\nabla(u - v \pm M) \cdot \nu)|_{\Gamma_N} \\ &= ((K(x, t, \nabla u) \cdot \nu - K(x, t, \nabla v) \cdot \nu)|_{\Gamma_N} = 0. \end{aligned}$$

Thus these functions are nonnegative for w^+ and nonpositive for w^- .

Based on the fact that the operator (16) possesses the NNP and NPP properties, it follows that $w^+ \geq 0$ and $w^- \leq 0$. Altogether, we have verified (17), hence the theorem is proved. \square

3.2.4. Summary of the obtained relations

Figure 2 summarizes the obtained relations between the qualitative properties. The relations are given only for the maximum principles. For minimum principles a similar figure can be obtained changing M to m , NPP to NNP and the sign in condition (2). The solid arrows denote implications without any restrictions, while the implications indicated by dashed arrows are valid only by restricting the operator or the function u according to the indicated requirements. We may observe that most of these relations are in a natural analogy with the linear case (recalled in Figure 1), but now the implication of the MNC property follows not from the properties of the original nonlinear operator but from those of the linearized operators.

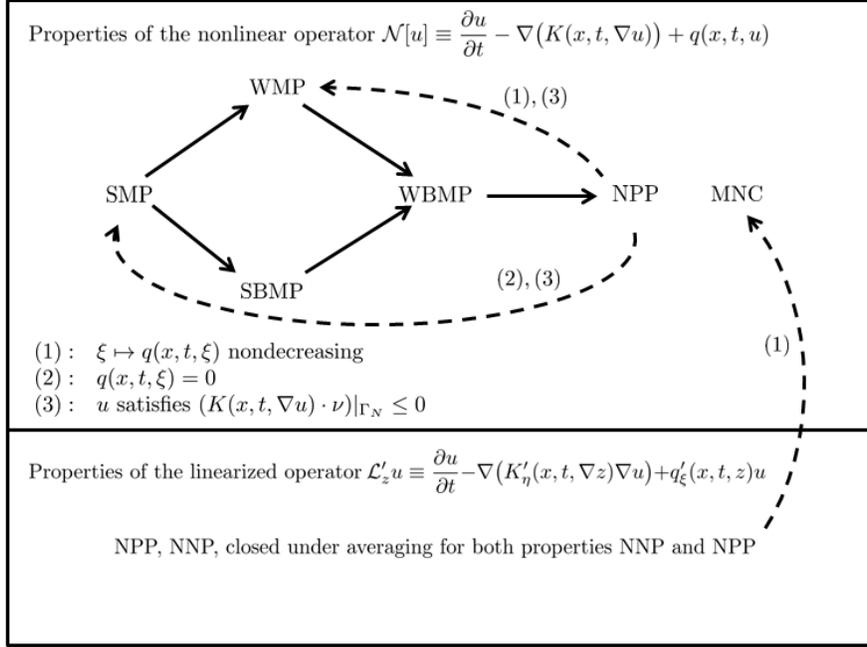


Figure 2: Connections between the qualitative properties of the nonlinear operator (7).

3.2.5. An example in one space dimension

As an example, let us consider the nonlinear operator

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\exp \left(\frac{\partial u}{\partial x} \right) \right) + u^3 + u,$$

where the space domain is the 1D interval $[0, 1]$. We suppose that $\partial\Omega_N = \emptyset$. Let us apply the general comparison theorem (e.g. [17], page 315) to estimate the values of u with its values on the boundary. Here the x -dependent term of $\mathcal{N}[u]$ is

$$-\exp \left(\frac{\partial u}{\partial x} \right) \frac{\partial^2 u}{\partial x^2} + \text{l.o. terms}, \quad \text{where} \quad -\exp \left(\frac{\partial u}{\partial x} \right) < 0,$$

hence by the mentioned comparison theorem the following estimation is true: if $\mathcal{N}[w] \leq \mathcal{N}[u] \leq \mathcal{N}[W]$ in Q_T and $w \leq u \leq W$ on the parabolic boundary Γ_{par} then the relation $w \leq u \leq W$ is true on the whole set \overline{Q}_T . In view of the fact that $\mathcal{N}[0] = 0$ is true, if $u \leq 0$ on Γ_{par} and $\mathcal{N}[u] \leq 0$ then $u \leq 0$ on \overline{Q}_T . This shows that for this operator the NPP property is satisfied (similarly, the NNP is also satisfied). Based on Theorem 4 and the fact that $\xi \mapsto q(x, t, \xi) = \xi^3 + \xi$ is nondecreasing, we obtain that the operator also satisfies the WMP (and similarly the WmP).

The linearized operator of \mathcal{N} is

$$\mathcal{L}_z u \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\exp \left(\frac{\partial z}{\partial x} \right) \frac{\partial u}{\partial x} \right) + (3z^2 + 1)u.$$

In view of the comparison theorem, this operator has both the NNP and NPP properties and it is closed under averaging for both properties NNP and NPP. Thus, by Theorem 7, the operator fulfils the maximum norm contractivity property.

Let us consider now the function $u(x, t) = -x \exp(-t)$. It can be checked easily that this function is nonpositive on the parabolic boundary and $\mathcal{N}[u] = -x^3 \exp(-3t) \leq 0$. The NPP property implies that $u \leq 0$ in \overline{Q}_T , which is trivially true. The MNC and the WMP are also satisfied.

3.2.6. Some classes of operators satisfying the proper qualitative properties

After establishing the above relations between qualitative properties and giving an example where the properties are satisfied, the next natural task is to give fairly wide sufficient conditions under which these properties in fact hold for given operators. As seen above, it suffices to verify the NNP (and/or the NPP, and the closedness under averaging for these properties) in order to derive the other qualitative properties. We follow these steps now for a fairly general class of operators.

Theorem 8. *Let us consider the nonlinear operator (7). Assume that*

- (i) $K(x, t, \eta) \cdot \eta \geq 0 \quad \forall (x, t, \eta) \in Q_T \times \mathbb{R}^n$,
- (ii) *if $\xi \geq 0$ then $q(x, t, \xi) \geq 0 \quad \forall (x, t) \in Q_T$.*

Then operator (7) possesses the nonpositivity property (NPP).

PROOF. Let $u \in \text{dom}(\mathcal{N})$ such that

$$\mathcal{N}[u] \leq 0, \quad u|_{\Gamma_{par}} \leq 0, \quad (K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0.$$

We must prove that $u \leq 0$ in \overline{Q}_T . Letting

$$u^+(x, t) := \max\{0, u(x, t)\},$$

we must prove that

$$u^+ \equiv 0. \tag{18}$$

Here we have $u \in C^1(\overline{Q}_T)$ (by the definition of $\text{dom}(\mathcal{N})$). Thus also $u \in H^1(Q_T)$, which implies $u^+ \in H^1(Q_T)$ (see [9]), hence we may set u^+ as a test function. Since $u^+|_{\Gamma_{par}} = 0$, we thus have

$$\begin{aligned} \int_{\Omega} \mathcal{N}[u] u^+ &= \int_{\Omega} \left(\frac{\partial u}{\partial t} u^+ + K(x, t, \nabla u) \cdot \nabla u^+ + q(x, t, u) u^+ \right) \\ &\quad - \int_{\partial\Omega_N} (K(x, t, \nabla u) \cdot \nu) u^+. \end{aligned}$$

Here $u^+ \geq 0$, hence, using the assumed sign conditions, we have

$$\int_{\Omega} \mathcal{N}[u] u^+ \leq 0, \quad \int_{\partial\Omega_N} (K(x, t, \nabla u) \cdot \nu) u^+ \leq 0.$$

Thus we obtain

$$\int_{\Omega} \left(\frac{\partial u}{\partial t} u^+ + K(x, t, \nabla u) \cdot \nabla u^+ + q(x, t, u) u^+ \right) \leq 0.$$

Since either $u^+ = u$ or $u^+ = 0$ at a fixed point, we can replace u by u^+ in the whole integral, understanding derivatives almost everywhere:

$$\int_{\Omega} \left(\frac{\partial u^+}{\partial t} u^+ + K(x, t, \nabla u^+) \cdot \nabla u^+ + q(x, t, u^+) u^+ \right) \leq 0.$$

However, here our assumptions (i)–(ii) yield

$$K(x, t, \nabla u^+) \cdot \nabla u^+ + q(x, t, u^+) u^+ \geq 0,$$

hence

$$\int_{\Omega} \frac{\partial u^+}{\partial t} u^+ \leq 0. \quad (19)$$

Now let us study the function

$$t \mapsto \int_{\Omega} (u^+(\cdot, t))^2 \quad (20)$$

on the interval $[0, T]$. One can show with elementary analysis that $(u^+)^2 \in C^1(\overline{Q}_T)$, in fact, we only need this w.r.t. variable t . Namely, since $u \in C^1(\overline{Q}_T)$, for points $(x_0, t_0) \in \overline{Q}_T$ where u attains a positive (or negative) value, u^+ coincides with u (or 0, respectively) in a neighbourhood, hence u^+ itself is C^1 there. If $u(x_0, t_0) = 0$ then the property $u \in C^1(\overline{Q}_T)$ implies

$$u^+(x, t)^2 \leq u(x, t)^2 \leq c(|x - x_0|^2 + (t - t_0)^2),$$

hence $\frac{\partial}{\partial t}(u^+)^2(x_0, t_0) = 0$. To show the continuity of $\frac{\partial}{\partial t}(u^+)^2$ at (x_0, t_0) , we note that by the above, $\frac{\partial}{\partial t}(u^+)^2$ exists on all \overline{Q}_T and $|\frac{\partial}{\partial t}(u^+)^2| \leq |\frac{\partial}{\partial t} u^2|$, hence

$$\left| \lim_{(x_0, t_0)} \frac{\partial}{\partial t}(u^+)^2 \right| \leq \left| \lim_{(x_0, t_0)} \frac{\partial}{\partial t} u^2 \right| = 2 \left| \lim_{(x_0, t_0)} \frac{\partial u}{\partial t} u \right| = 0.$$

Based on the above, one can differentiate under the integral in the function (20). Further, since $u^+ \in H_D^1(Q_T)$, we have

$$\frac{\partial}{\partial t}(u^+)^2 = 2 \frac{\partial u^+}{\partial t} u^+$$

almost everywhere. Altogether, also using (19),

$$\frac{\partial}{\partial t} \left(\int_{\Omega} (u^+)^2 \right) = \int_{\Omega} \frac{\partial}{\partial t}(u^+)^2 = 2 \int_{\Omega} \frac{\partial u^+}{\partial t} u^+ \leq 0.$$

That is, the function (20) is nondecreasing, which yields for all $t \geq 0$ that

$$\int_{\Omega} (u^+(\cdot, t))^2 \leq \int_{\Omega} (u^+(\cdot, 0))^2 = 0.$$

The latter fact is due to $u^+(\cdot, 0) \equiv 0$, which follows from assumption $u|_{\Gamma_{par}} \leq 0$. This shows that $u^+ \equiv 0$, i.e. (18) holds. Altogether, NPP is proved. \square

A similar theorem can be formulated for the NNP property as follows.

Theorem 9. *Let us consider the nonlinear operator (7). Assume that*

- (i) $K(x, t, \eta) \cdot \eta \geq 0 \quad \forall (x, t, \eta) \in Q_T \times \mathbb{R}^n$,
- (ii) if $\xi \leq 0$ then $q(x, t, \xi) \leq 0 \quad \forall (x, t) \in Q_T$.

Then operator (7) possesses the nonnegativity property (NNP).

PROOF. The nonnegativity property (NNP) follows in the same way as the property NPP in the previous theorem. Now we let $u \in \text{dom}(\mathcal{N})$ such that

$$\mathcal{N}[u] \geq 0, \quad u|_{\Gamma_{par}} \geq 0, \quad (K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0$$

and must prove that $u \geq 0$ in \overline{Q}_T . Exchanging u^+ by the function

$$u^-(x, t) := \min\{0, u(x, t)\},$$

we must prove that

$$u^- \equiv 0, \tag{21}$$

which goes on in the same way as above by exchanging u^+ by u^- in the proof as well. \square

As a special case of the above, let us consider the linear parabolic operator (4):

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \text{div}(A(x, t)\nabla u) + h(x, t)u \tag{22}$$

which falls into the type (7) with $K(x, t, \eta) = A(x, t)\eta$, $q(x, t, \xi) = h(x, t)\xi$. Clearly, if the matrices $A(x, t)$ are positive semidefinite and $h \geq 0$, then the corresponding K and q satisfy assumptions (i)–(ii) of Theorem 8 and Theorem 9.

Corollary 10. *Let us consider the linear operator (22). Assume that*

- (i) the matrix $A(x, t) \succeq 0$ (i.e. it is positive semidefinite) $\quad \forall (x, t) \in Q_T$,
- (ii) $h(x, t) \geq 0 \quad \forall (x, t) \in Q_T$.

Then the operator (22) possesses both the nonpositivity property (NPP) and the nonnegativity property (NNP).

Theorem 11. *Let us consider the nonlinear operator (7). Assume that*

- (i) $K(x, t, \eta) \cdot \eta \geq 0 \quad \forall (x, t, \eta) \in Q_T \times \mathbb{R}^n$,
- (ii) $\xi \mapsto q(x, t, \xi)$ is nondecreasing $\forall (x, t) \in Q_T$,
- (iii) $q(x, t, 0) \geq 0 \quad \forall (x, t) \in Q_T$.

Then the operator (7) satisfies the WMP for functions $u \in \text{dom}(\mathcal{N})$ satisfying

$$(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0.$$

(There is no restriction on u if $\partial\Omega_N = \emptyset$).

If $q \equiv 0$ then the same is true for SMP instead of WMP.

Finally, under the same assumptions (i)–(ii) and the modified assumption

$$(iii)' \quad q(x, t, 0) \leq 0 \quad (\forall (x, t) \in Q_T),$$

the operator (7) satisfies the WmP for functions $u \in \text{dom}(\mathcal{N})$ satisfying $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \geq 0$ (and there is no restriction on u if $\partial\Omega_N = \emptyset$), and if $q \equiv 0$ then the same is true for SmP instead of WmP.

PROOF. Assumptions (ii)–(iii) imply that for positive ξ we have

$$q(x, t, \xi) \geq q(x, t, 0) \geq 0.$$

Together with assumption (i), we obtain that Theorem 8 can be applied to the operator (7), i.e. it possesses the NPP. Then, again by Assumption (ii) and using Theorem 4, the NPP implies the WMP for functions u with the property $(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0$ (and there is no restriction on u if $\partial\Omega_N = \emptyset$). If $q \equiv 0$, then Theorem 6 yields the SMP for similar functions u . The final similar statements are obtained by reversing signs. \square

Theorem 12. *Let us consider the nonlinear operator (7). Assume that*

- (a) the matrix $K'_\eta(x, t, \eta) \succeq 0$ (i.e. it is positive semidefinite) $\forall (x, t, \eta) \in Q_T \times \mathbb{R}^n$,
- (b) $q'_\xi(x, t, \xi) \geq 0 \quad \forall (x, t, \xi) \in Q_T \times \mathbb{R}$.

Then the operator (7) satisfies the MNC property.

PROOF. Our goal is to apply Theorem 7. In order that the nonlinear operator (7) satisfies the maximum norm contractivity (MNC) property, we must therefore check that

- (i) $\xi \mapsto q(x, t, \xi)$ is nondecreasing,

and the linearized operators (8)

- (ii) possess the nonnegativity property (NNP),
- (iii) possess the nonpositivity property (NPP), and
- (iv) are closed under averaging for both properties NNP and NPP.

Property (i) holds since assumption (b) just ensures that $\xi \mapsto q(x, t, \xi)$ is nondecreasing.

Now we study the linearized operators (8), i.e.

$$\mathcal{L}'_z u \equiv \frac{\partial u}{\partial t} - \operatorname{div} (K'_\eta(x, t, \nabla z) \nabla u) + q'_\xi(x, t, z) u. \quad (23)$$

Properties (ii)-(iii) hold since owing to assumptions (a)–(b), Corollary 10 yields that the operator (23) possesses the nonpositivity property (NPP) and nonnegativity property (NNP).

Property (iv) holds for the following reason. For any family of linearized operators (23), the coefficients $K(x, t, \eta; s)$ and $q(x, t; s)$ satisfy the relations $K'_\eta(x, t, \eta; s) \succeq 0$ and $q'_\xi(x, t, \xi; s) \geq 0$ (since this holds for any arguments of these functions). Hence the averaged coefficient (6) also satisfy these non-negativities, and thus Corollary 10 also applies to the operator with averaged coefficients, i.e. it also possesses NPP and NNP. \square

The above results can be summarized with one set of sufficient conditions:

Corollary 13. *Let us consider the nonlinear operator (7). Assume that*

- (a) $K'_\eta(x, t, \eta) \succeq 0 \quad \forall (x, t, \eta) \in Q_T \times \mathbb{R}^n,$
- (b) $K(x, t, 0) = 0 \quad \forall (x, t) \in Q_T,$
- (c) $q'_\xi(x, t, \xi) \geq 0 \quad \forall (x, t, \xi) \in Q_T \times \mathbb{R},$
- (d) $q(x, t, 0) = 0 \quad \forall (x, t) \in Q_T.$

Then the operator (7) satisfies the the nonpositivity property (NPP), the nonnegativity property (NNP) and the maximum norm contractivity (MNC) property. Further, for functions $u \in \operatorname{dom}(\mathcal{N})$ satisfying

$$(K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0 \quad (\text{or } \geq 0)$$

the operator (7) satisfies the WMP (or WmP, respectively). (There is no restriction on u if $\partial\Omega_N = \emptyset$). If $q \equiv 0$ then the same is true for SMP instead of WMP (or SMP instead of WmP, respectively).

PROOF. Using the Newton–Leibniz theorem, our assumptions (a)–(b) and (c)–(d) imply assumptions (i) and (ii) of Theorem 8, respectively, hence NPP and NNP hold. Further, our assumptions include those of Theorem 12, hence MNC holds too. Finally, our assumptions include (ii)–(iii) of Theorem 11, and again by the Newton–Leibniz theorem, they imply assumption (i) of Theorem 11. Hence, for the given types of functions u , the WMP holds, and if $q \equiv 0$ then the SMP also holds. Finally, the same can be told of WmP and SmP using the opposite signs. \square

Remark 3. In the above theorems one may relax the nonnegativity type conditions on q . First, whenever nonnegativity is assumed for q , one may allow

an at most linearly decreasing negative bound. For instance, in Theorem 8 it suffices to assume, for some constant $\mu > 0$,

$$(ii)' \quad \text{if } \xi \geq 0 \text{ then } q(x, t, \xi) \geq -\mu\xi \quad \forall (x, t) \in Q_T$$

instead of (ii), and the operator (7) still possesses the nonpositivity property (NPP). Namely, let

$$\mathcal{N}[u] \leq 0, \quad u|_{\Gamma_{par}} \leq 0, \quad (K(x, t, \nabla u) \cdot \nu)|_{\Gamma_N} \leq 0,$$

we must prove that $u \leq 0$ in \overline{Q}_T . Let

$$w := e^{-\mu t} u.$$

Let us define the coefficients

$$\tilde{K}(x, t, \eta) := e^{-\mu t} K(x, t, e^{\mu t} \eta), \quad \tilde{q}(x, t, \xi) := e^{-\mu t} q(x, t, e^{\mu t} \xi) + \mu\xi,$$

and operator

$$\tilde{\mathcal{N}}[u] := \frac{\partial u}{\partial t} - \operatorname{div}(\tilde{K}(x, t, \nabla u)) + \tilde{q}(x, t, u).$$

Then $\tilde{\mathcal{N}}$ satisfies the conditions of Theorem 8:

- (i) $\tilde{K}(x, t, \eta) \cdot \eta = e^{-2\mu t} K(x, t, e^{\mu t} \eta) \cdot e^{\mu t} \eta \geq 0 \quad \forall (x, t, \eta) \in Q_T \times \mathbb{R}^n$,
- (ii) if $\xi \geq 0$ then $\tilde{q}(x, t, \xi) = e^{-\mu t} (q(x, t, e^{\mu t} \xi) + \mu e^{\mu t} \xi) \geq 0 \quad \forall (x, t) \in Q_T$.

Hence operator $\tilde{\mathcal{N}}$ possesses the nonpositivity property (NPP). Here

$$\begin{aligned} \tilde{\mathcal{N}}[w] &= \frac{\partial w}{\partial t} - \operatorname{div}(e^{-\mu t} K(x, t, e^{\mu t} \nabla w)) + e^{-\mu t} q(x, t, e^{\mu t} w) + \mu w \\ &= e^{-\mu t} \left(e^{\mu t} \frac{\partial w}{\partial t} + \mu e^{\mu t} w - \operatorname{div}(K(x, t, e^{\mu t} \nabla w)) + q(x, t, e^{\mu t} w) \right) \\ &= e^{-\mu t} \left(\frac{\partial u}{\partial t} - \operatorname{div}(K(x, t, \nabla u)) + q(x, t, u) \right) = e^{-\mu t} \mathcal{N}[u]. \end{aligned}$$

Hence

$$\tilde{\mathcal{N}}[w] = e^{-\mu t} \mathcal{N}[u] \leq 0,$$

further,

$$w|_{\Gamma_{par}} = e^{-\mu t} u|_{\Gamma_{par}} \leq 0, \quad (\tilde{K}(x, t, \nabla w) \cdot \nu)|_{\Gamma_N} = e^{-\mu t} K(x, t, \nabla u) \cdot \nu|_{\Gamma_N} \leq 0,$$

hence by the NPP for $\tilde{\mathcal{N}}$, we have $w \leq 0$ in \overline{Q}_T , which implies

$$u \leq 0 \text{ in } \overline{Q}_T.$$

Similarly, whenever $q'_\xi(x, t, \xi) \geq 0$ is assumed (in particular, in Corollary 13 above), one may allow instead that

$$\text{if } \xi \geq 0 \text{ then } q'_\xi(x, t, \xi) \geq -\mu \quad \forall (x, t) \in Q_T$$

for some constant $\mu > 0$.

Remark 4. (i) In the above result we assumed $K(x, t, 0) = 0$, which does not cover examples like in Subsection 3.2.5. However, if $K(x, t, 0) \neq 0$ but $\operatorname{div} K(x, t, 0) = 0$ and $K(x, t, 0) \cdot \nu \geq 0$ on $\partial\Omega_N$ (for instance, if $K(x, t, \eta) = B(t, \eta)$ is independent of x and $\partial\Omega_N = \emptyset$), then we can replace $K(x, t, \eta)$ by

$$\hat{K}(x, t, \eta) := K(x, t, \eta) - K(x, t, 0)$$

in the operator, since

$$\hat{K}(x, t, 0) = 0$$

and at the same time the operator remains unchanged: owing to $\operatorname{div} K(x, t, 0) = 0$, we have

$$\begin{aligned} \hat{\mathcal{N}}[u] &:= \frac{\partial u}{\partial t} - \operatorname{div}(\hat{K}(x, t, \nabla u)) + q(x, t, u) \\ &= \mathcal{N}[u] := \frac{\partial u}{\partial t} - \operatorname{div}(K(x, t, \nabla u)) + q(x, t, u). \end{aligned}$$

For instance, the operator \mathcal{N} in Subsection 3.2.5 remains unchanged if the term $\exp\left(\frac{\partial u}{\partial x}\right)$ is replaced by $(\exp\left(\frac{\partial u}{\partial x}\right) - 1)$, i.e. the coefficient $K(x, t, \eta) = \exp(\eta)$ is replaced by $\hat{K}(x, t, \eta) = \exp(\eta) - 1$ that already satisfies $\hat{K}(x, t, 0) = 0$.

(ii) For nonlinear parabolic operators in one space dimension, where Ω is an interval, a comparison principle has been derived in [17], which is an extension of the weak maximum principle.

Another type of extension of the weak maximum principle is the existence of invariant rectangles, see [3] for systems in the case of semilinear Dirichlet problems, both on continuous and discrete level.

(iii) For linear problems the NNP and NPP properties are equivalent. For nonlinear operators this only holds for special cases as in Proposition 3, but is not true in general. For instance, let us consider the operator

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(\left(\frac{\partial u}{\partial x} \right)^3 \right) + 10 \exp(u),$$

where $K(x, t, \eta) = \eta^3$ and $q(x, t, \xi) = 10 \exp(\xi)$. Because $K(x, t, \eta) \cdot \eta = \eta^4 \geq 0$ and $q(x, t, \xi) = \exp(\xi) \geq 0$ for positive values of ξ , the NPP is satisfied for the operator (Theorem 8).

Let $u = -t \sin x$ be defined on the domain $Q_T = (0, \pi) \times (0, 1/2)$. Then

$$\begin{aligned} \mathcal{N}[u] &= -\sin x - 3t^3 \cos^2 x \sin x + 10 \exp(-t \sin x) \\ &\geq -1 - \frac{3}{8} + 10 \exp(-1/2) \approx 4.69 \geq 0, \end{aligned}$$

moreover $0 = u|_{\Gamma_{par}} \geq 0$. But the condition $u \geq 0$ is violated, actually $u \leq 0$ is satisfied. This shows that the NNP property is not valid for this operator.

3.2.7. Some further examples

Based on the previous section, we can give some further typical examples or real-life classes of equations for which our results can be applied, in particular, when Corollary 13 holds.

- (i) Certain nonlinear diffusion operators have the form

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \operatorname{div} (a(x, |\nabla u|^2) \nabla u),$$

where $a : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and

$$0 \leq a(x, r^2) \leq \frac{\partial}{\partial r} (a(x, r^2) r).$$

Then it is easy to see that the function

$$K(x, t, \eta) := a(x, |\eta|^2) \eta$$

satisfies

$$K'_\eta(x, t, \eta) \geq 0,$$

and $q \equiv 0$, hence Corollary 13 holds.

- (ii) In particular, one may have a degenerate coefficient similar to the above form in the p -Laplace operator:

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \operatorname{div} (|\nabla u|^{p-2} \nabla u)$$

for some constant $p \geq 2$.

- (iii) Semilinear reaction-diffusion processes involve the operator

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \operatorname{div} (A(x, t) \nabla u) + q(x, t, u)$$

where $A(x, t)$ is a positive definite matrix; $q'_\xi(x, t, \xi) \geq 0$ for autocatalytic reactions or more generally let $q'_\xi(x, t, \xi)$ be bounded from below (see Remark 3); finally, such reactions are commonly described by the so-called mass action type kinetics, which implies that $q(x, t, 0) = 0$ for all x, t . This operator obviously satisfies all conditions of Corollary 13 and thus all the listed qualitative properties as well. For instance, \mathcal{N} often has the form

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \kappa \Delta u + |u|^\beta u$$

for some $\kappa, \beta \geq 0$. In addition, instead of $|u|^\beta u$ we may also allow proper non-monotone nonlinearities based on Remark 3, e.g. $|u|^\beta u - u$, such as in the operator in the Chaffee-Infante equation [3]:

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \kappa \Delta u + u^3 - u.$$

3.3. *Relations between the qualitative properties of nonlinear parabolic operators with heat conduction coefficient*

3.3.1. *The considered type of operators*

As a counterpart of the previous section, we now study operators where the nonlinearity in the principal part depends on u but not on ∇u . Such nonlinearities typically arise in nonlinear heat conduction problems.

Let $p : Q_T \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $q : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ be given coefficients. We will analyze the relations between the qualitative properties of the operator (3). At the present state we cannot give conditions that guarantee the MNC from the NNP and NPP properties. We formulate only relations between the maximum-minimum principles and the nonnegativity-nonpositivity properties.

3.3.2. *Connections between the maximum-minimum principles and the nonnegativity-nonpositivity preservations*

Theorem 14. *Let us consider the nonlinear operator (3). If*

- (i) $p(x, t, \xi) > 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$,
- (ii) $\xi \mapsto q(x, t, \xi)$ is nondecreasing,
- (iii) $q(x, t, 0) = 0$ for all $(x, t) \in Q_T$

then the NPP implies the WMP for functions u with the property $p(x, t, \nabla u) \frac{\partial u}{\partial \nu} \leq 0$ on $\partial\Omega_N$. (There is no restriction on u if $\partial\Omega_N = \emptyset$.)

PROOF. Let the NPP hold for the operator (3), we must prove the WMP. Let $u \in \text{dom}(\mathcal{N})$ be an arbitrary function. Using notation

$$M := \max\{0, \sup_{Q_T} \mathcal{N}[u]\},$$

the desired WMP reads as

$$u(x, t) \leq Mt + \max\{0, \max u|_{\Gamma_{par}}\} \quad \forall (x, t) \in \overline{Q}_T.$$

Assume indirectly that the WMP is violated, i.e. for some $(x_0, t_0) \in \overline{Q}_T$

$$u(x_0, t_0) - Mt_0 = \max(u - Mt)|_{Q_T} > \max\{0, \max u|_{\Gamma_{par}}\}. \quad (24)$$

We proceed in two steps.

(i) Assume that for u the conditions $\mathcal{N}[u] \leq 0$ and $u|_{\Gamma_{par}} \leq 0$ are fulfilled. Then $M = 0$ and (24) becomes

$$u(x_0, t_0) > \max\{0, \max u|_{\Gamma_{par}}\} = 0.$$

This contradicts the relation $u \leq 0$, which means that the NPP is violated too.

(ii) Assume that $\mathcal{N}[u]$ or $u|_{\Gamma_{par}}$ has a positive maximum. We prove that in this case the indirect assumption (24) leads again to a contradiction.

Let us notice that $u(x_0, t_0) - Mt_0 > 0$ since the r.h.s. of (24) is at least zero. We first verify that

$$(x_0, t_0) \in \overline{Q}_T \setminus \Gamma_{par}. \quad (25)$$

In other words, the maximizer (x_0, t_0) is either in the interior of \overline{Q}_T or it satisfies $x_0 \in \partial\Omega_N$ or $t_0 = T$. To see this, let

$$m := \max(u(x, t) - Mt)|_{\Gamma_{par}}.$$

For $(x_0, t_0) \in \overline{Q}_T \setminus \Gamma_{par}$ to hold, we must prove that $u(x_0, t_0) - Mt_0 > m$. First we consider the case when only $\mathcal{N}[u]$ may have a positive maximum but $u|_{\Gamma_{par}} \leq 0$. Then also $u(x, t) - Mt \leq 0$ on Γ_{par} . Hence $m \leq 0$, thus we get the desired relation

$$u(x_0, t_0) - Mt_0 > 0 \geq m.$$

Second, it remains to consider the case when $\max u|_{\Gamma_{par}} > 0$. Then from (24)

$$u(x_0, t_0) - Mt_0 > \max\{0, \max u|_{\Gamma_{par}}\} = \max u|_{\Gamma_{par}} \geq \max(u - Mt)|_{\Gamma_{par}} = m.$$

Altogether, we have thus verified (25).

Now we are able to prove that the indirect assumption (24) leads to a contradiction. As seen before, we must consider three cases for the point (x_0, t_0) where the function $u(x, t) - Mt$ attains its maximum on \overline{Q}_T .

- Let the (positive) maximum of $u(x, t) - Mt$ lie in the interior of \overline{Q}_T , which is Q_T . Then we may assume that this maximum is strictly greater than the maximum of $u(x, t) - Mt$ on ∂Q_T , otherwise we are recast to the remaining cases. In this case there exists $\varepsilon > 0$ such that the function

$$v_\varepsilon(x, t) := u(x, t) - Mt - \varepsilon t$$

also attains its (positive) maximum in Q_T . Denote its maximizer by (x_1, t_1) , i.e.

$$v_\varepsilon(x_1, t_1) = \max_{(x,t) \in Q_T} v_\varepsilon(x, t) > 0,$$

where $0 < t_1 < T$ and x_1 lies in Ω . This is a local maximum, hence

$$\frac{\partial v_\varepsilon}{\partial t}(x_1, t_1) = 0, \quad \nabla v_\varepsilon(x_1, t_1) = \mathbf{0}, \quad \Delta v_\varepsilon(x_1, t_1) \leq 0. \quad (26)$$

Following the idea of [25], now let us expand operator (3):

$$\mathcal{N}[u] = \frac{\partial u}{\partial t} - p(x, t, u) \Delta u - (\nabla p)(x, t, u) \cdot \nabla u - (\partial_u p)(x, t, u) |\nabla u|^2 + q(x, t, u).$$

Since $u = v_\varepsilon + Mt + \varepsilon t$, we obtain

$$\begin{aligned} & \frac{\partial v_\varepsilon}{\partial t} + M + \varepsilon - p(x, t, u) \Delta v_\varepsilon \\ & - (\nabla p)(x, t, u) \cdot \nabla v_\varepsilon - (\partial_u p)(x, t, u) |\nabla v_\varepsilon|^2 + q(x, t, v_\varepsilon + Mt + \varepsilon). \end{aligned}$$

Evaluating at (x_1, t_1) and using (26) and that $p \geq 0$, we have

$$\mathcal{N}[u] \geq M + \varepsilon + q(x_1, t_1, v_\varepsilon(x_1, t_1) + Mt_1 + \varepsilon). \quad (27)$$

Here $\mathcal{N}[u] \leq M$, hence

$$q(x_1, t_1, v_\varepsilon(x_1, t_1) + Mt_1 + \varepsilon) \leq -\varepsilon < 0.$$

On the other hand, $v_\varepsilon(x_1, t_1) + Mt_1 + \varepsilon \geq v_\varepsilon(x_1, t_1) > 0$, hence the monotonicity of q and assumption (iii) imply that

$$q(x_1, t_1, v_\varepsilon(x_1, t_1) + Mt_1 + \varepsilon) \geq q(x_1, t_1, 0) = 0,$$

which is a contradiction.

- Let the (positive) maximum of $u(x, t) - Mt$ lie on the time level $t = T$ but in the interior w.r.t the space domain Ω . Then the above derivation works with minor differences. Namely, then v_ε may have its maximum either in Q_T or for $t_1 = T$. In the first case the proof is the same; in the second case we have $\frac{\partial v_\varepsilon}{\partial t}(x_1, T) \geq 0$ instead of being equal to 0, but this inequality is in the right direction such that the estimate (27) remains true.
- Let the (positive) maximum of $u(x, t) - Mt$ lie on the Neumann space boundary Γ_N at some point (x_1, t_1) . Then the above derivation works again with some differences. Now v_ε may have its maximum either in Q_T or for $x_1 \in \partial\Omega_N$, and in the first case the proof is the same again as firstly. In the second case this is a local maximum w.r.t. $\partial\Omega_N$, since we have assumed at the beginning of the paper that $\partial\Omega_N$ is a relatively open subset of $\partial\Omega$. Thus the directional space derivatives of v_ε tangential to $\partial\Omega$ are zero at (x_1, t_1) , and (since this is a global maximum) the normal derivative $\frac{\partial v_\varepsilon}{\partial \nu}$ is nonnegative at (x_1, t_1) . On the other hand, boundary condition

$$p(x, t, u) \frac{\partial u}{\partial \nu} \leq 0$$

and assumption $p > 0$ yield that $\frac{\partial u}{\partial \nu} \leq 0$, i.e. that $\frac{\partial v_\varepsilon}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0$ at (x_1, t_1) . Hence the gradient of v_ε vanishes at (x_1, t_1) . Similarly, we get $\Delta v_\varepsilon \leq 0$ and (as above) $\frac{\partial v_\varepsilon}{\partial t} \geq 0$ at (x_1, t_1) , thus the proof can be continued as above to obtain the desired contradiction. \square

With a similar proof as for the previous theorem, the following statement can be shown:

Corollary 15. *If*

- (i) $p(x, t, \xi) > 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$,
- (ii) $\xi \mapsto q(x, t, \xi)$ is nondecreasing,
- (iii) $q(x, t, 0) = 0$ for all $(x, t) \in Q_T$

then the NNP implies the WmP for functions u with the property $p(x, t, \nabla u) \frac{\partial u}{\partial \nu} \geq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$.)

In order to give the analogue of Theorem 6, we need the following definition for operators without lower-order terms:

Definition 3. Let us consider the nonlinear operator

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \operatorname{div} (p(x, t, u) \nabla u) \quad \text{in } Q_T. \quad (28)$$

We define the class of shifted coefficient operators as

$$\mathcal{N}^+ := \{\mathcal{N}_r[u] : r \in \mathbb{R}\},$$

where

$$\mathcal{N}_r[u] := \frac{\partial u}{\partial t} - \operatorname{div} (p(x, t, u - r) \nabla u) \quad (\forall r \in \mathbb{R}).$$

We accordingly say that *the NPP holds on the class \mathcal{N}^+* if each operator \mathcal{N}_r satisfies the NPP as in Definition 1, which now means that for all $r \in \mathbb{R}$

$$\mathcal{N}_r[u] \leq 0, \quad u|_{\Gamma_{par}} \leq 0, \quad p(x, t, u - r) \frac{\partial u}{\partial \nu}|_{\Gamma_N} \leq 0 \quad \Rightarrow \quad u \leq 0 \text{ in } \overline{Q}_T.$$

Theorem 16. *Let us consider the nonlinear operator N in (3) with $q(x, t, \xi) \equiv 0$. Then the validity of the NPP on the class \mathcal{N}^+ implies the SMP for functions u with the property $p(x, t, \nabla u) \frac{\partial u}{\partial \nu} \leq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$).*

PROOF. Assume that the NPP holds on the class \mathcal{N}^+ . Let $u \in \operatorname{dom}(\mathcal{N})$ satisfy $p(x, t, \nabla u) \frac{\partial u}{\partial \nu} \leq 0$, we must prove that the SMP

$$u(x, t) \leq t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\} + \max u|_{\Gamma_{par}}$$

holds. If $\max u|_{\Gamma_{par}} \geq 0$ then the desired SMP coincides with the WMP, and the latter holds by Theorem 14 since $\mathcal{N} = \mathcal{N}_0$ itself satisfies the NPP. It remains to consider the case when $\max u|_{\Gamma_{par}} < 0$, i.e. when

$$\exists m > 0 : \quad \max u|_{\Gamma_{par}} = -m.$$

Let

$$w := u + m.$$

By assumption the operator \mathcal{N}_m satisfies the NPP, hence by Theorem 14 \mathcal{N}_m satisfies the WMP as well. Hence, in particular, for the above function w , using that it satisfies

$$p(x, t, \nabla w) \frac{\partial w}{\partial \nu} = p(x, t, \nabla u) \frac{\partial u}{\partial \nu} \leq 0,$$

the WMP yields

$$w(x, t) \leq t \cdot \max\{0, \sup_{Q_T} \mathcal{N}_m[w]\} + \max\{0, \max w|_{\Gamma_{par}}\}.$$

Here

$$\mathcal{N}_m[w] := \frac{\partial w}{\partial t} - \operatorname{div} (p(x, t, w - m) \nabla w) = \frac{\partial u}{\partial t} - \operatorname{div} (p(x, t, u) \nabla u) = \mathcal{N}[u]$$

and

$$\max\{0, \max w|_{\Gamma_{par}}\} = 0,$$

hence

$$w(x, t) \leq t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\}.$$

Adding $-m = \max u|_{\Gamma_{par}}$, we obtain

$$w(x, t) - m \leq t \cdot \max\{0, \sup_{Q_T} \mathcal{N}[u]\} + \max u|_{\Gamma_{par}}.$$

Since $u = w - m$, we have obtained the desired statement. \square

3.3.3. An example in one space dimension

As an example, let us consider the nonlinear operator

$$\mathcal{N}[u] \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left((1 + u^2) \frac{\partial u}{\partial x} \right) + (1 + 3 \exp(-2t))u$$

where the space domain is the 1D interval $[0, 1]$ and we suppose that $\partial\Omega_N = \emptyset$. The general comparison theorem can be applied again to show that for this operator the NPP and NNP properties are satisfied. Based on Theorem 14 and the fact that $\xi \mapsto q(x, t, \xi) = (1 + 3 \exp(-2t))\xi$ is nondecreasing, $q(x, t, 0) = 0$ and $p(x, t, u) = 1 + u^2 > 0$ we obtain that the operator also satisfies the WMP (and similarly the WmP).

Let us consider now the function $u(x, t) = -x \exp(-t)$. It can be checked easily that this function is nonpositive on the parabolic boundary and $\mathcal{N}[u] = -x \exp(-3t) \leq 0$. The NPP property implies that $u \leq 0$ in \bar{Q}_T , which is trivially true. The WMP is also satisfied.

3.3.4. Some classes of heat conduction type operators satisfying the proper qualitative properties

We follow the line of Subsection 3.2.6 to obtain conditions for NPP and NNP, and to derive WMP.

Theorem 17. *Let us consider the nonlinear operator (3). Assume that*

- (i) $p(x, t, \xi) \geq 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$,
- (ii) $q(x, t, \xi)\xi \geq 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$.

Then the operator (3) possesses the nonpositivity property (NPP) and the non-negativity property (NNP).

PROOF. It goes on just similarly to that of Theorem 17. Briefly, for the nonpositivity property (NPP), if $u \in \text{dom}(\mathcal{N})$ such that

$$\mathcal{N}[u] \leq 0, \quad u|_{\Gamma_{par}} \leq 0, \quad (p(x, t, \nabla u) \frac{\partial u}{\partial \nu})|_{\Gamma_N} \leq 0$$

then we must prove that $u \leq 0$ in $\overline{Q_T}$. The divergence theorem now yields

$$\begin{aligned} \int_{\Omega} \mathcal{N}[u] u^+ &= \int_{\Omega} \left(\frac{\partial u}{\partial t} u^+ + p(x, t, u) \nabla u \cdot \nabla u^+ + q(x, t, u) u^+ \right) \\ &\quad - \int_{\partial\Omega_N} p(x, t, \nabla u) \frac{\partial u}{\partial \nu} u^+, \end{aligned}$$

the definition of u^+ and the sign conditions then imply

$$\int_{\Omega} \left(\frac{\partial u^+}{\partial t} u^+ + p(x, t, u^+) |\nabla u^+|^2 + q(x, t, u^+) u^+ \right) \leq 0.$$

Again, our assumptions (i)–(ii) imply that the second and third terms are non-negative, hence

$$\int_{\Omega} \frac{\partial u^+}{\partial t} u^+ \leq 0 \tag{29}$$

which shows that

$$\int_{\Omega} (u^+(\cdot, t))^2 \leq \int_{\Omega} (u^+(\cdot, 0))^2 = 0,$$

hence $u^+ \equiv 0$, i.e. $u \leq 0$. The NNP follows in the same way. \square

Theorem 18. *If*

- (i) $p(x, t, \xi) > 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$,
- (ii) $\xi \mapsto q(x, t, \xi)$ is nondecreasing,
- (iii) $q(x, t, 0) = 0$ for all $(x, t) \in Q_T$

then the operator (3) possesses the WMP for functions u with the property $(p(x, t, \nabla u) \frac{\partial u}{\partial \nu})|_{\Gamma_N} \leq 0$. (There is no restriction on u if $\partial\Omega_N = \emptyset$.)

If $q \equiv 0$ then the same is true for SMP instead of WMP.

PROOF. Assumptions (ii)–(iii) imply (as seen in the proof of Theorem 11) that $q(x, t, \xi) \geq 0$ for all $(x, t, \xi) \in Q_T \times \mathbb{R}$, hence, together with assumptions (i), Theorem 17 yields that the NPP holds. Then Theorem 14 shows that WMP also holds for the given functions u . Finally, if $q \equiv 0$, then we can apply Theorem 16 since if p is positive then each shifted coefficient of the operators in the class \mathcal{N}^+ is also positive. \square

3.3.5. Some further examples

Similarly to the gradient-dependent case, we can give some further typical examples where our results can be applied.

- (i) Nonlinear heat conduction operators typically have the form

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \operatorname{div}(k(x, u) \nabla u),$$

where $k > 0$. Here $q \equiv 0$. In particular, Theorem 18 obviously holds for these operators such that SMP holds.

- (ii) In particular, the operator in the porous medium equation [26] can also be written in the form of the above type:

$$\mathcal{N}[u] := \frac{\partial u}{\partial t} - \Delta(|u|^m) = \frac{\partial u}{\partial t} - \operatorname{div}(m|u|^{m-1}\nabla u).$$

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