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On some properties of a system of nonlinear partial functional differential equations

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Abstract. We consider a system of a semilinear hyperbolic functional differential equation (where the lower order terms contain functional dependence on the unknown function) with initial and boundary conditions and a quasilinear elliptic functional differential equation (containing t as a parameter) with boundary conditions. Existence and some qualitative properties of weak solutions for $t \in (0, \infty)$ are proved.

Keywords: partial functional-differential equations, nonlinear systems of partial differential equations, nonlinear systems of mixed type, qualitative properties.

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1 Introduction

In the present paper we consider weak solutions of the following system of equations:

$$u''(t) + Q(u(t)) + \varphi(x)h'(u(t)) + H(t, x; u, z) + \psi(x)u'(t) = F_1(t, x; z), \tag{1.1}$$

$$-\sum_{j=1}^{n} D_j[a_j(t, x, Dz(t), z(t); u)] + a_0(t, x, Dz(t), z(t); u, z) = F_2(t, x; u), \tag{1.2}$$

$$(t, x) \in Q_T = (0, T) \times \Omega$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and we use the notations $u(t) = u(t, x)$, $u' = D_t u$, $u'' = D_t^2 u$, $z(t) = z(t, x)$, $Dz = \left(\frac{\partial z}{\partial x_1}, \dots \frac{\partial z}{\partial x_n}\right)$, Q may be e.g. a linear second order symmetric elliptic differential operator in the variable x; h is a C^2 function having certain polynomial growth, H contains nonlinear functional (non-local) dependence on u and z, with some polynomial growth and F_1 contains some functional dependence on z. Further, the functions a_j define a quasilinear elliptic differential operator in x (for fixed t) with functional dependence on u for $i = 1, \dots, n$ and on u, z for $i = 0$, respectively. Finally, F_2 may non-locally depend on u. The system (1.1), (1.2) consists of a semilinear hyperbolic functional equation and an elliptic functional equation (containing the time t as a parameter).

This paper was motivated by some problems which were modelled by systems consisting of (functional) differential equations of different types. In [4] S. Cinca investigated a

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model, consisting of an elliptic, a parabolic and an ordinary nonlinear differential equation, which arise when modelling diffusion and transport in porous media with variable porosity. In [6] J. D. Logan, M. R. Petersen and T. S. Shores considered and numerically studied a similar system which describes reaction-mineralogy-porosity changes in porous media with one-dimensional space variable. J. H. Merkin, D. J. Needham and B. D. Sleeman considered in [7] a system, consisting of a nonlinear parabolic and an ordinary differential equation, as a mathematical model for the spread of morphogens with density dependent chemosensitivity. In [3,8,9] the existence of solutions of such systems were studied.

In [12] existence of weak solutions was proved for $t \in (0,T)$. In this paper existence and some qualitative properties of weak solutions for $t \in (0,\infty)$ are proved.

In Section 2 the existence theorem in $(0,T)$ will be formulated and in Section 3 we shall prove existence and certain properties of solutions for $t \in (0,\infty)$.

2 Solutions in $(0,T)$

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain having the uniform C^1 regularity property (see [1]), $Q_T = (0,T) \times \Omega$. Denote by $W^{1,p}(\Omega)$ the Sobolev space of real valued functions with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^n |D_j u|^p + |u|^p\right) dx\right]^{1/p} \qquad \left(2 \le p < \infty, \quad D_j u = \frac{\partial u}{\partial x_j}\right).$$

The number q is defined by $1/p + 1/q = 1$. Further, let $V_1 \subset W^{1,2}(\Omega)$ and $V_2 \subset W^{1,p}(\Omega)$ be closed linear subspaces containing $C_0^\infty(\Omega)$, $V_j^\star$ the dual spaces of V_j, the duality between $V_j^\star$ and V_j will be denoted by $\langle \cdot, \cdot \rangle$, the scalar product in $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. Finally, denote by $L^p(0,T;V_j)$ the Banach space of the set of measurable functions $u : (0,T) \to V_j$ with the norm

$$\|u\|_{L^p(0,T;V_j)} = \left[\int_0^T \|u(t)\|_{V_j}^p dt\right]^{1/p}$$

and $L^\infty(0,T;V_j)$, $L^\infty(0,T;L^2(\Omega))$ the set of measurable functions $u : (0,T) \to V_j$, $u : (0,T) \to L^2(\Omega)$, respectively, with the $L^\infty(0,T)$ norm of the functions $t \mapsto \|u(t)\|_{V_j}$, $t \mapsto \|u(t)\|_{L^2(\Omega)}$, respectively.

First we formulate the existence theorem for $t \in (0,T)$ which was proved in [12], by using the results of [11], the theory of monotone operators (see, e.g., [14,15]) and Schauder's fixed point theorem.

Now we formulate the assumptions on the functions in (1.1), (1.2).

(A_1) $Q : V_1 \to V_1^\star$ is a linear continuous operator such that

$$\langle Qu, v\rangle = \langle Qv, u\rangle, \qquad \langle Qu, u\rangle \ge c_0 \|u\|_{V_1}^2$$

for all $u, v \in V_1$ with some constant $c_0 > 0$.

(A_2) $\varphi, \psi : \Omega \to \mathbb{R}$ are measurable functions satisfying

$$c_1 \le \varphi(x) \le c_2, \qquad c_1 \le \psi(x) \le c_2 \quad \text{for a.a. } x \in \Omega$$

with some positive constants c_1, c_2.

(A_3) $h : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function satisfying

$$h(\eta) \geq 0, \qquad |h''(\eta)| \leq \text{const}|\eta|^{\lambda-1} \quad \text{for } |\eta| > 1 \text{ where}$$
$$1 < \lambda \leq \lambda_0 = \frac{n}{n-2} \quad \text{if } n \geq 3, \qquad 1 < \lambda < \infty \quad \text{if } n = 2.$$

(A_4) $H : Q_T \times L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}$ is a function for which $(t,x) \mapsto H(t,x;u,z)$ is measurable for all fixed $u \in L^2(\Omega)$, $z \in L^p(0,T;V_2)$, H has the Volterra property, i.e. for all $t \in [0,T]$, $H(t,x;u,z)$ depends only on the restriction of u and z to $(0,t)$. Further, the following inequality holds for all $t \in [0,T]$ and $u, u_j \in L^2(\Omega)$, $z \in L^p(0,T;V_2)$:

$$\int_\Omega |H(t,x;u,z)|^2 dx \leq \text{const}\left[\|z\|^2_{L^p(0,T;V_2)} + 1\right]\left[\int_0^t \int_\Omega h(u)dxd\tau + \int_\Omega h(u)dx + 1\right];$$

$$\int_0^t \left[\int_\Omega |H(\tau,x;u_1,z) - H(\tau,x;u_2,z)|^2 dx\right] d\tau \leq M(K,z) \int_0^t \left[\int_\Omega |u_1 - u_2|^2 dx\right] d\tau$$
$$\text{if } \|u_j\|_{L^\infty(0,T;V_1)} \leq K$$

where for all fixed number $K > 0$, $z \mapsto M(K,z) \in \mathbb{R}^+$ is a bounded (nonlinear) operator.

Finally, $(z_k) \to z$ in $L^p(0,T;V_2)$ implies

$$H(t,x;u_k,z_k) - H(t,x;u_k,z) \to 0 \text{ in } L^2(Q_T) \text{ uniformly if } \|u_k\|_{L^2(Q_T)} \leq \text{const}.$$

(A_5) $F_1 : Q_T \times L^p(0,T;V_2) \to \mathbb{R}$ is a function satisfying $(t,x) \mapsto F_1(t,x;z) \in L^2(Q_T)$ for all fixed $z \in L^p(0,T;V_2)$ and $(z_k) \to z$ in $L^p(0,T;V_2)$ implies that $F_1(t,x;z_k) \to F_1(t,x;z)$ in $L^2(Q_T)$.

Further,
$$\int_0^T \|F_1(\tau,x;z)\|^2_{L^2(\Omega)} d\tau \leq \text{const}\left[1 + \|z\|^{\beta_1}_{L^p(0,T;V_2)}\right]$$
with some constant $\beta_1 > 0$.

(B_1) The functions

$$a_j : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \to \mathbb{R} \quad (j = 1, \ldots n),$$
$$a_0 : Q_T \times \mathbb{R}^{n+1} \times L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}$$

are such that $a_j(t,x,\xi;u)$, $a_0(t,x,\xi;u,z)$ are measurable functions of variable $(t,x) \in Q_T$ for all fixed $\xi \in \mathbb{R}^{n+1}$, $u \in L^2(Q_T)$, $z \in L^p(0,T;V_2)$ and continuous functions of variable $\xi \in \mathbb{R}^{n+1}$ for all fixed $u \in L^2(Q_T)$, $z \in L^p(0,T;V_2)$ and a.a. fixed $(t,x) \in Q_T$.

Further, if $(u_k) \to u$ in $L^2(Q_T)$ then for all $z \in L^p(0,T;V_2)$, $\xi \in \mathbb{R}^{n+1}$, a.a. $(t,x) \in Q_T$, for a subsequence

$$a_j(t,x,\xi;u_k) \to a_j(t,x,\xi;u) \qquad (j = 1,\ldots,n),$$
$$a_0(t,x,\xi;u_k,z_k) - a_0(t,x,\xi;u,z_k) \to 0.$$

(B_2) For $j = 1,\ldots,n$
$$|a_j(t,x,\xi;u)| \leq g_1(u)|\xi|^{p-1} + [k_1(u)](t,x),$$
where $g_1 : L^2(Q_T) \to \mathbb{R}^+$ is a bounded, continuous (nonlinear) operator,

$$k_1 : L^2(Q_T) \to L^q(Q_T) \text{ is continuous and } \|k_1(u)\|_{L^q(Q_T)} \leq \text{const}(1 + \|u\|^\gamma_{L^2(Q_T)});$$

$$|a_0(t,x,\xi;u,z)| \le g_2(u,z)|\xi|^{p-1} + [k_2(u,z)](t,x)$$

where

$$g_2 : L^2(Q_T) \times L^p(0,T;V_2) \to \mathbb{R}^+ \quad \text{and} \quad k_2 : L^2(Q_T) \times L^p(0,T;V_2) \to L^q(Q_T)$$

are continuous bounded operators such that

$$\|k_2(u,z)\|_{L^q(Q_T)} \le \text{const}\left[1 + \|u\|^{\gamma}_{L^2(Q_T)}\right]$$

with some constant $\gamma > 0$.

(B_3) The following inequality holds for all $t \in [0,T]$ with some constants $c_2 > 0$, $\beta > 0$ (not depending on t, u):

$$\int_{Q_T} \sum_{j=1}^{n} [a_j(t,x,Dz(t),z(t);u) - a_j(t,x,Dz^\star(t),z^\star(t);u)][D_j z(t) - D_j z^\star(t)]dxdt$$
$$+ \int_{Q_T} [a_0(t,x,Dz(t),z(t);u,z) - a_0(t,x,Dz^\star(t),z^\star(t);u,z^\star)][z(t) - z^\star(t)]dxdt$$
$$\ge \frac{c_2}{1 + \|u\|^{\beta}_{L^2(Q_T)}} \|z - z^\star\|^p_{L^p(0,T;V_2)}.$$

(B_4) For all fixed $u \in L^2(Q_T)$ the function

$$F_2 : Q_T \times L^2(Q_T) \to \mathbb{R} \text{ satisfies } (t,x) \mapsto F_2(t,x;u) \in L^q(Q_T),$$
$$\|F_2(t,x;u)\|_{L^q(Q_T)} \le \text{const}\left[1 + \|u\|^{\gamma}_{L^2(Q_T)}\right]$$

(see (B_2)) and

$$(u_k) \to u \text{ in } L^2(Q_T) \text{ implies } F_2(t,x;u_k) \to F_2(t,x;u) \text{ in } L^q(Q_T).$$

Finally,

$$\frac{\beta_1}{2}\frac{\beta + \gamma}{p-1} < 1.$$

Theorem 2.1. *Assume (A_1)–(A_5) and (B_1)–(B_4). Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$ there exists $u \in L^\infty(0,T;V_1)$ such that*

$$u' \in L^\infty(0,T;L^2(\Omega)), \qquad u'' \in L^2(0,T;V_1^\star) \quad \text{and} \quad z \in L^p(0,T;V_2)$$

such that u,z satisfy (1.1) in the sense: for a.a. $t \in [0,T]$, all $v \in V_1$

$$\langle u''(t), v\rangle + \langle Q(u(t)), v\rangle + \int_\Omega \varphi(x)h'(u(t))v dx + \int_\Omega H(t,x;u,z)v dx + \int_\Omega \psi(x)u'(t)v dx$$
$$= \int_\Omega F_1(t,x;z)v)dx \tag{2.1}$$

and the initial conditions

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{2.2}$$

Further, u,z satisfy (1.2) in the sense: for a.a. $t \in (0,T)$, all $w \in V_2$

$$\int_\Omega \left[\sum_{j=1}^{n} a_j(t,x,Dz(t),z(t);u)\right] D_j w dx + \int_\Omega a_0(t,x,Dz(t),z(t);u,z) w dx$$
$$= \int_\Omega F_2(t,x;u) w dx. \tag{2.3}$$

Remark 2.2. Examples, satisfying the assumptions of Theorem 2.1 can be found in [12].

Main steps of the proof

Now we formulate the main steps in the proof in Theorem 2.1 which will be applied in the next section. (For the detailed proof , see [12].)

Consider the problem (2.1), (2.2) for u with an arbitrary fixed $z = \tilde{z} \in L^p(0,T;V_2)$. According to [11] assumptions (A_1)–(A_5) imply that there exists a unique solution $u = \tilde{u} \in L^\infty(0,T;V_1)$ with the properties $\tilde{u}' \in L^\infty(0,T;L^2(\Omega))$, $\tilde{u}'' \in L^2(0,T;V_1^\star)$ satisfying (2.1) and the initial condition (2.2). Then consider problem (2.3) for z with the above $u = \tilde{u}$. According to the theory of monotone operators (see, e.g., [14, 15]) there exists a unique solution $z \in L^p(0,T;V_2)$ of (2.3). By using the notation $S(\tilde{z}) = z$, it is shown that the operator $S : L^p(0,T;V_2) \to L^p(0,T;V_2)$ satisfies the assumptions of Schauder's fixed point theorem: it is continuous, compact and there exists a closed ball $B_0(R) \subset L^p(0,T;V_2)$ such that

$$S(B_0(R)) \subset B_0(R). \tag{2.4}$$

Then Schauder's fixed point theorem implies that S has a fixed point $z^\star \in L^p(0,T;V_2)$. Defining $u^\star$ by the solution of (2.1), (2.2) with $z = z^\star$, functions $u^\star$, $z^\star$ satisfy (2.1)–(2.3).

Now we formulate some details of the proof which will be used in the next section.

According to [11] the solution $\tilde{u}$ of (2.1), (2.2) with $z = \tilde{z}$ we obtain as the weak limit in $L^p(0,T;V_1)$ of Galerkin approximations

$$\tilde{u}_m(t) = \sum_{l=1}^{m} g_{lm}(t) w_l \quad \text{where} \quad g_{lm} \in W^{2,2}(0,T)$$

and $w_1, w_2, \ldots$ is a linearly independent system in V_1 such that the linear combinations are dense in V_1, further, the functions $\tilde{u}_m$ satisfy (for $j = 1, \ldots, m$)

$$\begin{aligned} &\langle \tilde{u}_m''(t), w_j\rangle + \langle Q(\tilde{u}_m(t)), w_j\rangle + \int_\Omega \varphi(x) h'(\tilde{u}_m(t)) w_j dx \\ &\qquad + \int_\Omega H(t,x;\tilde{u}_m,\tilde{z}) w_j dx + \int_\Omega \psi(x)\tilde{u}_m'(t) w_j dx = \int_\Omega F_1(t,x;\tilde{z}) w_j dx, \end{aligned} \tag{2.5}$$

$$\tilde{u}_m(0) = u_{m0}, \qquad \tilde{u}_m'(0) = u_{m1} \tag{2.6}$$

where u_{m0}, u_{m1} $(m = 1,2,\ldots)$ are linear combinations of $w_1, w_2, \ldots, w_m$, satisfying $(u_{m0}) \to u_0$ in V_1 and $(u_{m1}) \to u_1$ in $L^2(\Omega)$ as $m \to \infty$.

Multiplying (2.5) by $(g_{jm})'(t)$, summing with respect to j and integrating over $(0,t)$, by Young's inequality we find

$$\begin{aligned} &\frac{1}{2}\|\tilde{u}_m'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\langle Q(\tilde{u}_m(t)), \tilde{u}_m(t)\rangle + \int_\Omega \varphi(x) h(\tilde{u}_m(t)) dx \\ &+ \int_0^t \left[\int_\Omega H(\tau,x;\tilde{u}_m,\tilde{z}_k)\tilde{u}_m'(\tau)dx\right] d\tau + \int_0^t \left[\int_\Omega \psi(x)|\tilde{u}_m'(\tau)|^2 dx\right] d\tau \\ &\quad = \int_0^t \left[\int_\Omega F_1(\tau,x;\tilde{z})\tilde{u}_m'(\tau)dx\right] d\tau + \frac{1}{2}\|\tilde{u}_m'(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\langle Q(\tilde{u}_m(0)), \tilde{u}_m(0)\rangle \\ &\qquad + \int_\Omega \varphi(x) h(\tilde{u}_m(0)) dx \le \frac{1}{2}\int_0^T \|F_1(\tau,x;\tilde{z})\|^2_{L^2(\Omega)} d\tau + \frac{1}{2}\int_0^T \|\tilde{u}_m'(\tau)\|^2_{L^2(\Omega)} + \text{const} \end{aligned} \tag{2.7}$$

where the constant is not depending on m, k, t. (See [11].)

By using (A_2), (A_4), (A_5) and the Cauchy–Schwarz inequality, we obtain from (2.7)

$$\frac{1}{2}\|\tilde{u}'_m(t)\|^2_{L^2(\Omega)} d\tau +^2 + \frac{c_0}{2}\|\tilde{u}_m(t)\|^2_{V_1} + c_1 \int_\Omega h(\tilde{u}_m(t))dx \tag{2.8}$$
$$\leq \int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)} d\tau + \text{const}\left\{1 + \int_0^t \|\tilde{u}'_m(\tau)\|^2_{L^2(\Omega)} d\tau + \int_0^t \left[\int_\Omega h(\tilde{u}_m(\tau))dx\right] d\tau\right\}$$

where the constants are not depending on $m, t, \tilde{z}$. Hence, by Gronwall's lemma one obtains

$$\|\tilde{u}'_m(t)\|^2_{L^2(\Omega)} + \int_\Omega h(\tilde{u}_m(t))dx$$
$$\leq \text{const} \int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)} d\tau + \text{const} \int_0^t \left[\int_0^T \left[1 + \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)} d\tau\right] \cdot \mathrm{e}^{t-s}\right] ds$$
$$= \text{const} \int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)} d\tau \tag{2.9}$$

where the constants are independent of $m, t, \tilde{z}$. Thus by (2.8) and (A_5) we find

$$\|\tilde{u}_m(t)\|^2_{V_1} \leq \text{const} \int_0^T \|F_1(\tau, x; \tilde{z})\|^2_{L^2(\Omega)} d\tau \leq \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right]$$

which implies (for the limit of $(\tilde{u}_m)$)

$$\|\tilde{u}\|^2_{L^2(Q_T)} \leq \text{const}\left[1 + \|\tilde{z}\|^{\beta_1}_{L^p(0,T;V_2)}\right]. \tag{2.10}$$

On the other hand, by (B_3), (B_4) we have for the solution z of (2.3) with $u = \tilde{u}$

$$\frac{c_2}{1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)}} \|z\|^p_{L^p(0,T;V_2)}$$
$$\leq \|F_2(t, x; \tilde{u})\|_{L^2(Q_T)} \|z\|_{L^p(0,T;V_2)} + \text{const}\left[\|k_1(\tilde{u})\|_{L^q(Q_T)} + c(\tilde{u})\right] \|z\|_{L^p(0,T;V_2)} \tag{2.11}$$

where the constant is not depending on $\tilde{u}$, further, by (B_2)

$$\|k_1(\tilde{u})\|_{L^q(Q_T)} \leq \text{const}\left[1 + \|\tilde{u}\|^{\gamma}_{L^2(Q_T)}\right] \quad \text{and} \quad c(\tilde{u}) \leq \text{const}\left[1 + \|\tilde{u}\|^{\gamma}_{L^2(Q_T)}\right]. \tag{2.12}$$

The inequalities (2.11), (2.12) imply

$$\|z\|^{p-1}_{L^p(0,T;V_2)} \leq \text{const}\left[1 + \|\tilde{u}\|^{\beta}_{L^2(Q_T)}\right] \cdot \left[\|F_2(t, x; \tilde{u})\|_{L^2(Q_T)} + 1 + \|\tilde{u}\|^{\gamma}_{L^2(Q_T)}\right] \tag{2.13}$$

thus by (2.10) and (B_4)

$$\|z\|_{L^p(0,T;V_2)} \leq \text{const}\left[1 + \|\tilde{u}\|^{\frac{\beta+\gamma}{p-1}}_{L^2(Q_T)}\right] \leq \text{const}\left[1 + \|\tilde{z}\|^{\frac{\beta_1(\beta+\gamma)}{2(p-1)}}_{L^p(0,T;V_2)}\right] \tag{2.14}$$

where the constants are not depending on $\tilde{u}$ and $\tilde{z}$.

According to the assumption (B_4)

$$\frac{\beta_1(\beta + \gamma)}{2(p-1)} < 1, \tag{2.15}$$

so (2.14) implies that there is a closed ball $B_0(R) \subset L^p(0, T; V_2)$ such that $S(B_0(R)) \subset B_0(R)$.

3 Solutions in $(0, \infty)$

Now we formulate an existence theorem with respect to solutions for $t \in (0, \infty)$. Denote by $L^p_{\text{loc}}(0, \infty; V_1)$ the set of functions $u : (0, \infty) \to V_1$ such that for each fixed finite $T > 0$, their restrictions to $(0, T)$ satisfy $u|_{(0,T)} \in L^p(0, T; V_1)$ and let $Q_\infty = (0, \infty) \times \Omega$, $L^\alpha_{\text{loc}}(Q_\infty)$ the set of functions $u : Q_\infty \to \mathbb{R}$ such that $u|_{Q_T} \in L^\alpha(Q_T)$ for any finite T.

Now we formulate assumptions on H, F_1, a_j, F_2.

$(\tilde{A}_4)$ The function $H : Q_\infty \times L^2_{\text{loc}}(Q_\infty) \times L^p_{\text{loc}}(0, \infty; V_2) \to \mathbb{R}$ is such that for all fixed $u \in L^2_{\text{loc}}(Q_\infty)$, $z \in L^p_{loc}(0, \infty; V_2)$ the function $(t, x) \mapsto H(t, x; u, z)$ is measurable, H has the Volterra property (see (A_4)) and for each fixed finite $T > 0$, the restriction H_T of H to $Q_T \times L^2(Q_T) \times L^p(0, T; V_2)$ satisfies (A_4).

Remark 3.1. Since H has the Volterra property, this restriction H_T is well defined by the formula

$$H_T(t, x; \tilde{u}, \tilde{z}) = H(t, x; u, z), \qquad (t, x) \in Q_T, \quad \tilde{u} \in L^2(Q_T), \quad \tilde{z} \in L^p(0, T; V_2)$$

where $u \in L^2_{\text{loc}}(Q_\infty)$, $z \in L^p_{\text{loc}}(0, \infty; V_2)$ may be any functions satisfying $u(t, x) = \tilde{u}(t, x)$, $z(t, x) = \tilde{z}(t, x)$ for $(t, x) \in Q_T$.

$(\tilde{A}_5)$ $F_1 : Q_\infty \times L^p_{\text{loc}}(0, \infty; V_2) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_1 to $(0, T)$ satisfies (A_5).

$(\tilde{B})$ $a_j : Q_\infty \times \mathbb{R}^{n+1} \times L^2_{\text{loc}}(Q_\infty) \to \mathbb{R}$ $(j = 1, \dots, n)$ and $a_0 : Q_\infty \times \mathbb{R}^{n+1} \times L^2_{\text{loc}}(Q_\infty) \times L^p_{\text{loc}}(0, \infty; V_2) \to \mathbb{R}$ have the Volterra property and for each finite $T > 0$, their restrictions to $(0, T)$ satisfy (B_1)–(B_3).

$(\tilde{B}_4)$ $F_2 : Q_\infty \times L^2_{\text{loc}}(Q_\infty) \to \mathbb{R}$ has the Volterra property and for each fixed finite $T > 0$, the restriction of F_2 to $(0, T)$ satisfies (B_4).

Theorem 3.2. *Assume (A_1)–(A_3), $(\tilde{A}_4)$, $(\tilde{A}_5)$, $(\tilde{B})$, $(\tilde{B}_4)$. Then for all $u_0 \in V_1$, $u_1 \in L^2(\Omega)$ there exist*

$$u \in L^\infty_{\text{loc}}(0, \infty; V_1), \qquad z \in L^p_{\text{loc}}(0, \infty; V_2) \quad \textit{such that}$$
$$u' \in L^\infty_{\text{loc}}(0, \infty; L^2(\Omega)), \qquad u'' \in L^2_{\text{loc}}(0, \infty; V_1^\star),$$

(2.1) and (2.3) hold for a.a. $t \in (0, \infty)$ and the initial condition (2.2) is fulfilled.

Assume that the following additional conditions are satisfied: there exist H^∞, $F_1^\infty \in L^2(\Omega)$, $u_\infty \in V_1$, a bounded function $\tilde{\beta}$, belonging to $L^2(0, \infty; L^2(\Omega))$ such that

$$Q(u_\infty) = F_1^\infty - H^\infty, \tag{3.1}$$

$$|H(t, x; u, z) - H^\infty(x)| \le \tilde{\beta}(t, x), \qquad |F_1(t, x; z) - F_1^\infty(x)| \le \tilde{\beta}(t, x) \tag{3.2}$$

for all fixed $u \in L^2_{\text{loc}}(Q_\infty)$, $z \in L^p_{\text{loc}}(0, \infty; V_2)$). Further, there exist functions

$$a_j^\infty : \Omega \times \mathbb{R}^{n+1} \times V_1 \to \mathbb{R}, \qquad j = 1, \dots, n$$
$$a_0^\infty : \Omega \times \mathbb{R}^{n+1} \times V_1 \times V_2 \to \mathbb{R}, \quad F_2^\infty : \Omega \times V_1 \to \mathbb{R}$$

such that for each fixed $z_0 \in V_2$ and $w_0 \in V_1$ with the property

$$\lim_{t \to \infty} \|u(t) - w_0\|_{L^2(\Omega)} = 0,$$

$$\lim_{t\to\infty} \|a_j(t,x,Dz_0,z_0;u) - a_j^\infty(x,Dz_0,z_0;w_0)\|_{L^q(\Omega)} = 0, \qquad j=1,\dots,n, \tag{3.3}$$

$$\lim_{t\to\infty} \|a_0(t,x,Dz_0,z_0;u,z_0) - a_0^\infty(x,Dz_0,z_0;w_0,z_0)\|_{L^q(\Omega)} = 0, \tag{3.4}$$

$$\lim_{t\to\infty} \|F_2(t,x;u) - F_2^\infty(x;w_0)\|_{L^q(\Omega)} = 0. \tag{3.5}$$

Finally, (B_3) is satisfied such that the following inequalities hold for all $t > 0$ with some constants $c_2 > 0$, $\beta > 0$ (not depending on t):

$$\begin{aligned} &\int_\Omega \sum_{j=1}^n [a_j(t,x,Dz(t),z(t);u) - a_j(t,x,Dz^\star(t),z^\star(t);u)][D_j z - D_j z^\star]dx \\ &+ \int_\Omega [a_0(t,x,Dz(t),z(t);u,z) - a_0(t,x,Dz^\star(t),z^\star(t);u,z^\star)][z(t)-z^\star(t)]dx \\ &\geq \frac{c_2}{1+\|u\|^\beta_{L^2(Q_t\setminus Q_{t-a})}} \|z-z^\star\|^p_{V_1} \end{aligned} \tag{3.6}$$

with some fixed $a > 0$ (finite delay).

Then for any solution u, z of (2.1)–(2.3) in $(0,\infty)$ we have

$$u \in L^\infty(0,\infty;V_1), \tag{3.7}$$

$$\|u'(t)\|_H \leq \text{const}\, e^{-c_1 T} \tag{3.8}$$

where c_1 is given in (A_2) and there exists $w_0 \in V_1$ such that

$$u(T) \to w_0 \text{ in } L^2(\Omega) \text{ as } T\to\infty, \qquad \|u(T)-w_0\|_H \leq \text{const}\, e^{-c_1 T} \tag{3.9}$$

and w_0 satisfies

$$Q(w_0) + \varphi h'(w_0) = F_1^\infty - H^\infty. \tag{3.10}$$

Finally, there exists a unique solution $z_0 \in V_2$ of

$$\begin{aligned} &\sum_{j=1}^n \int_\Omega a_j^\infty(x,Dz_0,z_0;w_0) D_j v dx + \int_\Omega a_0^\infty(x,Dz_0,z_0;w_0,z_0) v dx \\ &= \int_\Omega F_2^\infty(x;w_0) v dx \quad \text{for all } v \in V_2 \end{aligned} \tag{3.11}$$

(where w_0 is the solution of (3.10)) and

$$\lim_{t\to\infty} \|z(t) - z_0\|_{V_2} = 0. \tag{3.12}$$

Proof. Let $(T_k)_{k\in\mathbb{N}}$ be a monotone increasing sequence, converging to $+\infty$. According to Theorem 2.1, there exist solutions u_k, z_k of (2.1)–(2.3) for $t \in (0,T_k)$. The Volterra property of H, F_1, a_j, F_2 implies that the restrictions of u_k, z_k to $t \in (0,T_l)$ with $T_l < T_k$ satisfy (2.1)–(2.3) for $t \in (0,T_l)$.

Now consider the restrictions $u_k|_{(0,T_1)}$, $z_k|_{(0,T_1)}$, $k = 2,3,\dots$ Applying (2.14) to $T = T_1$ and $z = \tilde{z} = z_k|_{(0,T_1)}$, by (2.15) we obtain that the sequence

$$\left(z_k|_{(0,T_1)}\right)_{k\in\mathbb{N}} \text{ is bounded in } L^p(0,T_1;V_2). \tag{3.13}$$

The operator $S : L^p(0,T_1;V_2) \to L^p(0,T_1;V_2)$ is compact thus there is a subsequence $(z_{1k})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that the sequence of restrictions $(z_{1k}|_{(0,T_1)})_{k\in\mathbb{N}}$ is convergent in $L^p(0,T_1;V_2)$.

Now consider the restrictions $z_{1k}|_{(0,T_2)}$ By using the above arguments, we find that there exists a subsequence $(z_{2k})_{k\in\mathbb{N}}$ of $(z_{1k})_{k\in\mathbb{N}}$ such that $(z_{2k}|_{(0,T_2)})_{k\in\mathbb{N}}$ is convergent in $L^p(0,T_2;V_2)$.

Thus for all $l\in\mathbb{N}$ we obtain a subsequence $(z_{lk})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ such that $(z_{lk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(0,T_l;V_2)$. Then the diagonal sequence $(z_{kk})_{k\in\mathbb{N}}$ is a subsequence of $(z_k)_{k\in\mathbb{N}}$ such that for all fixed $l\in\mathbb{N}$, $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is convergent in $L^p(0,T_l;V_2)$ to some $z^\star\in L^p_{\mathrm{loc}}(0,\infty;V_2)$. Since z_{ll} is a fixed point of $S=S_l: L^p(0,T_l;V_2)\to L^p(0,T_l;V_2)$ and S_l is continuous thus the limit $z^\star|_{(0,T_l)}$ in $L^p(0,T_l;V_2)$ of $(z_{kk}|_{(0,T_l)})_{k\in\mathbb{N}}$ is a fixed point of $S=S_l$.

Consequently, the solutions $u_l^\star$ of (2.1), (2.2) when z is the restriction of $z^\star$ to $(0,T_l)$ and the restriction of $z^\star$ to $(0,T_l)$ satisfy (2.1)–(2.3) for $t\in(0,T_l)$. Since for $m<l$, $u_l^\star|_{(0,T_m)}=u_m^\star$ (by the Volterra property of H, F_1, a_j, F_2), we obtain $u^\star\in L^2_{loc}(Q_\infty)$ such that for all fixed l, $u^\star|_{(0,T_l)}$, $z^\star|_{(0,T_l)}$ satisfy (2.1)–(2.3) for $t\in(0,T_l)$, so the first part of Theorem 3.2 is proved.

Now assume that the additional conditions (3.1), (3.2) are satisfied. Then we obtain (3.7)–(3.10) for $u=u^\star$, $z=z^\star$ by using the arguments of the proof of Theorem 3.2 in [11]. For convenience we formulate the main steps of the proof.

Let u, z be arbitrary solutions of (2.1)–(2.3) for $t\in(0,\infty)$ and $z_{kk}=z|_{(0,T_k)}$, $u_{kk}=u|_{(0,T_k)}$. Then z_{kk}, u_{kk} are solutions of (2.1)–(2.3) for $t\in(0,T_l)$ if $k\ge l$, hence the sequence $(z_{kk})|_{k\in\mathbb{N}}$ is bounded in $L^p(0,T_l;V_2)$ for each fixed l (see, e.g., (3.13)), consequently, from (2.7) (with $\tilde z_k=z_{kk}$) we obtain for the solutions u_{kk} of (2.1), (2.2) with $\tilde z=z_{kk}$ (since u_{kk} is the limit of the Galerkin approximations)

$$\begin{aligned}
&\frac{1}{2}\|u'_{kk}(t)\|_H^2+\frac{1}{2}\langle Q(u_{kk}(t)),u_{kk}(t)\rangle+\int_\Omega\varphi(x)h(u_{kk}(t))dx\\
&+\int_0^t\left[\int_\Omega\psi(x)|u'_{kk}(\tau)|^2dx\right]d\tau+\int_0^t\left[\int_\Omega H(\tau,x;u_{kk},z_{kk})u'_{kk}(\tau)dx\right]d\tau\\
&\quad=\int_0^t\left[\int_\Omega F_1(\tau,x;z_{kk})u'_{kk}(\tau)dx\right]d\tau+\frac{1}{2}\|u'_{kk}(0)\|_H^2+\frac{1}{2}\langle Q(u_{kk}(0)),u_{kk}(t)\rangle\\
&\qquad+\int_\Omega\varphi(x)h(u_{kk}(0))dx
\end{aligned}\tag{3.14}$$

for all $t>0$. Hence we find by (3.1), (3.2) and Young's inequality for $w_{kk}=u_{kk}-u_\infty$

$$\begin{aligned}
&\frac{1}{2}\|w'_{kk}(t)\|^2_{L^2(\Omega)}+\frac{c_0}{2}\|u_{kk}(t))\|^2_{V_1}+c_1\int_\Omega h(u_{kk}(t))dx+\mathrm{const}\int_0^t\left[\int_\Omega|w'_{kk}|^2dx\right]d\tau\\
&\le\mathrm{const}\left\{\int_0^t\|F_1(\tau,x;z_{kk})-F_1^\infty\|^2_{L^2(\Omega)}d\tau+\int_0^t\|H(\tau,x;u_{kk}z_{kk})-H^\infty\|^2_{L^2(\Omega)}d\tau\right\}\\
&\quad+\varepsilon\int_0^t\left[\int_\Omega|w'_{kk}|^2dx\right]d\tau+\frac{1}{2}\|u'_{kk}(0)\|^2_{L^2(\Omega)}+\frac{1}{2}\langle Q(u_{kk}(0)),u_{kk}(0)\rangle+c_2\int_\Omega h(u_{kk}(0))dx\\
&\le\varepsilon\int_0^t\left[\int_\Omega|w'_{kk}|^2dx\right]d\tau+\mathrm{const}+C(\varepsilon)\|\tilde\beta\|^2_{L^2(0,\infty;L^2(\Omega))}.
\end{aligned}\tag{3.15}$$

Choosing sufficiently small $\varepsilon>0$, we obtain

$$\int_0^t\left[\int_\Omega|w'_{kk}|^2dx\right]d\tau\le\mathrm{const}\tag{3.16}$$

and thus by (3.15)

$$\|u'_{kk}(t)\|^2_{L^2(\Omega)}+\tilde c\int_0^t\|u'_{kk}(\tau)\|^2_{L^2(\Omega)}d\tau\le c^\star$$

with some positive constants $\tilde c$ and $c^\star$ not depending on k and $t\in(0,\infty)$. Hence by Gronwall's lemma we obtain (3.8) for the weak limit of the sequence (u_{kk}) and by (3.15) we find (3.7).

It is not difficult to show that

$$\|u(T_2) - u(T_1)\|_{L^2(\Omega)} \le \int_{T_1}^{T_2} \|u'(t)\|_{L^2(\Omega)} dt \tag{3.17}$$

(see [11]), thus (3.8) implies (3.9) and by $u \in L^\infty(0,\infty;V_1)$, the limit w_0 of $u(t)$ as $t \to \infty$ must belong to V_1.

In order to prove (3.10) we apply equation (1.1) to $v\chi_{T_k}(t)$ with arbitrary fixed $v \in V_1$ where $\lim_{k\to\infty}(T_k) = +\infty$ and

$$\chi_{T_k}(t) = \chi(t - T_k), \quad \chi \in C_0^\infty, \quad \operatorname{supp}\chi \subset [0,1], \quad \int_0^1 \chi(t)dt = 1.$$

Then by (3.8) one obtains (3.10) as $k \to \infty$.

Now we show that there exists a unique solution $z_0 \in V_2$ of (3.11). This statement follows from the fact that the operator (applied to $z_0 \in V_2$) on the left-hand side of (3.11) is bounded, demicontinuous and uniformly monotone (see, e.g. [14,15]) by (B_1), (B_2), (3.9) (3.3), (3.4), (3.6).

Finally, we show (3.12). By (3.6) we have

$$\begin{aligned}
&\frac{c_2}{1 + \|u\|_{L^2(Q_t\setminus Q_{t-a})}} \|z(t) - z_0\|_{V_2}^p \\
&\le \int_\Omega \sum_{j=1}^n [a_j(t,x,Dz,z;u) - a_j(t,x,Dz_0,z_0;u)](D_jz - D_jz_0)dx \\
&\quad + \int_\Omega [a_0(t,x,Dz,z;u,z) - a_0(t,x,Dz_0,z_0;u,z_0)](z - z_0)dx \\
&= \int_\Omega [F_2(t,x;u) - F_2^\infty(x,w_0)](z - z_0)dx \\
&\quad - \int_\Omega \sum_{j=1}^n [a_j(t,x,Dz_0,z_0;u) - a_j^\infty(x,Dz_0,z_0;w_0)](D_jz - D_jz_0)dx \\
&\quad - \int_\Omega [a_0(t,x,Dz_0,z_0;u,z_0) - a_0^\infty(t,x,Dz_0,z_0;w_0,z_0)](z - z_0)dx \\
&\le \|F_2(t,x;u) - F_2^\infty(x,w_0)\|_{L^q(\Omega)} \|z(t) - z_0\|_{L^p(\Omega)} \\
&\quad + \sum_{j=1}^n \|a_j(t,x,Dz_0,z_0;u) - a_j^\infty(x,Dz_0,z_0;w_0)\|_{L^q(\Omega)} \|D_jz(t) - D_jz_0\|_{L^p(\Omega)} \\
&\quad + \|a_0(t,x,Dz_0,z_0;u,z_0) - a_0^\infty(x,Dz_0,z_0;w_0,z_0)\|_{L^q(\Omega)} \|z(t) - z_0\|_{L^p(\Omega)}.
\end{aligned} \tag{3.18}$$

Since $p > 1$ and $\|u\|_{L^2(Q_t\setminus Q_{t-a})}^\beta$ is bounded for $t \in (0,\infty)$ by (3.9), thus (3.3)–(3.5), (3.18) imply (3.12).

Remark 3.3. Assume that the inequalities (3.3)–(3.5) hold such that for $j = 1,\dots,n$

$$|a_j(t,x,\xi;u) - a_j^\infty(x,\xi;u)| \le \text{const}\left[\|u(t) - w_0\|_{L^p(Q_t\setminus Q_{t-a})} + \eta(t)\right]\left[1 + |\xi|^{p-1}\right],$$
$$|a_0(t,x,\xi;u,z_0) - a_0^\infty(x,\xi;u,z)| \le \text{const}\left[\|u(t) - w_0\|_{L^p(Q_t\setminus Q_{t-a})} + \eta(t)\right]\left[1 + |\xi|^{p-1}\right],$$
$$|F_2(t,x;u) - F_2^\infty(x;w_0)| \le \text{const}\left[\|u(t) - w_0\|_{L^p(q_t\setminus Q_{t-a})} + \eta(t)\right].$$

Then

$$\|z(t) - z_0\|_{V_2}^{p-1} \le \text{const}\left[e^{-c_1 t} + \eta(t)\right], \qquad t > 0.$$

The above inequalities are satisfied e.g. if

$$a_j(t,x,\xi;u) = g_j(x,\xi)\left[1 + \int_{t-a}^{t} |u(\tau,x)|d\tau + \eta(t)\right], \qquad j = 1,\dots,n$$

$$a_0(t,x,\xi;u,z) = g_0(x,\xi)\left[1 + \int_{t-a}^{t} |u(\tau,x)|d\tau + \eta(t)\right]$$

where

$$|g_j(x,\xi)| \le \text{const}[|\xi|^{p-1} + \tilde{g}(x)], \qquad \tilde{g} \in L^q(\Omega), \qquad \eta \ge 0,\ \lim_{\infty} \eta = 0,$$

$$\sum_{j=0}^{n} [g_j(x,\xi) - g_j(x,\xi^\star)](\xi_j - \xi_j^\star) \ge c_2|\xi - \xi^\star|^p$$

with some constant $c_2 > 0$. □

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