A pure probabilistic interpretation of possibilistic expected value, variance, covariance and correlation *

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Abstract: In this work we shall give a pure probabilistic interpretation of possibilistic expected value, variance, covariance and correlation.

Keywords: Possibility distribution, expected value, variance, covariance, correlation

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1 Probability and possibility

In probability theory, the dependency between two *random variables* can be characterized through their joint probability density function. Namely, if X and Y are two random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively, then the density function, $f_{X,Y}(x,y)$, of their joint random variable (X,Y), should satisfy the following properties

$$\int_{\mathbb{R}} f_{X,Y}(x,t)dt = f_X(x), \quad \int_{\mathbb{R}} f_{X,Y}(t,y)dt = f_Y(y),$$

for all $x, y \in \mathbb{R}$. Furthermore, $f_X(x)$ and $f_Y(y)$ are called the the marginal probability density functions of random variable (X,Y). X and Y are said to be independent if the relationship

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

holds for all x, y. The expected value of random variable X is defined as

$$E(X) = \int_{\mathbb{R}} x f_X(x)dx.$$

The covariance between two random variables X and Y is defined as

$$\text{Cov}(X,Y) = E\big((X - E(X))(Y - E(Y))\big) = E(XY) - E(X)E(Y),$$

and if X and Y are independent then $\text{Cov}(X,Y) = 0$. The variance of random variable X is defined by

$$\sigma_X^2 = E(X^2) - (E(X))^2.$$

The correlation coefficient between X and Y is defined by

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

and it is clear that $-1 \leq \rho(X,Y) \leq 1$.

A *fuzzy number* A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathcal{F}$. Fuzzy numbers can be considered as possibility distributions [11, 15]. If $A \in \mathcal{F}$ is a fuzzy number and $x \in \mathbb{R}$

a real number then $A(x)$ can be interpreted as the degree of possiblity of the statement "x is A". A fuzzy set C in $\mathbb{R}^n$ is said to be a joint possibility distribution of fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, n$, if it satisfies the relationship

$$\max_{x_j \in \mathbb{R},\ j \neq i} C(x_1, \dots, x_n) = A_i(x_i)$$

for all $x_i \in \mathbb{R}$, $i = 1, \dots, n$. Furthermore, A_i is called the i-th marginal possibility distribution of C, and the projection of C on the i-th axis is A_i for $i = 1, \dots, n$.

Fuzzy numbers $A_i \in \mathcal{F}$, $i = 1, \dots, n$ are said to be non-interactive if their joint possibility distribution C satisfies the relationship

$$C(x_1, \dots, x_n) = \min\{A_1(x_1), \dots, A_n(x_n)\},$$

or, equivalently,

$$[C]^\gamma = [A_1]^\gamma \times \cdots \times [A_n]^\gamma$$

holds for all $x_1, \dots, x_n \in \mathbb{R}$ and $\gamma \in [0, 1]$. Marginal probability distributions are determined from the joint one by the principle of 'falling integrals' and marginal possibility distributions are determined from the joint possibility distribution by the principle of 'falling shadows'.

If $A, B \in \mathcal{F}$ are non-interactive then their joint membership function is defined by

$$C = A \times B,$$

where

$$C(x, y) = (A \times B)(x, y) = \min\{A(x), B(y)\}$$

for any $x, y \in \mathbb{R}$.

It is clear that in this case for any $u \in [A]^\gamma$ and for all $v \in [B]^\gamma$ we have

$$(u, v) \in [C]^\gamma,$$

since from $A(u) \geq \gamma$ and $B(v) \geq \gamma$ it follows that

$$\min\{A(u), B(v)\} \geq \gamma,$$

that is $(u, v) \in [C]^\gamma$.

On the other hand, A and B are said to be interactive if they can not take their values independently of each other.

It is clear that in this case any change in the membership function of A does not effect the second marginal possibility distribution and vice versa. On the other hand, A and B are said to be interactive if they can not take their values independently of each other [11].

Let $A \in \mathcal{F}$ be fuzzy number with $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0,1]$. A function $f\colon [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monoton increasing and satisfies the following normalization condition

$$\int_0^1 f(\gamma) d\gamma = 1.$$

Different weighting functions can give different (case-dependent) importances to γ-levels sets of fuzzy numbers. It is motivated in part by the desire to give less importance to the lower levels of fuzzy sets [14] (it is why f should be monotone increasing).

2 A pure probabilistic interpretation of possibilistic expected value, variance, covariance and correlation

The f-weighted *possibilistic expected value* of $A \in \mathcal{F}$, defined in [12], can be written as

$$E_f(A) = \int_0^1 E(U_\gamma) f(\gamma) d\gamma = $$
$$= \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) d\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ for all $\gamma \in [0,1]$.

The f-weighted *possibilistic variance* of $A \in \mathcal{F}$, defined in [12], can be written as

$$\mathrm{Var}_f(A) = \int_0^1 \sigma^2_{U_\gamma} f(\gamma) d\gamma$$
$$= \int_0^1 \frac{(a_2(\gamma) - a_1(\gamma))^2}{12} f(\gamma) d\gamma.$$

The f-weighted *measure of possibilistic covariance* between $A, B \in \mathcal{F}$, (with respect to their joint distribution C), defined by [13], can be written as

$$\text{Cov}_f(A, B) = \int_0^1 \text{Cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma,$$

where X_γ and Y_γ are random variables whose joint distribution is uniform on $[C]^\gamma$ for all $\gamma \in [0, 1]$.

The f-weighted *possibilistic correlation* of $A, B \in \mathcal{F}$, (with respect to their joint distribution C), defined in [9], can be written as

$$\rho_f(A, B) =$$

$$\frac{\int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma}{\left(\int_0^1 \sigma^2_{U_\gamma} f(\gamma) d\gamma\right)^{1/2} \left(\int_0^1 \sigma^2_{V_\gamma} f(\gamma) d\gamma\right)^{1/2}}.$$

where V_γ is a uniform probability distribution on $[B]^\gamma$. Thus, the possibilistic correlation represents an average degree to which X_γ and Y_γ are linearly associated as compared to the dispersions of U_γ and V_γ.

It is clear that we do not run a standard probabilistic calculation here. A standard probabilistic calculation might be the following

$$\frac{\int_0^1 \text{cov}(X_\gamma, Y_\gamma) f(\gamma) d\gamma}{\left(\int_0^1 \sigma^2_{X_\gamma} f(\gamma) d\gamma\right)^{1/2} \left(\int_0^1 \sigma^2_{Y_\gamma} f(\gamma) d\gamma\right)^{1/2}}.$$

That is, the standard probabilistic approach would use the marginal distributions, X_γ and Y_γ, of a uniformly distributed random variable on the level sets of $[C]^\gamma$.

Theorem 2.1 ([9]). *If $[C]^\gamma$ is convex for all $\gamma \in [0, 1]$ then $-1 \leq \rho_f(A, B) \leq 1$ for any weighting function f.*

The possibilistic expected value, variance, covariance and correlation have been extensively used for real option valuation [4, 8], project selection [2, 5, 6, 10], capital budgeting [1] and optimal portfolio selection [7].

3 Examples

First, let us assume that A and B are non-interactive, i.e. $C = A \times B$.

Then $[C]^\gamma = [A]^\gamma \times [B]^\gamma$ for any $\gamma \in [0,1]$ and we have $\mathrm{Cov}_f(A,B) = 0$ (see [13]) and $\rho_f(A,B) = 0$ for any weighting function f.

In the case, the covariance of A and B with respect to their joint possibility distribution C is (see [13])

$$\mathrm{Cov}_f(A,B) =$$
$$\frac{1}{12}\int_0^1 [a_2(\gamma) - a_1(\gamma)][b_2(\gamma) - b_1(\gamma)]f(\gamma)d\gamma,$$

and

$$\rho_f(A,B) = 1,$$

for any weighted function f.

If $u \in [A]^\gamma$ for some $u \in \mathbb{R}$ then there exists a unique $v \in \mathbb{R}$ that B can take. Furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will also move to the left (right). This property can serve as a justification of the principle of (complete positive) correlation of A and B.

In the case, the covariance of A and B with respect to their joint possibility distribution D is (see [13])

$$\mathrm{Cov}_f(A,B) =$$
$$-\frac{1}{12}\int_0^1 [a_2(\gamma) - a_1(\gamma)][b_2(\gamma) - b_1(\gamma)]f(\gamma)d\gamma,$$

and

$$\rho_f(A,B) = -1,$$

for any weighted function f.

If $u \in [A]^\gamma$ for some $u \in \mathbb{R}$ then there exists a unique $v \in \mathbb{R}$ that B can take. Furthermore, if u is moved to the left (right) then the corresponding value (that B can take) will move to the right (left). This property can serve as a justification of the principle of (complete negative) correlation of A and B.

Zero covariance does not always imply non-interactivity. Really, let G be a joint possibility distribution with a symmetrical γ-level set, i.e., there exist $a, b \in \mathbb{R}$ such that

$$G(x,y) = G(2a - x, y)$$

$$= G(x, 2b - y) = G(2a - x, 2b - y),$$

for all $x, y \in [G]^\gamma$, where (a, b) is the center of the set $[G]^\gamma$.

Theorem 3.1 ([13]). *If all γ-level sets of G are symmetrical then the covariance between its marginal distributions A and B becomes zero for any weighting function f, that is,*

$$\mathrm{Cov}_f(A, B) = 0,$$

even though A and B may be interactive.

Now consider the case when

$$A(x) = B(x) = (1 - x) \cdot \chi_{[0,1]}(x)$$

for $x \in \mathbb{R}$, that is, $[A]^\gamma = [B]^\gamma = [0, 1 - \gamma]$ for $\gamma \in [0, 1]$.
Suppose that their joint possibility distribution is given by

$$F(x, y) = (1 - x - y) \cdot \chi_T(x, y),$$

where

$$T = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

After some calculations we get

$$\mathrm{Cov}_f(A, B) = -\frac{1}{36}\int_0^1 (1 - \gamma)^2 f(\gamma) d\gamma,$$

and

$$\rho_f(A, B) = -1/3,$$

for any weighting function f.
Now consider the case when

$$A(1 - x) = B(x) = x \cdot \chi_{[0,1]}(x)$$

for $x \in \mathbb{R}$, that is, $[A]^\gamma = [0, 1 - \gamma]$ and $[B]^\gamma = [\gamma, 1]$, for $\gamma \in [0, 1]$.
Let

$$E(x, y) = (y - x) \cdot \chi_S(x, y),$$

where

$$S = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \leq 1, y - x \geq 0\}.$$

A γ-level set of E is computed by

$$[E]^\gamma = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \leq 1, y - x \geq \gamma\}.$$

We can easily see that

$$\max_x E(x, y) = y \cdot \chi_{[0,1]}(y) = B(y),$$
$$\max_y E(x, y) = (1 - x) \cdot \chi_{[0,1]}(x) = A(x).$$

After some calculations we get

$$\mathrm{Cov}_f(A, B) = \frac{1}{36} \int_0^1 (1 - \gamma)^2 f(\gamma) d\gamma,$$

and

$$\rho_f(A, B) = 1/3,$$

for any weighting function f.

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