On fuzzy reasoning schemes *

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Abstract

In this work we provide a short survey of the most frequently used fuzzy reasoning schemes. The paper is organized as follows: in the first section we introduce the basic notations and definitions needed for fuzzy inference systems; in the second section we explain how the GMP works under Mamdani, Larsen and Gödel implications, furthermore we discuss the properties of compositional rule of inference with several fuzzy implications; and in the third section we describe Tsukamoto's, Sugeno's and the simplified fuzzy inference mechanisms in multi-input-single-output fuzzy systems.

1 FUZZY SETS AND LOGIC

The use of fuzzy logic and fuzzy reasoning methods are becoming more and more popular in intelligent information systems [30, 31], especially in hyperknowledge support systems [6, 7, 8]; knowledge formation processes in knowledge-based systems [26]; active decision support systems [1, 3, 4, 5, 27]; medical support systems [9, 11, 12, 13, 14, 15, 18, 23]; robotics [16]; financial analysis [2]; control [19, 20, 28, 29] and pattern recognition [22].

Fuzzy sets were introduced by Zadeh [32] as a means of representing and manipulating data that was not precise, but rather fuzzy.

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Definition 1.1 *Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$\mu_A: X \to [0,1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$. Frequently we will write simply $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy (sub)sets in X is denoted by $\mathcal{F}(X)$. The degree to which the statement "x is A" is true is defined as $A(x)$ - the degree of membership of x in A.

The use of fuzzy sets provides a basis for a systematic way for the manipulation of vague and imprecise concepts. In particular, we can employ fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.

Definition 1.2 *A linguistic variable is characterized by a quintuple*

$$(x, T(x), U, G, M)$$

in which x is the name of variable; $T(x)$ is the term set of x, that is, the set of names of linguistic values of x with each value being a fuzzy number defined on U; G is a syntactic rule for generating the names of values of x; and M is a semantic rule for associating with each value its meaning.

For example, if *speed* is interpreted as a linguistic variable, then its term set T (speed) could be

$$T = \{\text{slow, moderate, fast, very slow, more or less fast, sligthly slow}, \dots\}$$

where each term in T (speed) is characterized by a fuzzy set in a universe of discourse $U = [0, 100]$. We might interpret

- *slow* as "a speed below about 40 mph"
- *moderate* as "a speed close to 55 mph"
- *fast* as "a speed above about 70 mph"

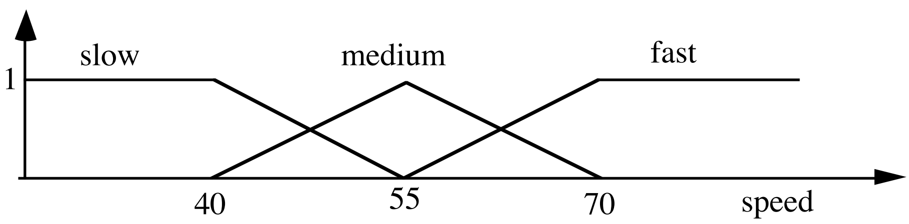

Figure 1: Values of linguistic variable *speed*.

These terms can be characterized as fuzzy sets whose membership functions are

$$\mathrm{slow}(v) = \begin{cases} 1 & \text{if } v \le 40 \\ 1 - (v - 40)/15 & \text{if } 40 \le v \le 55 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{moderate}(v) = \begin{cases} 1 - |v - 55|/30 & \text{if } 40 \le v \le 70 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{fast}(v) = \begin{cases} 1 & \text{if } v \ge 70 \\ 1 - (70 - v)/15 & \text{if } 55 \le v \le 70 \\ 0 & \text{otherwise} \end{cases}$$

In many practical applications we normalize the domain of inputs and use the following type of fuzzy partition: NB (Negative Big), NM (Negative Medium), NS (Negative Small), ZE (Zero), PS (Positive Small), PM (Positive Medium), PB (Positive Big).

Triangular norms were introduced by Schweizer and Sklar [24] to model the distances in probabilistic metric spaces. In fuzzy sets theory triangular norms are extensively used to model logical connective *and*. Triangular conorms are extensively used to model logical connective *or*.

Definition 1.3 *A mapping*

$$T\colon [0,1] \times [0,1] \to [0,1]$$

is a triangular norm (t-norm for short) iff it is symmetric, associative, non-decreasing in each argument and $T(a,1) = a$*, for all* $a \in [0,1]$.

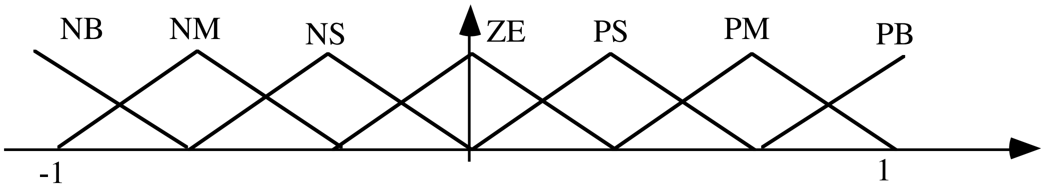

Figure 2: A usual fuzzy partition of $[-1, 1]$.

Definition 1.4 *A mapping*

$$S: [0, 1] \times [0, 1] \to [0, 1]$$

is a triangular co-norm (t-conorm for short) if it is symmetric, associative, non-decreasing in each argument and $S(a, 0) = a$, *for all* $a \in [0, 1]$.

The three basic t-norms and t-conorms pairs are

- minimum/maximum:

$$MIN(a, b) = \min\{a, b\} = a \wedge b, \quad MAX(a, b) = \max\{a, b\} = a \vee b$$

- Łukasiewicz:

$$LAND(a, b) = \max\{a + b - 1, 0\}, \quad LOR(a, b) = \min\{a + b, 1\}$$

- probabilistic: $PAND(a, b) = ab, \quad POR(a, b) = a + b - ab$

We can extend the classical set theoretic operations from ordinary set theory to fuzzy sets. We note that all those operations which are extensions of crisp concepts reduce to their usual meaning when the fuzzy subsets have membership degrees that are drawn from $\{0, 1\}$. For this reason, when extending operations to fuzzy sets we use the same symbol as in set theory. Let A and B are fuzzy subsets of a nonempty (crisp) set X.

Definition 1.5 *The intersection of* A *and* B *is defined as*

$$(A \cap B)(t) = T(A(t), B(t)) = A(t) \wedge B(t),$$

where T *is a t-norm. If* $T = \min$ *then we get*

$$(A \cap B)(t) = \min\{A(t), B(t)\},$$

for all $t \in X$.

If p is a proposition of the form "u is A" where A is a fuzzy set, for example, "big pressure" and q is a proposition of the form "v is B" for example, "small volume" then the membership function of the fuzzy implication $A \to B$ is defined *pointwise* as

$$(A \to B)(u, v) = I(A(u), B(v))$$

where I is properly chosen function. We shall use the notation

$$(A \to B)(u, v) = A(u) \to B(v).$$

In our interpretation $A(u)$ is considered as the truth value of the proposition "u is big pressure", and $B(v)$ is considered as the truth value of the proposition "v is small volume".

There are three important classes of fuzzy implication operators:

- S**-implications**: defined by

  $$x \to y = S(n(x), y)$$

  where S is a t-conorm and n is a negation on $[0, 1]$. These implications arise from the Boolean formalism $p \to q = \neg p \vee q$. Typical examples of S-implications are the Łukasiewicz and Kleene-Dienes implications.

- R**-implications**: obtained by residuation of continuous t-norm T, i.e.

  $$x \to y = \sup\{z \in [0, 1] \mid T(x, z) \le y\}$$

  These implications arise from the *Intutionistic Logic* formalism. Typical examples of R-implications are the Gödel and Gaines implications.

- **t-norm implications**: if T is a t-norm then

  $$x \to y = T(x, y)$$

  Although these implications do not verify the properties of material implication they are used as model of implication in many applications of fuzzy logic. Typical examples of t-norm implications are the Mamdani ($x \to y = \min\{x, y\}$) and Larsen ($x \to y = xy$) implications.

The most often used fuzzy implication operators are listed in the following table.

Name	Definition
Łukasiewicz	$x \to y = \min\{1, 1 - x + y\}$
Mamdani	$x \to y = \min\{x, y\}$
Larsen	$x \to y = xy$
Standard Strict	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$
Gödel	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Gaines	$x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Kleene-Dienes	$x \to y = \max\{1 - x, y\}$
Kleene-Dienes-Łukasiewicz	$x \to y = 1 - x + xy$

Table 1.1 Fuzzy implication operators.

2 THE THEORY OF APPROXIMATE REASONING

In 1979 *Zadeh* introduced the theory of approximate reasoning [35]. This theory provides a powerful framework for reasoning in the face of imprecise and uncertain information. Central to this theory is the representation of propositions as statements assigning fuzzy sets as values to variables. Suppose we have two interactive variables $x \in X$ and $y \in Y$ and the causal relationship between x and y is completely known. Namely, we know that y is a function of x, that is $y = f(x)$. Then we can make inferences easily

$$"y = f(x)" \ \& \ "x = x_1" \longrightarrow "y = f(x')".$$

This inference rule says that if we have $y = f(x)$, for all $x \in X$ and we observe that $x = x_1$ then y takes the value $f(x_1)$. More often than not we do not know the complete causal link f between x and y, only we now the values of $f(x)$ for some particular values of x, that is

$$\Re_i: \text{ if } x = x_i \text{ then } y = y_i, \text{ for } i = 1, \dots, m.$$

Suppose that we are given an $x' \in X$ and want to find an $y' \in Y$ which correponds to x' under the rule-base $\Re = \{\Re_1, \dots, \Re_m\}$. This problem is frequently quoted as interpolation.

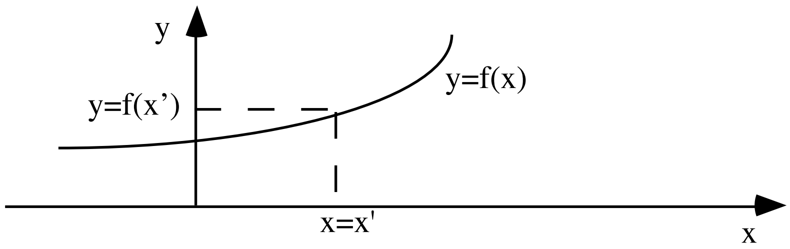

Figure 3: Simple crisp inference.

Let x and y be linguistic variables, e.g. "x is high" and "y is small". The basic problem of approximate reasoning is to find the membership function of the consequence C from the rule-base $\{\Re_1, \dots, \Re_n\}$ and the fact A, where $\Re_i$ is of the form

$$\Re_1 : \text{if } x \text{ is } A_i \text{ then } y \text{ is } C_i.$$

In [35] Zadeh introduced a number of translation rules which allow us to represent some common linguistic statements in terms of propositions in our language. In the following we describe some of these translation rules.

Entailment rule:

$$\begin{array}{c} x \text{ is } A \\ A \subset B \\ \hline x \text{ is } B \end{array} \qquad \begin{array}{l} \text{Mary is very young} \\ \text{very young} \subset \text{young} \\ \hline \text{Mary is young} \end{array}$$

Conjuction rule:

$$\begin{array}{c} x \text{ is } A \\ x \text{ is } B \\ \hline x \text{ is } A \cap B \end{array} \qquad \begin{array}{l} \text{pressure is not very high} \\ \text{pressure is not very low} \\ \hline \text{pressure is not very high and not very low} \end{array}$$

Disjunction rule:

$$\begin{array}{ll} & x \text{ is } A \\ \text{or} & x \text{ is } B \\ \hline & x \text{ is } A \cup B \end{array} \qquad \begin{array}{ll} & \text{pressure is not very high vspace2pt} \\ \text{or} & \text{pressure is not very low} \\ \hline & \text{pressure is not very high or not very low} \end{array}$$

Projection rule:

$$\frac{(x,y) \text{ have relation } R}{x \text{ is } \Pi_X(R)} \qquad \frac{(x,y) \text{ have relation } R}{y \text{ is } \Pi_Y(R)}$$

$$\frac{(x,y) \text{ is close to } (3,2)}{x \text{ is close to } 3} \qquad \frac{(x,y) \text{ is close to } (3,2)}{y \text{ is close to } 2}$$

Negation rule:

$$\frac{\text{not } (x \text{ is } A)}{x \text{ is } \neg A} \qquad \frac{\text{not } (x \text{ is high})}{x \text{ is not high}}$$

In fuzzy logic and approximate reasoning, the most important fuzzy implication inference rule is the *Generalized Modus Ponens* (GMP). The classical *Modus Ponens* inference rule says:

$$\begin{array}{lll} \text{premise} & \text{if } p \text{ then} & q \\ \text{fact} & \quad p & \\ \hline \text{consequence} & & q \end{array}$$

This inference rule can be interpreted as: If p is true and $p \to q$ is true then q is true. The fuzzy implication inference is based on the compositional rule of inference for approximate reasoning suggested by Zadeh [33]. It says

$$\begin{array}{lll} \text{premise} & \text{if } x \text{ is } A \text{ then} & y \text{ is } B \\ \text{fact} & \quad x \text{ is } A' & \\ \hline \text{consequence:} & & y \text{ is } B' \end{array}$$

where the consequence B' is determined as a composition of the fact and the fuzzy implication operator $B' = A' \circ (A \to B)$ that is,

$$B'(v) = \sup_{u \in U} \min\{A'(u), (A \to B)(u,v)\},$$

for all $v \in V$. In many practical cases instead of sup-min composition we use sup-T composition, where T is a t-norm,

$$B'(v) = \sup_{u \in U} T(A'(u), (A \to B)(u,v)),$$

for all $v \in V$. It is clear that T can not be chosen independently of the implication operator.

Suppose that A, B and A' are fuzzy numbers. The Generalized Modus Ponens should satisfy some rational properties

Property 2.1 *Basic property:*

if x *is* A *then* y *is* B		*if* pressure is big *then* volume is small	
x *is* A		pressure is big	
y *is* B		volume is small	

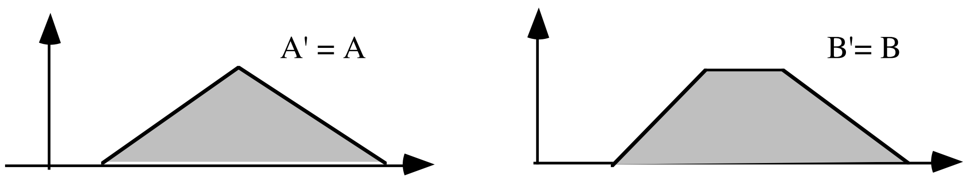

Figure 4: Basic property.

Property 2.2 *Total indeterminance:*

if x *is* A *then* y *is* B	*if* pres. is big *then* volume is small
x *is* $\neg A$	pres. is not big
y *is unknown*	volume is unknown

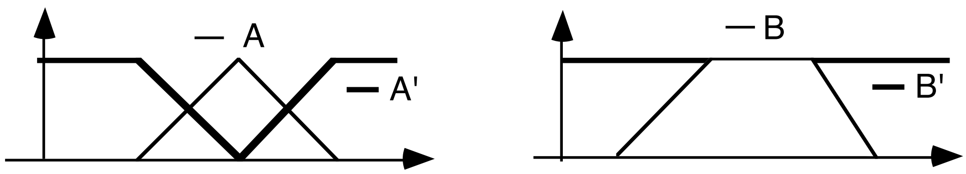

Figure 5: Total indeterminance.

Property 2.3 *Subset:*

$$\begin{array}{ll} \textit{if } x \textit{ is } A \textit{ then} & y \textit{ is } B \\ x \textit{ is } A' \subset A & \\ \hline & y \textit{ is } B \end{array} \qquad \begin{array}{lll} \textit{if} & \text{pres. is big } \textit{then} & \text{volume is small} \\ & \text{pres. is very big} & \\ \hline & & \text{volume is small} \end{array}$$

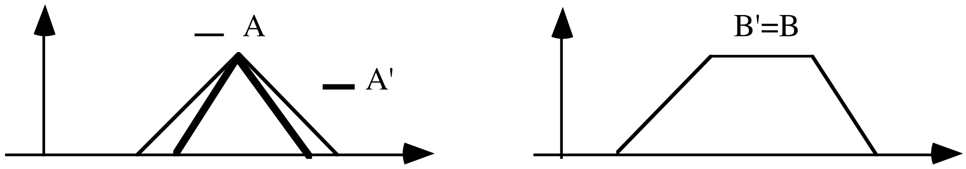

Figure 6: Subset property.

Property 2.4 *Superset:*

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A' & \\ \hline & & y \textit{ is } B' \supset B \end{array}$$

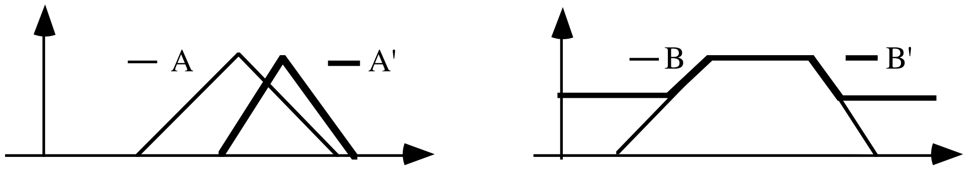

Figure 7: Superset property.

Suppose that A, B and A' are fuzzy numbers. We show that the Generalized Modus Ponens with Mamdani implication operator does not satisfy all the four properties listed above.

Example 2.1 *The GMP with Mamdani implication operator. where the membership function of the consequence B' is defined by*

$$B'(y) = \sup\{A'(x) \wedge A(x) \wedge B(y)\},$$

for all $y \in I\!R$.

Basic property: Let $A' = A$ and let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A(x), \min\{A(x), B(y)\} = \sup_x \min\{A(x), B(y)\} =$$

$$\min\{B(y), \sup_x A(x)\} = \min\{B(y), 1\} = B(y).$$

So the basic property is satisfied. *Total indeterminance:* Let $A' = \neg A = 1 - A$ and let $y \in I\!R$ be arbitrarily fixed. Then we get

$$B'(y) = \sup_x \min\{1 - A(x), \min\{A(x), B(y)\} = \sup_x \min\{A(x), 1 - A(x), B(y)\} =$$

$$\min\{B(y), \sup_x \min\{A(x), 1 - A(x)\}\} = \min\{B(y), 1/2\} = 1/2B(y) < 1$$

this means that the total indeterminance property is not satisfied. *Subset:* Let $A' \subset A$ and let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), \min\{A(x), B(y)\} = \sup_x \min\{A(x), A'(x), B(y)\} =$$

$$\min\{B(y), \sup_x A'(x)\} = \min\{B(y), 1\} = B(y)$$

So the subset property is satisfied. *Superset:* Let $y \in I\!R$ be arbitrarily fixed. Then we get

$$B'(y) = \sup_x \min\{A'(x), \min\{A(x), B(y)\} = \sup_x \min\{A(x), A'(x), B(y)\} \le B(y).$$

So the superset property of GMP is not satisfied by Mamdani implication operator.

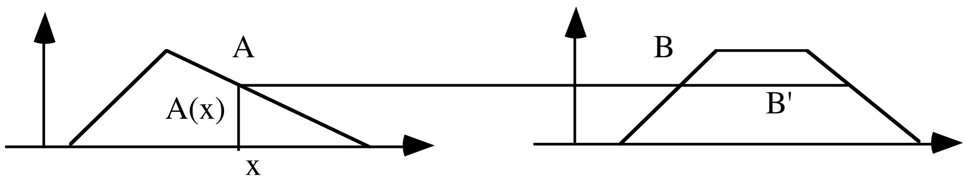

Figure 8: The GMP with Mamdani's implication operator.

Example 2.2 *The GMP with Larsen's product implication. where the membership function of the consequence B' is defined by*

$$B'(y) = \sup \min\{A'(x), A(x)B(y)\},$$

for all $y \in I\!R$.

Basic property: Let $A' = A$ and let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A(x), A(x)B(y)\} = B(y).$$

So the basic property is satisfied. *Total indeterminance:* Let $A' = \neg A = 1 - A$ and let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{1 - A(x), A(x)B(y)\} = \frac{B(y)}{1 + B(y)} < 1$$

this means that the total indeterminance property is not satisfied. *Subset:* Let $A' \subset A$ and let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), A(x)B(y)\} = \sup_x \min\{A(x), A'(x)B(y)\} = B(y)$$

So the subset property is satisfied. *Superset:* Let $y \in I\!R$ be arbitrarily fixed. Then we have

$$B'(y) = \sup_x \min\{A'(x), A(x)B(y)\} \leq B(y).$$

So, the superset property is not satisfied in the GMP with Larsen's product implication.

Suppose we are given one block of fuzzy rules of the form

$\Re_1$:	if x is A_1 then z is C_1,	
$\Re_2$:	if x is A_2 then z is C_2,	
	$\cdots\cdots\cdots$	
$\Re_n$:	if x is A_n then z is C_n	
fact:	x is A	
consequence:		z is C

The i-th fuzzy rule from this rule-base, $\Re_i$, is implemented by a *fuzzy implication* R_i and is defined as

$$R_i(u, w) = A_i(u) \to C_i(w)$$

There are two main approaches to determine the membership function of consequence C:

1. *Combine the rules first*. In this approach, we first combine all the rules by an aggregation operator **Agg** into one rule which used to obtain C from A.

$$R = \mathbf{Agg}\left(\Re_1, \Re_2, \cdots, \Re_n\right)$$

If the implicite sentence connective *also* is interpreted as *and* then we get

$$R(u,w) = \bigcap_{i=1}^{n} R_i(u,w) = \min(A_i(u) \to C_i(w))$$

or by using a t-norm T for modeling the connective *and*

$$R(u,w) = T(R_1(u,w), \dots, R_n(u,w))$$

If the implicite sentence connective *also* is interpreted as *or* then we get

$$R(u,w) = \bigcup_{i=1}^{n} R_i(u,v,w) = \max(A_i(u) \to C_i(w))$$

or by using a t-conorm S for modeling the connective *or*

$$R(u,w) = S(R_1(u,w), \dots, R_n(u,w))$$

Then we compute C from A by the compositional rule of inference as

$$C = A \circ R = A \circ \mathbf{Agg}\,(R_1, R_2, \cdots, R_n)$$

2. *Fire the rules first*. Fire the rules individually, given A, and then combine their results into C. We first compose A with each R_i producing intermediate result
$$C_i' = A \circ R_i$$
for $i = 1, \dots, n$ and then combine the C_i' component wise into C' by some aggregation operator **Agg**
$$C' = \mathbf{Agg}\,(C_1', \dots, C_n') = \mathbf{Agg}\,(A \circ R_1, \dots, A \circ R_n).$$

We show that the sup-min compositional operator and the connective *also* interpreted as the union operator are commutative. Thus the consequence, C, inferred from the complete set of rules is equivalent to the aggregated result, C', derived from individual rules.

Lemma 2.1 *Let*

$$C = A \circ \bigcup_{i=1}^{n} R_i$$

be defined by standard sup-min composition as

$$C(w) = \sup_u \min\{A(u), \max\{R_1(u,w), \dots, R_n(u,w)\}\}$$

and let

$$C' = \bigcup_{i=1}^{n} A \circ R_i$$

defined by the sup-min composition as

$$C'(w) = \max_{i=1,\dots,n} \{\sup_u A(u) \wedge R_i(u,w)\}.$$

Then $C(w) = C'(w)$ *for all* w *from the universe of discourse* W*.*

Proof. Using the distributivity of $\wedge$ over $\vee$ we get

$C(w) = \sup_u\{A(u) \wedge (R_1(u,w) \vee \dots \vee R_n(u,w))\} = \sup_u\{(A(u) \wedge R_1(u,w)) \vee \dots \vee$

$(A(u) \wedge R_n(u,w))\} = \max\{\sup_u A(u) \wedge R_1(u,w), \dots, \sup_u A(u) \wedge R_n(u,w)\} = C'(w).$

Which ends the proof.

Similar statement holds for the sup-product compositional rule of inference, i.e the sup-product compositional operator and the connective *also* as the union operator are commutative:

Lemma 2.2 *Let*

$$C = A \circ \bigcup_{i=1}^{n} R_i$$

be defined by sup-product composition as

$$C(w) = \sup_u A(u) \max\{R_1(u,w), \dots, R_n(u,w)\}$$

and let

$$C' = \bigcup_{i=1}^{n} A \circ R_i$$

defined by the sup-product composition as

$$C'(w) = \max\{\sup_u A(u) R_i(u,w), \dots, \sup_u A(u) R_n(u,w)\}$$

Then $C(w) = C'(w)$ *holds for each* w *from the universe of discourse* W*.*

However, the sup-min compositional operator and the connective *also* interpreted as the intersection operator are not usually commutative. In this case, the consequence, C, inferred from the complete set of rules is included in the aggregated result, C', derived from individual rules.

Lemma 2.3 *Let*

$$C = A \circ \bigcap_{i=1}^{n} R_i$$

be defined by standard sup-min composition as

$$C(w) = \sup_u \min\{A(u), \min\{R_1(u,w), \dots, R_n(u,w)\}\}$$

and let

$$C' = \bigcap_{i=1}^{n} A \circ R_i$$

defined by the sup-min composition as

$$C'(w) = \min\{\sup_u \{A(u) \wedge R_i(u,w)\}, \dots, \sup_u \{A(u) \wedge R_n(u,w)\}\}.$$

Then $C \subset C'$, i.e $C(w) \leq C'(w)$ holds for all w from the universe of discourse W.

Proof. From the relationship

$$A \circ \bigcap_{i=1}^{n} R_i \subset A \circ R_i$$

for each $i = 1, \dots, n$, we get

$$A \circ \bigcap_{i=1}^{n} R_i \subset \bigcap_{i=1}^{n} A \circ R_i.$$

Similar statement holds for the sup-t-norm compositional rule of inference, i.e the sup-product compositional operator and the connective *also* interpreted as the intersection operator are not commutative. In this case, the consequence, C, inferred from the complete set of rules is included in the aggregated result, C', derived from individual rules.

Lemma 2.4 *Let*

$$C = A \circ \bigcap_{i=1}^{n} R_i$$

be defined by sup-T composition as

$$C(w) = \sup_u T(A(u), \min\{R_1(u,w), \dots, R_n(u,w)\})$$

and let

$$C' = \bigcap_{i=1}^{n} A \circ R_i$$

defined by the sup-T composition. Then $C \subset C'$, i.e $C(w) \leq C'(w)$ holds for all w from the universe of discourse W.

If $X = \{x_1, \dots, x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \mu_1/x_1 + \dots + \mu_n/x_n$$

where the term $\mu_i/x_i,\ i = 1, \dots, n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Example 2.3 *We illustrate Lemma 2.3 by a simple example. Assume we have two fuzzy rules of the form*

$$\Re_1: \ \textit{if } x \textit{ is } A_1 \textit{ then } z \textit{ is } C_1$$
$$\Re_2: \ \textit{if } x \textit{ is } A_2 \textit{ then } z \textit{ is } C_2$$

where A_1, A_2 and C_1, C_2 are discrete fuzzy numbers of the universe of discourses $\{x_1, x_2\}$ and $\{z_1, z_2\}$, respectively. Suppose that we input a fuzzy set $A = a_1/x_1 + a_2/x_2$ to the system and let

$$R_1 = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{array}, \quad R_2 = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{array}$$

represent the fuzzy rules. We first compute the consequence C by

$$C = A \circ (R_1 \cap R_2).$$

Using the definition of intersection of fuzzy relations we get

$$C = (a_1/x_1 + a_2/x_2) \circ \left[\left(\begin{array}{ccc} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{array} \right) \bigcap \left(\begin{array}{ccc} & z_1 & z_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{array} \right) \right] =$$

$$(a_1/x_1 + a_2/x_2) \circ \left(\begin{array}{ccc} & z_1 & z_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{array} \right) = \emptyset$$

Let us compute now the membership function of the consequence C' by

$$C' = (A \circ R_1) \cap (A \circ R_2)$$

Using the definition of sup-min composition we get

$$A \circ R_1 = (a_1/x_1 + a_2/x_2) \circ \left(\begin{array}{ccc} & z_1 & z_2 \\ x_1 & 0 & 1 \\ x_2 & 1 & 0 \end{array} \right).$$

Plugging into numerical values

$(A \circ R_1)(z_1) = \max\{a_1 \wedge 0, a_2 \wedge 1\} = a_2$, $(A \circ R_1)(z_2) = \max\{a_1 \wedge 1, a_2 \wedge 0\} = a_1$,

So,

$$A \circ R_1 = a_2/z_1 + a_1/z_2$$

and from

$$A \circ R_2 = (a_1/x_1 + a_2/x_2) \circ \left(\begin{array}{ccc} & z_1 & z_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{array} \right) =$$

we get

$$A \circ R_2 = a_1/z_1 + a_2/z_2.$$

Finally,

$$C' = a_2/z_1 + a_1/z_2 \cap a_1/z_1 + a_2/z_2 = a_1 \wedge a_2/z_1 + a_1 \wedge a_2/z_2.$$

Which means that C is a proper subset of C' whenever $\min\{a_1, a_2\} \neq 0$.

Suppose now that the fact of the GMP is given by a fuzzy singleton, $\bar{x}_0, x_0 \in I\!R$. Then the process of computation of the membership function of the consequence becomes very simple. For example, if we use Mamdani's implication operator in the GMP then

rule 1:	if x is A_1 then	z is C_1
fact:	x is $\bar{x}_0$	
consequence:		z is C

where the membership function of the consequence C is computed as

$$C(w) = \sup_u \min\{\bar{x}_0(u), (A_1 \to C_1)(u,w)\} = \sup_u \min\{\bar{x}_0(u), \min\{A_1(u), C_1(w)\}\},$$

for all $w \in W$. Observing that $\bar{x}_0(u) = 0,\ \forall u \neq x_0$ the supremum turns into a simple minimum

$$C(w) = \min\{\bar{x}_0(x_0) \wedge A_1(x_0) \wedge C_1(w)s\} = \min\{A_1(x_0), C_1(w)\},$$

for all $w \in W$ (see Figure 8).

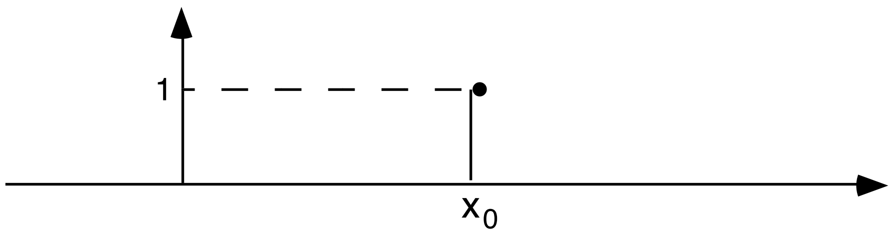

Figure 9: Fuzzy singleton.

If we use Gödel implication operator in the GMP then

$$C(w) = \sup_u \min\{\bar{x}_0(u), (A_1 \to C_1)(u,w)\} = A_1(x_0) \to C_1(w)$$

That is (see Figure 10):

$$C(w) = \begin{cases} 1 & \text{if } A_1(x_0) \leq C_1(w) \\ C_1(w) & \text{otherwise} \end{cases}$$

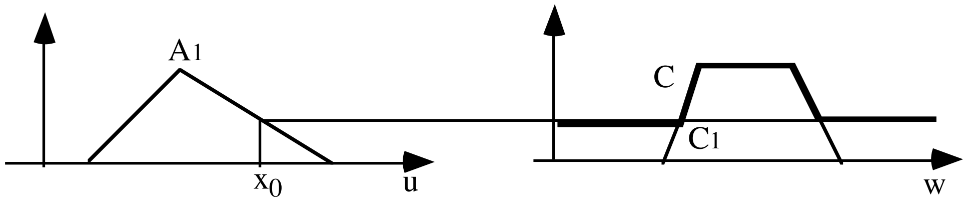

Figure 10: Inference with Gödel implication operator.

Lemma 2.5 *Consider one block of fuzzy rules of the form*

$$\Re_i\text{: if } x \text{ is } A_i \text{ then } z \text{ is } C_i,\ 1 \le i \le n$$

and suppose that the input to the system is a fuzzy singleton. Then the consequence, C, inferred from the complete set of rules is equal to the aggregated result, C', derived from individual rules. This statements holds for any kind of aggregation operators used to combine the rules.

Proof. Suppose that the input of the system $A = \bar{x}_0$ is a fuzzy singleton. On the one hand we have

$$C(w) = (A \circ \mathrm{Agg}\,\langle R_1, \dots, R_n \rangle)(w) = \mathrm{Agg}\,\langle R_1(x_0, w), \dots, R_n(x_0, w) \rangle.$$

On the other hand

$$C'(w) = \mathrm{Agg}\,\langle A \circ R_1, \dots, A \circ R_n \rangle(w) = \mathrm{Agg}\,\langle R_1(x_0, w), \dots, R_n(x_0, w) \rangle = C(w).$$

Consider one block of fuzzy rules of the form

$$\Re = \{A_i \to C_i,\ 1 \le i \le n\}$$

where A_i and C_i are fuzzy numbers.

Lemma 2.6 *Suppose that in $\Re$ the supports of A_i are pairwise disjunctive:*

$$\mathrm{supp} A_i \cap \mathrm{supp} A_j = \emptyset, \textit{ for } i \ne j.$$

If the Gödel implication operator is used in $\Re$ then we get

$$\bigcap_{i=1}^{n} A_i \circ (A_i \to C_i) = C_i$$

holds for $i = 1, \dots, n$

Proof. Since the GMP with Gödel implication satisfies the basic property we get

$$A_i \circ (A_i \to C_i) = A_i.$$

From $supp(A_i) \cap supp(A_j) = \emptyset,$ for $i \neq j$ it follows that

$$A_i \circ (A_j \to C_j) = \mathbf{1},\ i \neq j$$

where $\mathbf{1}$ is the universal fuzzy set. So,

$$\bigcap_{i=1}^{n} A_i \circ (A_i \to C_i) = C_i \cap \mathbf{1} = C_i.$$

This property means that deleting any of the rules from $\Re$ leaves a point $\hat{x}$ to which no rule applies. It means that every rule is useful.

Definition 2.1 *The rule-base $\Re$ is said to be separated (see Figure 11) if the core of A_i, defined by*

$$\mathrm{core}(A_i) = \{x \mid A_i(x) = 1\},$$

is not contained in

$$\bigcap_{j \neq i} \mathrm{supp} A_j$$

for $i = 1, \ldots, n$.

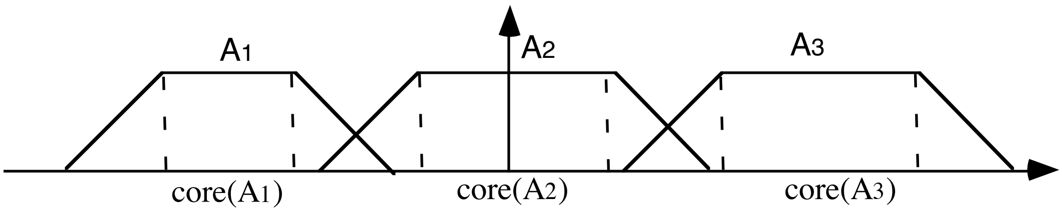

Figure 11: Separated rule-base.

The following theorem shows that Lemma 2.6 remains valid for separated rule-bases.

Theorem 2.1 *[10] Let $\Re$ be separated. If the implication is modelled by the Gödel implication operator then*

$$\bigcap_{i=1}^{n} A_i \circ (A_i \to C_i) = C_i$$

holds for $i = 1, \ldots, n$

Proof. Since the Gödel implication satisfies the basic property of the GMP we get

$$A_i \circ (A_i \to C_i) = A_i.$$

Since $\mathrm{core}(A_i) \cap \mathrm{supp}(A_j) \neq \emptyset,$ for $i \neq j$ there exists an element $\hat{x}$ such that $\hat{x} \in core(A_i)$ and $\hat{x} \notin supp(A_j),\ i \neq j$. That is $A_i(\hat{x}) = 1$ and $A_j(\hat{x}) = 0,\ i \neq j$. Applying the compositional rule of inference with Gödel implication operator we get

$$(A_i \circ A_j \to C_j)(z) = \sup_x \min\{A_i(x), A_j(x) \to C_j(x))\} \leq$$
$$\min\{A_i(\hat{x}), A_j(\hat{x}) \to C_j(\hat{x}))\} = 1,\ i \neq j$$

for any z. So,

$$\bigcap_{i=1}^{n} A_i \circ (A_i \to C_i) = C_i \cap \mathbf{1} = C_i$$

Which ends the proof.

3 MULTIPLE FUZZY REASONING SCHEMES

If several linguistic variables are involved in the antecedents and the conclusions of the rules then the system will be referred to as a multi-input-multi-output fuzzy system. For example, the case of two-input-single-output (MISO) fuzzy systems is of the form

$\Re_i$: if x is A_i and y is B_i then z is C_i

where x and y are the process state variables, z is the control variable, A_i, B_i, and C_i are linguistic values of the linguistic vatiables x, y and z in the universes of discourse U, V, and W, respectively, and an implicit sentence connective *also* links the rules into a rule set or, equivalently, a rule-base. The procedure for obtaining the fuzzy output of such a knowledge base consists from the following three steps:

- Find the firing level of each of the rules.
- Find the output of each of the rules.
- Aggregate the individual rule outputs to obtain the overall system output.

To infer the output z from the given process states x, y and fuzzy relations R_i, we apply the compositional rule of inference:

$$
\begin{array}{lll}
\Re_1: & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then} & z \text{ is } C_1 \\
\Re_2: & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then} & z \text{ is } C_2 \\
 & \cdots\cdots\cdots\cdots & \\
\Re_n: & \text{if } x \text{ is } A_n \text{ and } y \text{ is } B_n \text{ then} & z \text{ is } C_n \\
\text{fact}: & \quad x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 & \\
\hline
\text{consequence}: & & z \text{ is } C
\end{array}
$$

where the consequence is computed by

$$\text{consequence} = \text{Agg}\,\langle \text{fact} \circ \Re_1, \dots, \text{fact} \circ \Re_n \rangle.$$

That is,

$$C = \text{Agg}(\bar{x}_0 \times \bar{y}_0 \circ R_1, \dots, \bar{x}_0 \times \bar{y}_0 \circ R_n)$$

taking into consideration that $\bar{x}_0(u) = 0,\ u \neq x_0$ and $\bar{y}_0(v) = 0,\ v \neq y_0$, the computation of the membership function of C is very simple:

$$C(w) = \text{Agg}\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \dots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all $w \in W$. The procedure for obtaining the fuzzy output of such a knowledge base can be formulated as

- The firing level of the i-th rule is determined by

$$A_i(x_0) \times B_i(y_0).$$

- The output of of the i-th rule is calculated by

$$C'_i(w) := A_i(x_0) \times B_i(y_0) \rightarrow C_i(w)$$

  for all $w \in W$.

- The overall system output, C, is obtained from the individual rule outputs C'_i by

$$C(w) = \text{Agg}\{C'_1, \dots, C'_n\}$$

  for all $w \in W$.

Example 3.1 *If the sentence connective* also *is interpreted as anding the rules by using minimum-norm then the membership function of the consequence is computed as*

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cap \ldots \cap (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \min\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \ldots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all $w \in W$.

Example 3.2 *If the sentence connective* also *is interpreted as oring the rules by using minimum-norm then the membership function of the consequence is computed as*

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cup \ldots \cup (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \max\{A_1(x_0) \times B_1(y_0) \rightarrow C_1(w), \ldots, A_n(x_0) \times B_n(y_0) \rightarrow C_n(w)\}$$

for all $w \in W$.

Example 3.3 *Suppose that the Cartesian product and the implication operator are implemented by the t-norm* $T(u, v) = uv$. *If the sentence connective* also *is interpreted as oring the rules by using minimum-norm then the membership function of the consequence is computed as*

$$C = (\bar{x}_0 \times \bar{y}_0 \circ R_1) \cup \ldots \cup (\bar{x}_0 \times \bar{y}_0 \circ R_n).$$

That is,

$$C(w) = \max\{A_1(x_0)B_1(y_0)C_1(w), \ldots, A_n(x_0)B_n(y_0)C_n(w)\}$$

for all $w \in W$.

We present three well-known inference mechanisms in MISO fuzzy systems. For simplicity we assume that we have two fuzzy rules of the form

$\Re_1$:	if x is A_1 and y is B_1 then	z is C_1
$\Re_2$:	if x is A_2 and y is B_2 then	z is C_2
fact :	x is $\bar{x}_0$ and y is $\bar{y}_0$	
consequence :		z is C

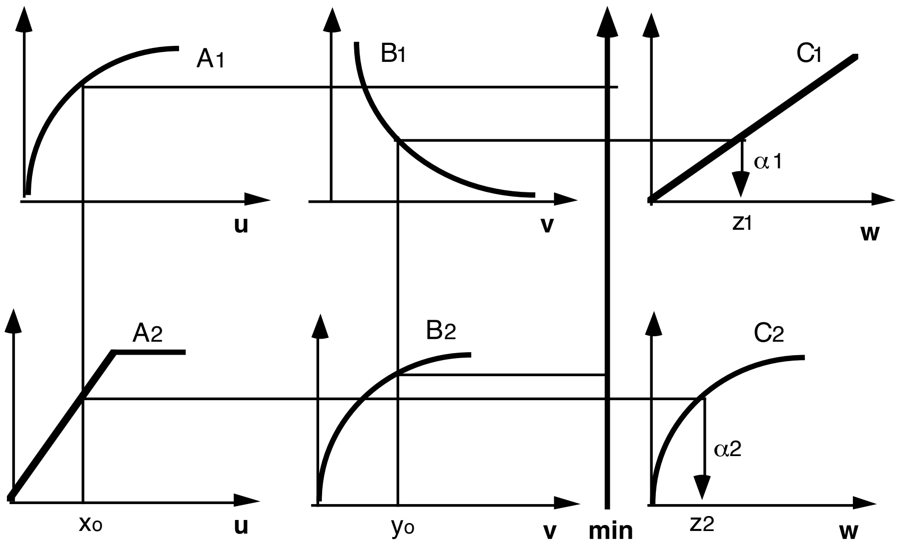

Figure 12: Tsukamoto's inference mechanism.

Tsukamoto. All linguistic terms are supposed to have monotonic membership functions.

The firing levels of the rules are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),\ \alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

In this mode of reasoning the individual crisp control actions z_1 and z_2 are computed from the equations

$$\alpha_1 = C_1(z_1), \quad \alpha_2 = C_2(z_2)$$

and the overall crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1 + \alpha_2 z_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1 C_1^{-1}(\alpha_1) + \alpha_2 C_2^{-1}(\alpha_2)}{\alpha_1 + \alpha_2}$$

i.e. z_0 is computed by the discrete Center-of-Gravity method. If we have m rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\alpha_1 z_1 + \cdots + \alpha_m z_m}{\alpha_1 + \cdots + \alpha_m},$$

where α_i is the firing level and z_i is the (crisp) output of the i-th rule, $i = 1, \dots, m$.
Sugeno and Takagi use the following architecture [25]

$$\begin{array}{lll} \Re_1: & \text{if } x \text{ is } A_1 \text{ and } y \text{ is } B_1 \text{ then} & z_1 = a_1x + b_1y \\ \Re_2: & \text{if } x \text{ is } A_2 \text{ and } y \text{ is } B_2 \text{ then} & z_2 = a_2x + b_2y \\ \text{fact}: & x \text{ is } \bar{x}_0 \text{ and } y \text{ is } \bar{y}_0 & \\ \hline \text{consequence}: & & z_0 \end{array}$$

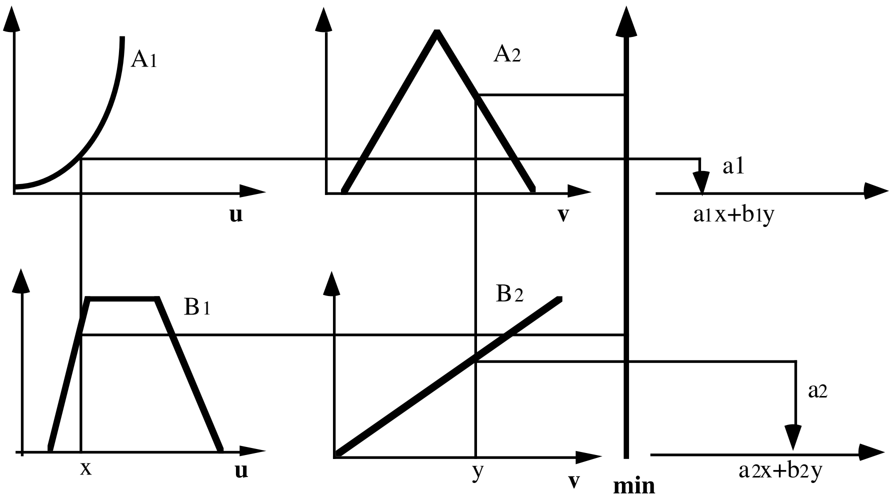

Figure 13: Sugeno's inference mechanism.

The firing levels of the rules are computed by

$$\alpha_1 = A_1(x_0) \wedge B_1(y_0),\ \alpha_2 = A_2(x_0) \wedge B_2(y_0)$$

then the individual rule outputs are derived from the relationships

$$z_1^* = a_1x_0 + b_1y_0,\ z_2^* = a_2x_0 + b_2y_0$$

and the crisp control action is expressed as

$$z_0 = \frac{\alpha_1 z_1^* + \alpha_2 z_2^*}{\alpha_1 + \alpha_2}$$

If we have m rules in our rule-base then the crisp control action is computed as

$$z_0 = \frac{\alpha_1 z_1^* + \cdots + \alpha_m z_m^*}{\alpha_1 + \cdots + \alpha_m},$$

where α_i denotes the firing level of the i-th rule, $i = 1, \ldots, m$.

Example 3.4 *We illustrate Sugeno's reasoning method by the following simple example*

if	x *is SMALL*	*and*	y *is BIG*	*then*	$z = x - y$
if	x *is BIG*	*and*	y *is SMALL*	*then*	$z = x + y$
if	x *is BIG*	*and*	y *is BIG*	*then*	$z = x + 2y$

where the membership functions $SMALL$ and BIG are defined by

$$SMALL(v) = \begin{cases} 1 & \textit{if } v \leq 1 \\ 1 - (v-1)/4 & \textit{if } 1 \leq v \leq 5 \\ 0 & \textit{otherwise} \end{cases}$$

$$BIG(u) = \begin{cases} 1 & \textit{if } u \geq 5 \\ 1 - (5-u)/4 & \textit{if } 1 \leq u \leq 5 \\ 0 & \textit{otherwise} \end{cases}$$

Suppose we have the inputs $x_0 = 3$ and $y_0 = 3$. What is the output of the system?

The firing level of the first rule is

$$\alpha_1 = \min\{SMALL(3), BIG(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the first rule is $z_1 = x_0 - y_0 = 3 - 3 = 0$. The firing level of the second rule is

$$\alpha_1 = \min\{BIG(3), SMALL(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the second rule is $z_2 = x_0 + y_0 = 3 + 3 = 6$. The firing level of the third rule is

$$\alpha_1 = \min\{BIG(3), BIG(3)\} = \min\{0.5, 0.5\} = 0.5$$

the individual output of the third rule is $z_3 = x_0 + 2y_0 = 3 + 6 = 9$. and the system output, z_0, is computed from the equation $z_0 = (0 \times 0.5 + 6 \times 0.5 + 9 \times 0.5)/1.5 = 5.0$.

Simplified fuzzy reasoning. In this context, the word *simplified* means that the individual rule outputs are given by crisp numbers, and therefore, we can use their weighted sum (where the weights are the firing strengths of the corresponding rules) to obtain the overall system output:

$$
\begin{array}{llll}
\Re_1: & \text{if} & x_1 \text{ is } A_{11} \text{ and } \dots \text{and } x_n \text{ is } A_{1n} & \text{then } y = z_1 \\
 & & \dots\dots\dots\dots & \\
\Re_m: & \text{if} & x_1 \text{ is } A_{m1} \text{ and } \dots \text{and } x_n \text{ is } A_{mn} & \text{then } y = z_m \\
\text{fact:} & & x_1 \text{ is } u_1 \quad \text{and } \dots \text{and } x_n \text{ is } u_n & \\
\hline
\text{consequence:} & & & y \text{ is } z_0
\end{array}
$$

where A_{ij} are values of the linguistc variables $x_1, \dots, x_n$. We derive z_0 from the initial content of the data base, $\{u_1, \dots, u_n\}$, and from the fuzzy rule base $\Re = \{\Re_1, \dots, \Re_m\}$. by the simplified fuzzy reasoning scheme as

$$z_0 = \frac{z_1\alpha_1 + \cdots + z_m\alpha_m}{\alpha_1 + \cdots + \alpha_m}$$

where $\alpha_i = (A_{i1} \times \cdots \times A_{in})(u_1, \dots, u_n),\ i = 1, \dots, m.$

Remark 3.1 *Jang [21] showed that fuzzy inference systems with simplified fuzzy IF-THEN rules (and, consequently, Sugeno's and Tsukamoto's systems as well) are universal approximators, i.e. they can approximate any continuous function on a compact set to arbitrary accuracy. It means that the more fuzzy terms (and consequently more rules) are used in the rule base, the closer is the output of the fuzzy system to the desired values of the function to be approximated.*

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