

SPECTRAL CONDITIONS FOR JORDAN *-ISOMORPHISMS ON OPERATOR ALGEBRAS

OSAMU HATORI AND LAJOS MOLNÁR

ABSTRACT. In this paper we study non-linear transformations between the unitary groups of von Neumann algebras and the twisted subgroups of positive invertible elements in unital C^* -algebras with various preserver properties concerning the spectrum, spectral radius, and generalized distance measures. We present several results which show that those transformations are closely related to the Jordan *-isomorphisms between the underlying full algebras. In fact, our results can easily be used for characterizations of that sort of isomorphisms.

1. INTRODUCTION

In this paper all algebras are assumed to be complex and unital, the unit usually being denoted by 1. Let A_1, A_2 be algebras and let $\sigma(\cdot)$ stand for the spectrum. A map (no linearity is assumed) $\phi: A_1 \rightarrow A_2$ is called spectrally multiplicative if it satisfies

$$\sigma(\phi(a)\phi(b)) = \sigma(ab)$$

for all pairs $a, b \in A_1$. There has recently been considerable interest in the study of such transformations since in many cases it turns out that they are closely related to isomorphisms, hence the spectral condition displayed above may faithfully compress the linearity and multiplicativity properties of maps into one two-variable equality between sets of scalars. For a typical result we recall that any spectrally multiplicative bijection between the algebras of all continuous complex valued functions over compact Hausdorff spaces is an algebra isomorphism followed by multiplication by a fixed real valued continuous function of modulus 1. In fact, for first countable spaces this was proved in the paper [7] (which was the starting point of that line of investigations) while in [11] the authors removed the first countability assumption. Concerning operator algebras we obtained in [7] that for an infinite dimensional Hilbert space H , any spectrally multiplicative bijection on the algebra of all bounded linear operators on H is either an algebra isomorphism or the negative of an algebra isomorphism. Hence in those cases we have that any spectrally multiplicative bijective map is a transformation which can be written as an algebra isomorphism multiplied by a central symmetry (which is a self-adjoint unitary which is in the center of the algebra in question). (We admit that presently Google Scholar pops up close to 90 citations to the paper [7].) For further reference we mention the survey paper [4] exhibiting a collection of recent results (mainly concerning function algebras) as well as the interesting paper [2] where a variant of spectrally multiplicative maps (involving three variables not only two) has been investigated on general algebras.

In this paper we continue that line of research and present results which can be viewed as characterizations of Jordan *-isomorphisms between operator algebras via their spectral multiplicativity properties of different kinds and their other characteristic invariance properties relating the spectral radius. However, there is significant difference between the previous investigations and the one we report on here. Namely, the properties we consider in this paper are assumed to be satisfied not on the whole algebra but only on certain subsets which are substructures of the general linear group. This means, and again it is the main novelty here, that the spectral multiplicativity and other related conditions are required only on some smaller sets (the so-called twisted subgroup of positive invertible elements, or the unitary group) but, as we shall see, they are still strong

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enough to imply that the transformations under considerations are closely related to Jordan $*$ -isomorphisms between the underlying full algebras. In addition to our conditions concerning the spectral radius we investigate transformations which preserve certain distance measures of very general kinds. Furthermore, we study spectral multiplicativity like and other conditions for pairs of maps defined on arbitrary sets with values in the above mentioned substructures of operator algebras.

Before presenting our results we collect the following facts concerning Jordan $*$ -isomorphisms between C^* -algebras. We first remind that by Proposition 1.3 in [12] every surjective Jordan homomorphism J between arbitrary algebras A_1, A_2 preserves invertibility and satisfies $J(a^{-1}) = J(a)^{-1}$ for any invertible element $a \in A_1$. This implies that a Jordan isomorphism maps the general linear group onto the general linear group and preserves the spectra of elements. We recall the important correspondence between the spectra of the elements ab and ba , where a, b belong to an algebra A . Namely, we always have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ and hence, if a, b are invertible, it follows that $\sigma(ab) = \sigma(ba)$. Let now A_1, A_2 be arbitrary C^* -algebras and $J: A_1 \rightarrow A_2$ be a Jordan isomorphism. By Theorem 6.3.4. in [1] there exists a central projection q (by a projection we always mean a self-adjoint idempotent) in the so-called bounded central closure of A_2 (a C^* -algebra that contains A_2 as a C^* -subalgebra) such that

$$J(ab) = qJ(a)J(b) + (1 - q)J(b)J(a), \quad a, b \in A.$$

Let $a, b \in A_1$ be invertible elements. Denote $x_1 = qJ(a), y_1 = qJ(b)$ and $x_2 = (1 - q)J(a), y_2 = (1 - q)J(b)$. We compute

$$\begin{aligned} \sigma(ab) \cup \{0\} &= \sigma(J(ab)) \cup \{0\} = \sigma(x_1 y_1) \cup \sigma(y_2 x_2) \\ &= \sigma(x_1 y_1) \cup \sigma(x_2 y_2) = \sigma(J(a)J(b)) \cup \{0\}. \end{aligned}$$

Since J preserves invertibility, $J(a)J(b)$ is invertible and by the above equality we have

$$\sigma(ab) = \sigma(J(a)J(b))$$

which proves that J is spectrally multiplicative on the general linear group.

For a C^* -algebra A_j , we denote by A_{js} the real linear subspace of all self-adjoint elements in A_j . The set of all positive elements (i.e., self-adjoint elements with non-negative spectrum) in A_j is denoted by A_{j+} . The set A_{j+}^{-1} of all invertibles in A_{j+} is a so-called twisted subgroup of the general linear group meaning that it is closed under the operation of the inverted Jordan triple product $ab^{-1}a$. For obvious reasons, it is also called the positive definite cone (or positive cone for short). Note that $A_{j+}^{-1} = \exp A_{js}$. The unitary group of A_j is denoted by U_j . Recall that we have $U_j = \exp i A_{js}$ if A_j is a von Neumann algebra. We repeat that by a symmetry we mean a self-adjoint unitary element (or, equivalently a unitary whose square is the identity).

Recall that for the spectral radius r we have the inequality $r(a) \leq \|a\|$ for every a in the C^* -algebra A_j and $r(a) = \|a\|$ holds for any normal element $a \in A_j$.

2. THE CASE OF THE SPACES OF POSITIVE INVERTIBLE ELEMENTS

Beside characterization via the spectral multiplicativity property, the first main result of the paper, Theorem 4 contains a sort of characterization of Jordan $*$ -isomorphisms in terms of a preserver property relating so-called generalized distance measures. For this we need a recently obtained very general Mazur-Ulam type result that we cite below as Theorem 3. To formulate it we need some preparation. From the paper [8] we recall the following.

Definition 1. Let X be a set equipped with a binary operation \diamond which satisfies the following conditions:

- (a1) $a \diamond a = a$ holds for every $a \in X$;
- (a2) $a \diamond (a \diamond b) = b$ holds for any $a, b \in X$;
- (a3) the equation $x \diamond a = b$ has a unique solution $x \in X$ for any given $a, b \in X$.

In this case the pair (X, \diamond) (or X itself) is called a point-reflection geometry.

For a C^* -algebra A and elements $a, b \in A_+^{-1}$ define $a \diamond b = ab^{-1}a$. In that way A_+^{-1} becomes a point-reflection geometry. Indeed, the conditions (a1), (a2) above are trivial to check. Concerning (a3) we recall that for any given $a, b \in A_+^{-1}$, the so-called Ricatti equation $xa^{-1}x = b$ has a unique solution $x = a \# b$ which is just the geometric mean of a and b defined by

$$a \# b = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}.$$

This result is usually termed as Anderson-Trapp theorem.

We need another concept, the one of so-called generalized distance measures.

Definition 2. Given an arbitrary non-empty set X , the function $d : X \times X \rightarrow [0, \infty[$ is called a generalized distance measure if it has the property that for an arbitrary pair $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$.

Hence, in this definition we require only the definiteness property of a metric but neither the symmetry nor the triangle inequality is assumed. Our general Mazur-Ulam type theorem in [8] reads as follows.

Theorem 3. Let X, Y be non-empty sets equipped with binary operations \diamond, \star , respectively, with which they form point-reflection geometries. Let $d : X \times X \rightarrow [0, \infty[$, $\rho : Y \times Y \rightarrow [0, \infty[$ be generalized distance measures. Pick $a, b \in X$, set

$$L_{a,b} = \{x \in X : d(a, x) = d(x, b \diamond a) = d(a, b)\}$$

and assume the following:

- (b1) $d(b \diamond x, b \diamond x') = d(x', x)$ holds for all $x, x' \in X$;
- (b2) $\sup\{d(x, b) : x \in L_{a,b}\} < \infty$;
- (b3) there exists a constant $K > 1$ such that $d(x, b \diamond x) \geq Kd(x, b)$ holds for every $x \in L_{a,b}$.

Let $\phi : X \rightarrow Y$ be a surjective map such that

$$\rho(\phi(x), \phi(x')) = d(x, x'), \quad x, x' \in X$$

and also assume that

- (b4) for the element $c \in Y$ with $c \star \phi(a) = \phi(b \diamond a)$ we have $\rho(c \star y, c \star y') = \rho(y', y)$ for all $y, y' \in Y$.

Then we have

$$\phi(b \diamond a) = \phi(b) \star \phi(a).$$

We shall also need the following properties defined for a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$:

- (c1) $h(t) = 0$ holds exactly for $t = 1$;
- (c2) for some $\theta > 0$ real number we have $|h(t)| \geq \theta$ for all $t \in]0, \infty[$ from outside a neighborhood of 1;
- (c3) h is differentiable at $t = 1$ and $f'(1) \neq 0$;
- (c4) $|h(t_0)| \neq |h(t_0^{-1})|$ holds for some $t_0 \in]0, \infty[$.

The first main result of the paper which involves the possibility of several characterizations of Jordan $*$ -isomorphisms reads as follows.

Theorem 4. Let A_j be a C^* -algebra for $j = 1, 2$. Suppose that ϕ is a surjection from A_{1+}^{-1} onto A_{2+}^{-1} . Consider the following statements:

- (4.1) $\sigma(ab^{-1}) = \sigma(\phi(a)\phi(b)^{-1})$, $a, b \in A_{1+}^{-1}$;
- (4.2) $r(ab^{-1} - 1) = r(\phi(a)\phi(b)^{-1} - 1)$, $a, b \in A_{1+}^{-1}$;
- (4.3) there exists a pair $h_1, h_2 :]0, \infty[\rightarrow \mathbb{R}$ of continuous functions which satisfy (c1)-(c3) and we have

$$\|h_1(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})\| = \|h_2(\phi(b)^{-\frac{1}{2}}\phi(a)\phi(b)^{-\frac{1}{2}})\|, \quad a, b \in A_{1+}^{-1};$$

- (4.4) there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 , an element $b_0 \in A_{2+}^{-1}$, a central projection $p \in \mathcal{A}_2$ and a positive real number c such that

$$\phi(a) = b_0(pJ(a)^c + (1-p)J(a)^{-c})b_0, \quad a \in A_{1+}^{-1};$$

(4.5) *there exists a Jordan *-isomorphism J from A_1 onto A_2 , an element $b_0 \in A_{2+}^{-1}$, a central projection $p \in \mathcal{A}_2$ such that*

$$\phi(a) = b_0(pJ(a) + (1-p)J(a)^{-1})b_0, \quad a \in A_{1+}^{-1};$$

(4.6) *there exists a Jordan *-isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\phi(a) = b_0J(a)b_0, \quad a \in A_{1+}^{-1}.$$

We have the implications (4.1) \Rightarrow (4.2) \Rightarrow (4.3) \Rightarrow (4.4). If $h_1 = h_2$, then we have (4.1) \Rightarrow (4.2) \Rightarrow (4.3) \Rightarrow (4.5). If $h_1 = h_2$ and satisfy (c4), then we have (4.1) \Leftrightarrow (4.2) \Leftrightarrow (4.3) \Leftrightarrow (4.6).

Before presenting the proof of Theorem 4 we make some useful remarks. First of all, whenever a normal element of a C^* -algebra is plugged in a continuous real function (with domain containing the spectrum of that element) that refers to the well-known continuous functional calculus. Let A be a C^* -algebra.

(R1) If a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ satisfies (c1), then for any $a \in A_+^{-1}$, the equality $h(a) = 0$ implies that $a = 1$. Indeed, by the spectral mapping theorem we have $h(\sigma(a)) = \sigma(h(a)) = 0$, which implies by (c1) that $\sigma(a) = \{1\}$, i.e., $\sigma(a - 1) = \{0\}$ which yields $a - 1 = 0$. It follows that the formula

$$d(a, b) = \|h(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})\|, \quad a, b \in A_+^{-1}$$

defines a generalized distance measure on A_+^{-1} .

(R2) Observe that if a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ satisfies (c1) and (c2), then for any sequence $t_n \in]0, \infty[$ with $h(t_n) \rightarrow 0$ we have $t_n \rightarrow 1$. This easily implies that, similarly, for any sequence $a_n \in A_+^{-1}$ with $h(a_n) \rightarrow 0$ in the norm topology, we have $a_n \rightarrow 1$.

(R3) Let $h :]0, \infty[\rightarrow \mathbb{R}$ be a continuous function which is differentiable at $t = 1$. Then the transformation $x \rightarrow h(x)$, $x \in A_+^{-1}$ is Fréchet-differentiable at $x = 1$ and its derivative $(Dh)(1)y = h'(1) \cdot y$, $y \in A_s$. Indeed, by the differentiability of the real function h we have a continuous function $\omega :]0, \infty[\rightarrow \mathbb{R}$ with $\omega(1) = 0$ such that $h(t) - h(1) - h'(1)(t - 1) = \omega(t)(t - 1)$ for all $t \in]0, \infty[$. It follows that $h(x) - h(1)1 - h'(1)(x - 1) = \omega(x)(x - 1)$ holds for all $x \in A_+^{-1}$ from which we obtain that $\|h(x) - h(1)1 - h'(1)(x - 1)\| \leq \|\omega(x)\| \|x - 1\|$. This implies

$$\frac{\|h(x) - h(1)1 - h'(1)(x - 1)\|}{\|x - 1\|} \rightarrow 0$$

as $x \rightarrow 1$ which proves the assertion.

(R4) If $h :]0, \infty[\rightarrow \mathbb{R}$ is a continuous function which satisfies (c1) and (c3), then we have a number $K > 1$ such that $|h(t^2)| \geq K|h(t)|$ holds for all t in an ϵ -neighbourhood of 1 with some $0 < \epsilon < 1$. Indeed, we compute

$$\frac{h(t^2)}{h(t)} = (t + 1) \frac{h(t^2)/(t^2 - 1)}{h(t)/(t - 1)} \rightarrow 2$$

as $t \rightarrow 1$ verifying our claim.

(R5) For any invertible element $x \in A$ and $a, b \in A_+^{-1}$ we have that $a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}}$ is unitarily equivalent to $b^{-\frac{1}{2}}ab^{-\frac{1}{2}}$ and $(xbx^*)^{-\frac{1}{2}}xax^*(xbx^*)^{-\frac{1}{2}}$ is unitarily equivalent to $b^{-\frac{1}{2}}ab^{-\frac{1}{2}}$. Indeed, as for the former statement, we have

$$a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}} = u^*(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})u,$$

where u is the unitary element in the polar decomposition of $b^{-\frac{1}{2}}a^{\frac{1}{2}} \in A$. As for the latter statement, we have

$$\begin{aligned} (xbx^*)^{-\frac{1}{2}}xax^*(xbx^*)^{-\frac{1}{2}} &= |b^{\frac{1}{2}}x^*|^{-1}(b^{\frac{1}{2}}x^*)^*(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})(b^{\frac{1}{2}}x^*)|b^{\frac{1}{2}}x^*|^{-1} \\ &= v^*(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})v, \end{aligned}$$

where v is the unitary element in the polar decomposition of $b^{\frac{1}{2}}x^*$.

(R6) For any scalar valued continuous function h on $]0, \infty[$, unitary $u \in A$ and positive invertible $a \in A_+^{-1}$ we have $h(ua u^{-1}) = uh(a)u^{-1}$. Indeed, it follows easily from the fact that h can be uniformly approximated by polynomials on any compact subinterval of $]0, \infty[$ and from the isometric property of the continuous functional calculus.

Proof of Theorem 4. It is obvious that (4.1) implies (4.2). To verify the implication (4.2) \Rightarrow (4.3) observe that for any $a, b \in A_{1+}^{-1}$ we have

$$\sigma(ab^{-1} - 1) = \sigma(ab^{-1}) - 1 = \sigma(b^{-\frac{1}{2}}ab^{-\frac{1}{2}}) - 1 = \sigma(b^{-\frac{1}{2}}ab^{-\frac{1}{2}} - 1)$$

which implies that

$$r(ab^{-1} - 1) = r(b^{-\frac{1}{2}}ab^{-\frac{1}{2}} - 1) = \|b^{-\frac{1}{2}}ab^{-\frac{1}{2}} - 1\|.$$

Therefore, assuming (4.2) and defining $h_1(t) = h_2(t) = t - 1$, $t \in]0, \infty[$ we plainly obtain (4.3).

The main part of the proof now follows. We assume that (4.3) holds with continuous functions $h_1, h_2 :]0, \infty[\rightarrow \mathbb{R}$ satisfying (c1)-(c3). Define

$$d(a, b) = \|h_1(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})\|, \quad a, b \in A_{1+}^{-1}$$

and

$$\rho(a, b) = \|h_2(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})\|, \quad a, b \in A_{2+}^{-1}.$$

By the remark (R1) we know that d, ρ are generalized distance measures and we have

$$(1) \quad \rho(\phi(a), \phi(b)) = d(a, b), \quad a, b \in A_{1+}^{-1}.$$

Observe the following. Applying (R5) and (R6) we have

$$d(za z^*, zb z^*) = d(a, b)$$

and

$$(2) \quad d(bx^{-1}b, bx'^{-1}b) = d(x^{-1}, x'^{-1}) = d(x', x)$$

for all $a, b, x \in A_{1+}^{-1}$ and invertible $z \in A_1$. Clearly, similar properties hold for the generalized distance measure ρ , too.

Now define the map $\phi_0 : A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ by $\phi_0(a) = \phi(1)^{-\frac{1}{2}}\phi(a)\phi(1)^{-\frac{1}{2}}$, $a \in A_{1+}^{-1}$. Plainly, ϕ_0 is a well defined and surjective map from A_{1+}^{-1} onto A_{2+}^{-1} , ϕ_0 is unital meaning that $\phi_0(1) = 1$, and the equality (1) holds also for ϕ_0 , i.e., we have $\rho(\phi_0(a), \phi_0(b)) = d(a, b)$, $a, b \in A_{1+}^{-1}$.

We are going to apply Theorem 3. In order to do that we need to check that the conditions in that theorem are satisfied. Firstly, we define the point-reflection geometry structures on A_{j+}^{-1} in the standard way, i.e., just as we did after Definition 1. The condition (b1) is fulfilled by (2).

To proceed, we claim the following. Let H be a subset of A_{1+}^{-1} with the property that there are positive numbers α, β such that $\alpha 1 \leq y \leq \beta 1$ holds for all $y \in H$. (This means that H is bounded away from zero and also from above with respect to the usual order \leq defined on the set of all self-adjoint elements coming from the notion of positivity. Recall that positive elements are the self-adjoint ones with spectrum within the set of non-negative reals.) Then we assert that there exists a number $\delta > 0$ with the property that whenever $a, b \in H$ are such that $\|a - b\| < \delta$, we necessarily have $\|b^{-\frac{1}{2}}xb^{-\frac{1}{2}} - 1\| < \epsilon$ (i.e., the spectrum of $b^{-\frac{1}{2}}xb^{-\frac{1}{2}}$ is in $]1 - \epsilon, 1 + \epsilon[$) for all $x \in L_{a,b}$, where ϵ is the number that appears in (R4) in relation with h_1 . Assume for a moment that this assertion is already proven. We can check rather easily that for $a, b \in A_{1+}^{-1}$ with $\|a - b\| < \delta$ the properties (b2) and (b3) are satisfied. Indeed, by the isometric property of the continuous functional calculus, the fulfillment of (b2) is clear since h_1 is bounded in the closed interval $[1 - \epsilon, 1 + \epsilon]$. As for (b3), applying the second equality in (2), the first part of (R5) and (R6), we can compute

$$\begin{aligned} d(x, bx^{-1}b) &= d(b^{-1}xb^{-1}, x^{-1}) = \|h_1(x^{\frac{1}{2}}(b^{-1}xb^{-1})x^{\frac{1}{2}})\| \\ &= \|h_1((x^{\frac{1}{2}}b^{-1}x^{\frac{1}{2}})^2)\| = \|h_1((b^{-\frac{1}{2}}xb^{-\frac{1}{2}})^2)\| \geq K\|h_1(b^{-\frac{1}{2}}xb^{-\frac{1}{2}})\| = Kd(x, b) \end{aligned}$$

for all $x \in L_{a,b}$ meaning that (b3) is also satisfied.

Now, in order to verify the starting assertion, assume on the contrary that we have sequences $a_n, b_n \in H, x_n \in L_{a_n, b_n}$ such that $\|a_n - b_n\| < \frac{1}{n}$ but $\|b_n^{-\frac{1}{2}} x_n b_n^{-\frac{1}{2}} - 1\| \geq \epsilon$. We compute

$$\|b_n^{-\frac{1}{2}} a_n b_n^{-\frac{1}{2}} - 1\| = \|b_n^{-\frac{1}{2}} (a_n - b_n) b_n^{-\frac{1}{2}}\| \leq \|b_n^{-1}\| \|a_n - b_n\|$$

and this last term converges to 0 since we have $\|b_n^{-1}\| \leq 1/\alpha$. Therefore, $b_n^{-\frac{1}{2}} a_n b_n^{-\frac{1}{2}} \rightarrow 1$ and by (c1) it follows that $d(a_n, b_n) = \|h_1(b_n^{-\frac{1}{2}} a_n b_n^{-\frac{1}{2}})\| \rightarrow 0$. Since $x_n \in L_{a_n, b_n}$, we have $d(a_n, x_n) = d(a_n, b_n) \rightarrow 0$ meaning that $h_1(x_n^{-\frac{1}{2}} a_n x_n^{-\frac{1}{2}}) \rightarrow 0$. Applying the observation in (R2) one can check rather easily that this implies $x_n^{-\frac{1}{2}} a_n x_n^{-\frac{1}{2}} \rightarrow 1$. Therefore, for an arbitrary scalar $0 < \gamma < 1$ we have an index n_0 such that for all $n \geq n_0$ we obtain $1 - \gamma \leq x_n^{-\frac{1}{2}} a_n x_n^{-\frac{1}{2}} \leq 1 + \gamma$ which yields $(1/(1 + \gamma))a_n \leq x_n \leq (1/(1 - \gamma))a_n$ for all $n \geq n_0$. Since we also have $b_n^{-\frac{1}{2}} a_n b_n^{-\frac{1}{2}} \rightarrow 1$, in a similar manner, we may also assume that $(1/(1 + \gamma))a_n \leq b_n \leq (1/(1 - \gamma))a_n$ holds for all $n \geq n_0$. These imply that

$$(1/(1 + \gamma) - 1/(1 - \gamma))a_n \leq x_n - b_n \leq (1/(1 - \gamma) - 1/(1 + \gamma))a_n$$

for all $n \geq n_0$. Since $\gamma > 0$ is arbitrary and we have $a_n \leq \beta 1$ for all n , we infer that $x_n - b_n \rightarrow 0$ which immediately yields $b_n^{-\frac{1}{2}} x_n b_n^{-\frac{1}{2}} \rightarrow 1$ contradicting to $\|b_n^{-\frac{1}{2}} x_n b_n^{-\frac{1}{2}} - 1\| \geq \epsilon$. This proves the above assertion and hence we have that (b2) and (b3) in Theorem 3 are satisfied. Observe that (b4) is fulfilled, too, which can be checked just as the condition (b1) above. By Theorem 3 it follows that there is a number $\delta > 0$ such that whenever $\|a - b\| < \delta$, $a, b \in A_{1+}^{-1}$, we necessarily have

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b).$$

Now pick arbitrary $a, b \in A_{1+}^{-1}$. We prove that the above displayed equality holds for a and b . To verify this, consider the curve

$$\Gamma(t) = a^{\frac{1}{2}} \left(\exp \left(t \log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) \right) \right) a^{\frac{1}{2}}, \quad t \in [0, 2]$$

connecting a and $ba^{-1}b$ and passing through b . The range of this curve is a norm-compact subset of A_{1+}^{-1} and hence it satisfies the condition we imposed on the subset H of A_{1+}^{-1} in the previous part of the proof. Therefore, there is a number $\delta > 0$ such that for any $a', b' \in \Gamma([0, 2])$ we have $\phi_0(b'a'^{-1}b') = \phi_0(b')\phi_0(a')^{-1}\phi_0(b')$. By the uniform continuity of Γ , for close enough $t, s \in [0, 2]$ we have $\|\Gamma(t) - \Gamma(s)\| < \delta$. Now, we can select a large enough n such that for the elements $a_k = \Gamma(k/2^n)$, $k = 0, 1, \dots, 2^{n+1}$ we have $\|a_k - a_{k+1}\| < \delta$. Clearly, $a_0 = a$, $a_{2^n} = b$, $a_{2^{n+1}} = ba^{-1}b$, and $a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$ holds for every $0 \leq k \leq 2^{n+1} - 2$. Moreover, by the closeness of a_k and a_{k+1} we have

$$\phi_0(a_{k+1}a_k^{-1}a_{k+1}) = \phi_0(a_{k+1})\phi_0(a_k)^{-1}\phi_0(a_{k+1})$$

for every $0 \leq k \leq 2^{n+1} - 1$. It requires purely algebraic computations to verify that then we have

$$\phi_0(a_{2^n}a_0^{-1}a_{2^n}) = \phi_0(a_{2^n})\phi_0(a_0)^{-1}\phi_0(a_{2^n}).$$

In fact, this is just the content of the technical Lemma 4.2 in [3]. As $a_0 = a$ and $a_{2^n} = b$, we get

$$(3) \quad \phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b)$$

for an arbitrary pair $a, b \in A_{1+}^{-1}$.

Putting $b = 1$ we deduce that $\phi_0(a^{-1}) = \phi_0(a)^{-1}$ for every $a \in A_{1+}^{-1}$ and then we obtain that

$$(4) \quad \phi_0(bab) = \phi_0(b)\phi_0(a)\phi_0(b), \quad a, b \in A_{1+}^{-1}.$$

One can trivially deduce using (3) and (4) that for any $a \in A_{1+}^{-1}$ we have $\phi_0(a^m) = \phi_0(a)^m$ first for any integer m and then for any rational number, too.

Pick $x \in A_{1s}$ and define $S: \mathbb{R} \rightarrow A_{2+}^{-1}$ by

$$S(t) = \phi_0(\exp(tx)), \quad t \in \mathbb{R}.$$

We assert that S is continuous with respect the norm-topology. To see this, first observe that applying (R2) for a sequence $x_n \in A_{1+}^{-1}$ we have

$$\begin{aligned} \|x_n - 1\| \rightarrow 0 &\Rightarrow \|h_1(x_n)\| \rightarrow 0 \Rightarrow d(x_n, 1) \rightarrow 0 \Rightarrow \\ &\rho(\phi_0(x_n), \phi_0(1)) = \rho(\phi_0(x_n), 1) \rightarrow 0 \Rightarrow \|h_2(\phi_0(x_n))\| \rightarrow 0 \Rightarrow \|\phi_0(x_n) - 1\| \rightarrow 0. \end{aligned}$$

Now, picking $t, t_0 \in \mathbb{R}$ we compute

$$\begin{aligned} \|S(t + t_0) - S(t_0)\| &= \|\phi_0(\exp((t + t_0)x)) - \phi_0(\exp(t_0x))\| \\ &\leq \|\phi_0(\exp(t_0x))^{\frac{1}{2}}\|^2 \|\phi_0(\exp(t_0x))^{-\frac{1}{2}} \phi_0(\exp((t + t_0)x)) \phi_0(\exp(t_0x))^{-\frac{1}{2}} - 1\| \\ &= \|\phi_0(\exp(t_0x))\| \|\phi_0\left(\exp\left(\frac{-t_0}{2}x\right) \exp((t + t_0)x) \exp\left(\frac{-t_0}{2}x\right)\right) - 1\| \\ &= \|\phi_0(\exp(t_0x))\| \|\phi_0(\exp(tx)) - 1\|. \end{aligned}$$

It follows that for $t \rightarrow 0$ we have $S(t + t_0) \rightarrow S(t_0)$ in the norm topology implying the norm-continuity of S . We next deduce that S is a one-parameter group in A_{2+}^{-1} . Indeed, let m, n, m', n' be integers with $m, m' \neq 0$. We calculate

$$\begin{aligned} S\left(\frac{n}{m} + \frac{n'}{m'}\right) &= \phi_0\left(\exp\left(\left(\frac{n}{m} + \frac{n'}{m'}\right)x\right)\right) = \phi_0\left(\exp\frac{1}{mm'}x\right)^{m'n+mn'} \\ &= \phi_0\left(\exp\frac{1}{mm'}x\right)^{m'n} \phi_0\left(\exp\frac{1}{mm'}x\right)^{mn'} = S\left(\frac{n}{m}\right) S\left(\frac{n'}{m'}\right). \end{aligned}$$

Since S is continuous, it follows that

$$S(t + t') = S(t)S(t'), \quad t, t' \in \mathbb{R}.$$

Therefore, S is a continuous one-parameter group in A_2 . By part (a) in Proposition 6.4.6 in [10] there exists $y \in A_2$ with

$$S(t) = \exp(ty), \quad t \in \mathbb{R}.$$

Since $S(t)$ is self-adjoint, we deduce that y is also self-adjoint using, e.g., (c) in Proposition 6.4.6 in [10].

Defining $f(x) = y$ we obtain a map $f : A_{1s} \rightarrow A_{2s}$ for which

$$\phi_0(\exp(tx)) = S(t) = \exp(tf(x)), \quad t \in \mathbb{R}, x \in A_{1s}.$$

As ϕ_0 preserves or, better say, respects the pair d, ρ of generalized distance measures, it is clearly injective. This implies that f is injective, too. Considering ϕ_0^{-1} in the place of ϕ_0 , we clearly have an injective map $g : A_{2s} \rightarrow A_{1s}$ such that $\phi_0^{-1}(\exp(ty)) = \exp(tg(y))$ holds for every $y \in A_{2s}$ and $t \in \mathbb{R}$. This easily implies that $y = f(g(y))$ holds for all $y \in A_{2s}$. Hence f is surjective and therefore it is a bijection from A_{1s} onto A_{2s} . Note that $f(0) = 0$ is true by the definition of f .

Our next claim is that f is a scalar multiple of a norm-isometry. To verify this, we assert that as $t \rightarrow 0$, we have that

$$(5) \quad \frac{d(\exp(tx), \exp(ty))}{|t|} \rightarrow |h'_1(1)| \|x - y\|$$

holds for all $x, y \in A_{1s}$. Clearly,

$$\begin{aligned} &\frac{\exp(-\frac{t}{2}y) \exp(tx) \exp(-\frac{t}{2}y) - 1}{t} \\ &= \exp\left(-\frac{t}{2}y\right) \frac{(\exp(tx) - 1) - (\exp(ty) - 1)}{t} \exp\left(-\frac{t}{2}y\right) \rightarrow x - y. \end{aligned}$$

Since

$$\frac{d(\exp(tx), \exp(ty))}{|t|} = \frac{\|h_1(\exp(-\frac{t}{2}y) \exp(tx) \exp(-\frac{t}{2}y)) - h_1(1)\|}{|t|},$$

the validity of (5) follows from (R3) and from the chain rule. Similarly, we obtain that

$$\frac{\rho(\exp(tx), \exp(ty))}{|t|} \rightarrow |h'_2(1)| \|x - y\|$$

holds for all $x, y \in A_{2s}$. Since ϕ_0 respects the pair d, ρ of generalized distance measures, it now follows that $|h'_1(1)| \|x - y\| = |h'_2(1)| \|f(x) - f(y)\|$, $x, y \in A_{1s}$. This implies that we have a positive scalar c such that $(1/c)f$ is a surjective isometry from A_{1s} onto A_{2s} . Since $f(0) = 0$, by the classical Mazur-Ulam theorem we infer that f is linear. The structure of linear isometries between the self-adjoint parts of C^* -algebras is well-known. In fact, according to a theorem of Kadison [6, Theorem 2] we obtain that $(1/c)f(1)$ is a central symmetry in A_2 and there is a Jordan $*$ -isomorphism J from A_1 onto A_2 such that

$$f(x) = f(1)J(x), \quad x \in A_{1s}.$$

Put $p = (1 + (1/c)f(1))/2$. Then p is a central projection in A_2 and

$$f(x) = c(pJ(x) - (1 - p)J(x)), \quad x \in A_{1s}.$$

We now calculate

$$\begin{aligned} \phi_0(\exp x) &= \exp(c(pJ(x) - (1 - p)J(x))) \\ &= \sum_{n=0}^{\infty} \frac{(c(pJ(x) - (1 - p)J(x)))^n}{n!} = \sum_{n=0}^{\infty} \frac{pJ((cx)^n) + (1 - p)J((-cx)^n)}{n!} \\ &= pJ(\exp(cx)) + (1 - p)J(\exp(-cx)) = pJ(\exp x)^c + (1 - p)J(\exp x)^{-c} \end{aligned}$$

for every $x \in A_{1s}$. Thus

$$(6) \quad \phi_0(a) = pJ(a)^c + (1 - p)J(a)^{-c}, \quad a \in A_{1+}^{-1},$$

and we arrive at (4.4). Observe further that if we assume $h_1 = h_2$ and that the central projection p above is non-trivial, then inserting $a = t1$, $t \in]0, \infty[$ and $b = 1$ into (6), and using the generalized distance measure preserving property of ϕ_0 , we easily obtain

$$|h_1(t)| = \max\{|h_1(t^c)|, |h_1(t^{-c})|\}$$

for all positive real number t . From this we first deduce that $|h_1(t)| = |h_1(t^{-1})|$ and then that $|h_1(t)| = |h_1(t^c)|$, $t \in]0, \infty[$. Differentiating h_1 at $t = 1$ we easily obtain that $c = 1$. Therefore, in the case where the projection p is non-trivial, we have $c = 1$. Similar argument applies when p is trivial, i.e., when $|h_1(t)| = |h_1(t^c)|$ or $|h_1(t)| = |h_1(t^{-c})|$, $t \in]0, \infty[$. This gives us the implication (4.3) \Rightarrow (4.5).

If above we also assume that $h_1 = h_2$ and $|h_1(t_0)| \neq |h_1(t_0^{-1})|$ holds for some $t_0 \in]0, \infty[$, then reconsidering the last part of the argument above, we see that p is necessarily trivial, in fact $p = 1$, and $c = 1$ verifying the implication (4.3) \Rightarrow (4.6).

To complete the proof, suppose now that (4.6) holds. For any $a, b \in A_{1+}^{-1}$ we infer that

$$\begin{aligned} \sigma(\phi(a)\phi(b)^{-1}) &= \sigma(b_0J(a)J(b)^{-1}b_0^{-1}) \\ &= \sigma(b_0J(a)J(b^{-1})b_0^{-1}) = \sigma(J(a)J(b^{-1})) = \sigma(ab^{-1}) \end{aligned}$$

and hence we obtain (4.1). \square

Observe that the implication (4.3) \Rightarrow (4.5) gives a substantial generalization of our former result Theorem 9 in [5] about the structure of Thompson isometries between the positive definite cones of C^* -algebras which is one of the main results in that paper. Indeed, one needs only to choose $h_1 = h_2 = \log$ to obtain that result from Theorem 4.

We also remark that in [8] we have presented structural results for surjective maps between the positive definite cones of factor von Neumann algebras which respect a pair of generalized distance measures of the form similar to what appears in (4.3) above with the difference that in [8] we have considered arbitrary unitarily invariant norms in the place of the unique C^* -algebra norm $\|\cdot\|$ (operator norm). So in a sense those results concern more general distance measures but in a more restricted context. Indeed, due to the (mainly algebraic) tools we have applied there, the

results [8] have been obtained only for factor von Neumann algebras and not, like here, for general C^* -algebras. Related results in the context of matrix algebras appeared in our former paper [9].

In what follows we present several sorts of extensions of our previous theorem.

Corollary 5. *Let A_j be a C^* -algebra for $j = 1, 2$ and suppose that ϕ and ψ are surjections from A_{1+}^{-1} onto A_{2+}^{-1} . Then the following assertions are equivalent:*

- (5.1) $\sigma(ab) = \sigma(\phi(a)\psi(b))$, $a, b \in A_{1+}^{-1}$;
- (5.2) $r(ab - 1) = r(\phi(a)\psi(b) - 1)$, $a, b \in A_{1+}^{-1}$;
- (5.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\|h(b^{\frac{1}{2}}ab^{\frac{1}{2}})\| = \|h(\psi(b)^{\frac{1}{2}}\phi(a)\psi(b)^{\frac{1}{2}})\|, \quad a, b \in A_{1+}^{-1};$$

- (5.4) *there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 and $b_0 \in A_{2+}^{-1}$ such that*

$$\phi(a) = b_0 J(a) b_0, \quad \psi(a) = b_0^{-1} J(a) b_0^{-1}, \quad a \in A_{1+}^{-1}.$$

Proof. The implication (5.1) \Rightarrow (5.2) is obvious, to see (5.2) \Rightarrow (5.3) set $h(t) = t - 1$, $t \in]0, \infty[$.

Suppose that (5.3) holds. For any $a \in A_{1+}^{-1}$ we have

$$0 = \|h(a^{-\frac{1}{2}}aa^{-\frac{1}{2}})\| = \|h(\psi(a^{-1})^{\frac{1}{2}}\phi(a)\psi(a^{-1})^{\frac{1}{2}})\|$$

which implies $\psi(a^{-1})^{\frac{1}{2}}\phi(a)\psi(a^{-1})^{\frac{1}{2}} = 1$, i.e., $\phi(a) = \psi(a^{-1})^{-1}$. It then follows that

$$\|h(b^{-\frac{1}{2}}ab^{-\frac{1}{2}})\| = \|h(\psi(b^{-1})^{\frac{1}{2}}\phi(a)\psi(b^{-1})^{\frac{1}{2}})\| = \|h(\phi(b)^{-\frac{1}{2}}\phi(a)\phi(b)^{-\frac{1}{2}})\|.$$

Applying Theorem 4 we obtain that there is a Jordan $*$ -isomorphism J from A_1 onto A_2 and $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \quad a \in A_{1+}^{-1}.$$

Moreover, we infer that

$$\psi(a) = \phi(a^{-1})^{-1} = b_0^{-1} J(a) b_0^{-1}, \quad a \in A_{1+}^{-1}$$

and obtain (5.4).

Suppose now that (5.4) holds. For any $a, b \in A_{1+}^{-1}$ we calculate

$$\sigma(\phi(a)\psi(b)) = \sigma(b_0 J(a) J(b) b_0^{-1}) = \sigma(J(a)J(b)) = \sigma(ab).$$

Thus (5.1) holds and the proof is complete. \square

From the above statement we immediately obtain the following corollary which presents a complete description of spectrally multiplicative maps between the positive definite cones of C^* -algebras.

Corollary 6. *Let A_j be a C^* -algebra for $j = 1, 2$. Suppose that ϕ is a surjection from A_{1+}^{-1} onto A_{2+}^{-1} . Then the following statements are equivalent:*

- (6.1) $\sigma(ab) = \sigma(\phi(a)\phi(b))$, $a, b \in A_{1+}^{-1}$;
- (6.2) $r(ab - 1) = r(\phi(a)\phi(b) - 1)$, $a, b \in A_{1+}^{-1}$;
- (6.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\|h(b^{\frac{1}{2}}ab^{\frac{1}{2}})\| = \|h(\phi(b)^{\frac{1}{2}}\phi(a)\phi(b)^{\frac{1}{2}})\|, \quad a, b \in A_{1+}^{-1};$$

- (6.4) *there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 such that*

$$\phi(a) = J(a), \quad a \in A_{1+}^{-1}.$$

Proof. In the light of the proofs of the previous results, the only implication we need to verify is (6.3) \Rightarrow (6.4). Assuming (6.3), by Corollary 6 there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 and $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \quad \psi(a) = b_0^{-1} J(a) b_0^{-1}, \quad a \in A_{1+}^{-1}.$$

Choosing $a = 1$, it follows that $b_0^2 = b_0^{-2}$ which implies $b_0 = 1$ and we are done. \square

With some extra efforts, from Corollary 5 we can deduce the following formally even more general result on the structure of maps defined on arbitrary sets with values in the positive definite cones of C^* -algebras with a specific property closely related to spectral multiplicativity.

Theorem 7. *Let A_j be a C^* -algebra for $j = 1, 2$ and F a non-empty set. Suppose that Φ_1 and Ψ_1 are surjections from F onto A_{1+}^{-1} and that Φ_2 and Ψ_2 are surjections from F onto A_{2+}^{-1} . The following statements are equivalent:*

- (7.1) $\sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y)), \quad x, y \in F;$
- (7.2) $r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_2(x)\Psi_2(y) - 1), \quad x, y \in F;$
- (7.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\|h(\Psi_1(y)^{\frac{1}{2}}\Phi_1(x)\Psi_1(y)^{\frac{1}{2}})\| = \|h(\Psi_2(y)^{\frac{1}{2}}\Phi_2(x)\Psi_2(y)^{\frac{1}{2}})\|, \quad x, y \in F;$$

- (7.4) *there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that*

$$\Phi_2(x) = b_0 J(\Phi_1(x)) b_0, \quad \Psi_2(x) = b_0^{-1} J(\Psi_1(x)) b_0^{-1}, \quad x, y \in F.$$

Proof. Again, in the light of the previous proofs the implications (7.1) \Rightarrow (7.2) \Rightarrow (7.3) are apparent.

Suppose that (7.3) holds. To prove (7.4), we first observe that $\Phi_1(x) = \Phi_1(x')$ implies $\Phi_2(x) = \Phi_2(x')$. Indeed, let $x, x' \in F$ and assume that $\Phi_1(x) = \Phi_1(x')$. Since $\Psi_1(F) = A_{1+}^{-1}$, there exists $y \in F$ with $\Psi_1(y) = \Phi_1(x)^{-1}$. Then we have

$$0 = \|h(\Psi_1(y)^{\frac{1}{2}}\Phi_1(x)\Psi_1(y)^{\frac{1}{2}})\| = \|h(\Psi_2(y)^{\frac{1}{2}}\Phi_2(x)\Psi_2(y)^{\frac{1}{2}})\|$$

implying that $\Psi_2(y)^{\frac{1}{2}}\Phi_2(x)\Psi_2(y)^{\frac{1}{2}} = 1$. Thus we have $\Psi_2(y)^{-1} = \Phi_2(x)$. In a similar way we obtain that $\Psi_2(y)^{-1} = \Phi_2(x')$ holds, too. It then follows that $\Phi_2(x) = \Phi_2(x')$. In the same way one can deduce that $\Psi_1(x) = \Psi_1(x')$ implies $\Psi_2(x) = \Psi_2(x')$. After this we define maps $\phi, \psi : A_{1+}^{-1} \rightarrow A_{2+}^{-1}$ by $\phi(\Phi_1(x)) = \Phi_2(x)$, $x \in F$ and by $\psi(\Psi_1(x)) = \Psi_2(x)$, $x \in F$. Apparently, ϕ, ψ are well defined and surjective. Rewriting the displayed equality in (7.3) we have

$$\|h(b^{\frac{1}{2}}ab^{\frac{1}{2}})\| = \|h(\psi(b)^{\frac{1}{2}}\phi(a)\psi(b)^{\frac{1}{2}})\|, \quad a, b \in A_{1+}^{-1}.$$

By Corollary 5 there exists a Jordan $*$ -isomorphism J from A_1 onto A_2 and an element $b_0 \in A_{2+}^{-1}$ such that

$$\phi(a) = b_0 J(a) b_0, \quad \psi(a) = b_0^{-1} J(a) b_0^{-1}, \quad a \in A_{1+}^{-1}.$$

In other words, we have

$$\Phi_2(x) = b_0 J(\Phi_1(x)) b_0, \quad \Psi_2(x) = b_0^{-1} J(\Psi_1(x)) b_0^{-1}, \quad x \in F$$

and this proves (7.4).

Finally, in a way similar to the proof of Corollary 5 one can check that (7.4) implies (7.1) which finishes the proof of the theorem. \square

We conclude the section with a few other corollaries which provide characterizations of Jordan $*$ -isomorphisms of the self-adjoint parts of C^* -algebras by means of their certain spectral multiplicativity properties.

Corollary 8. *Let A_j be a C^* -algebra for $j = 1, 2$. Suppose that f and g are surjections from A_{1s} onto A_{2s} . Then the following assertions are equivalent:*

- (8.1) $\sigma(\exp x \exp y) = \sigma(\exp f(x) \exp g(y)), \quad x, y \in A_{1s};$
- (8.2) $r(\exp x \exp y - 1) = r(\exp f(x) \exp g(y) - 1), \quad x, y \in A_{1s};$
- (8.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\begin{aligned} \|h(\exp(y/2) \exp(x) \exp(y/2))\| \\ = \|h(\exp(g(y)/2) \exp(f(x)) \exp(g(y)/2))\|, \quad x, y \in A_{1s}; \end{aligned}$$

- (8.4) *there exists a Jordan $*$ -isomorphism J from A_{1s} onto A_{2s} and $b_0 \in A_{2+}^{-1}$ such that*

$$\exp f(x) = b_0 (\exp J(x)) b_0, \quad \exp g(x) = b_0^{-1} (\exp J(x)) b_0^{-1}, \quad x, y \in A_{1s}.$$

Moreover, in any of the above cases, if $f(0) = 0$, then we $f = g = J$ on A_{1s} .

Proof. Define $\phi(a) = \exp(f(\log a))$, $a \in A_{1+}^{-1}$ and $\psi(b) = \exp(g(\log b))$, $b \in A_{1+}^{-1}$. Apply Corollary 5 to see the equivalence of the assertions (8.1)-(8.4). If $f(0) = 0$, we easily obtain $b_0 = 1$ which implies $f = g = J$ on A_{1s} . \square

If we have $f = g$ in the previous corollary, we trivially obtain the following statement.

Corollary 9. *Let A_j be a C^* -algebra for $j = 1, 2$. Suppose that f is a surjection from A_{1s} onto A_{2s} . Then the following assertions are equivalent:*

- (9.1) $\sigma(\exp x \exp y) = \sigma(\exp f(x) \exp f(y))$, $x, y \in A_{1s}$;
- (9.2) $r(\exp x \exp y - 1) = r(\exp f(x) \exp f(y) - 1)$, $x, y \in A_{1s}$;
- (9.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\begin{aligned} \|h(\exp(y/2) \exp(x) \exp(y/2))\| \\ = \|h(\exp(f(y)/2) \exp(f(x)) \exp(f(y)/2))\|, \quad x, y \in A_{1s}; \end{aligned}$$

- (9.4) *there exists a Jordan $*$ -isomorphism J from A_{1s} onto A_{2s} such that $f = J$ on A_{1s} .*

Similarly, putting $g(y) = -f(-y)$, $y \in \mathcal{A}_{1s}$ into Corollary 8 we have the following statement.

Corollary 10. *Let A_j be a C^* -algebra for $j = 1, 2$. Suppose that f is a surjection from A_{1s} onto A_{2s} . Then the following assertions are equivalent:*

- (10.1) $\sigma(\exp x (\exp y)^{-1}) = \sigma(\exp f(x) (\exp f(y))^{-1})$, $x, y \in A_{1s}$;
- (10.2) $r(\exp x (\exp y)^{-1} - 1) = r(\exp f(x) (\exp f(y))^{-1} - 1)$, $x, y \in A_{1s}$;
- (10.3) *there is a continuous function $h :]0, \infty[\rightarrow \mathbb{R}$ which satisfies (c1)-(c4) and we have*

$$\begin{aligned} \|h(\exp(y)^{-\frac{1}{2}} \exp(x) \exp(y)^{-\frac{1}{2}})\| \\ = \|h(\exp(f(y))^{-\frac{1}{2}} \exp(f(x)) \exp(f(y))^{-\frac{1}{2}})\|, \quad x, y \in A_{1s}; \end{aligned}$$

- (10.4) *there exists a Jordan $*$ -isomorphism J from A_{1s} onto A_{2s} and $b_0 \in A_{2+}^{-1}$ such that*

$$\exp f(x) = b_0 (\exp J(x)) b_0, \quad x, y \in A_{1s}.$$

Moreover, in any of the above cases, if $f(0) = 0$, then $f = J$ holds on A_{1s} .

3. THE CASE OF THE UNITARY GROUPS

In the last part of our paper we present spectral conditions for Jordan $*$ -isomorphisms between the unitary groups of von Neumann algebras. In the proof of our second main result Theorem 12 below we apply the next general Mazur-Ulam type result relating groups. It appeared as Proposition 20 in [8] (cf. Corollary 3.9 in [3]).

Theorem 11. *Suppose that G and H are groups equipped with generalized distance measures d and ρ , respectively. Pick $a, b \in G$, set*

$$L_{a,b} = \{x \in G : d(a, x) = d(x, ba^{-1}b) = d(a, b)\},$$

and assume the following:

- (d1) $d(bx^{-1}b, bx'^{-1}b) = d(x', x)$ holds for all $x, x' \in G$;
- (d2) $\sup\{d(x, b) : x \in L_{a,b}\} < \infty$;
- (d3) *there exists a constant $K > 1$ such that*

$$d(x, bx^{-1}b) \geq Kd(x, b), \quad x \in L_{a,b};$$

- (d4) $\rho(cy^{-1}c', cy'^{-1}c') = \rho(y', y)$ holds for all $c, c', y, y' \in H$.

Then for any surjective map $\phi : G \rightarrow H$ which satisfies

$$\rho(\phi(x), \phi(x')) = d(x, x'), \quad x, x' \in G$$

we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

One may ask if the above statement can be deduced from Theorem 3. The easy answer is "no", since the natural operation $ab^{-1}a$ called the inverted Jordan triple product in a group does not generally satisfy the uniqueness part of the condition in (a3).

Analogously to Section 2, below we shall consider generalized distance measures on unitary groups obtained from continuous functions defined on the unit circle \mathbb{T} .

For any continuous function $h : \mathbb{T} \rightarrow \mathbb{C}$ we shall consider the following properties:

- (e1) $h(z) = 0$ holds exactly for $z = 1$;
- (e2) h is differentiable at $z = 1$ meaning that the limit $\lim_{z \rightarrow 1} \frac{h(z) - h(1)}{z - 1}$ exists and we assume that it is non-zero.

Observe that just as in the remark (R4) one can prove that the condition imply that for any number $0 \leq K < 2$ we have $|h(z^2)| \geq K|h(z)|$ for all $z \in \mathbb{T}$ close enough to 1.

The second main result of the paper reads as follows.

Theorem 12. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ is a surjection from U_1 onto U_2 . The following two conditions are equivalent:*

- (12.1) $\sigma(ab^{-1}) = \sigma(\phi(a)\phi(b)^{-1})$, $a, b \in U_1$;
- (12.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and an element $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \quad a \in U_1.$$

Moreover, the following three conditions are also equivalent:

- (12.3) $r(ab^{-1} - 1) = r(\phi(a)\phi(b)^{-1} - 1)$, $a, b \in U_1$;
- (12.4) *there exists a pair $h_1, h_2 : \mathbb{T} \rightarrow \mathbb{C}$ of continuous functions which satisfy (e1)-(e2) and we have*

$$\|h_1(ab^{-1})\| = \|h_2(\phi(a)\phi(b)^{-1})\|, \quad a, b \in U_1;$$

- (12.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2 , an element $u_0 \in U_2$ and a central projection $p \in A_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1 - p)J(a)^{-1}), \quad a \in U_1.$$

Proof. We begin with the second part of the theorem. To see the implication (12.3) \Rightarrow (12.4) consider simply the functions $h_1(z) = h_2(z) = z - 1$, $z \in \mathbb{T}$.

In the rest of the proof we follow the proof of Theorem 4 rather closely in spirit. Assume that (12.4) holds with continuous functions $h_1, h_2 : \mathbb{T} \rightarrow \mathbb{C}$ satisfying (e1)-(e2). Define

$$d(a, b) = \|h_1(ab^{-1})\|, \quad a, b \in U_1$$

and

$$\rho(a, b) = \|h_2(ab^{-1})\|, \quad a, b \in U_2.$$

By the property (e1) we obtain that d, ρ are generalized distance measures and we have

$$(7) \quad \rho(\phi(a), \phi(b)) = d(a, b), \quad a, b \in U_1.$$

One can easily check that

$$d(zaw, zbw) = d(a, b)$$

and

$$(8) \quad d(bx^{-1}b, bx'^{-1}b) = d(x^{-1}, x'^{-1}) = d(x', x)$$

are satisfied for all $a, b, x, z, w \in U_1$. Clearly, similar properties hold for the generalized distance measure ρ , too.

Define the map $\phi_0 : U_1 \rightarrow U_2$ by $\phi_0(a) = \phi(1)^{-1}\phi(a)$, $a \in U_1$. Plainly, ϕ_0 is a well defined and surjective map from U_1 onto U_2 , it is unital meaning that $\phi_0(1) = 1$, and the equality (7) holds also for ϕ_0 , i.e., we have

$$(9) \quad \rho(\phi_0(a), \phi_0(b)) = d(a, b), \quad a, b \in U_1.$$

We are going to apply Theorem 11 for $G = U_1, H = U_2$, for the above defined distance measures d, ρ and for the surjective map ϕ_0 . We have seen in (8) that the conditions (d1), (d4) in Theorem 11

are satisfied. The condition (d2) is also fulfilled as a consequence of the boundedness of the continuous function h_1 . Now we show that (d3) is satisfied for such $a, b \in U_1$ which are close enough to each other in norm. To see this, we shall need the following simple observation: for any sequences a_n, b_n in U_1 we have

$$\|a_n - b_n\| = \|a_n b_n^{-1} - 1\| \rightarrow 0 \Leftrightarrow \|h_1(a_n b_n^{-1})\| = d(a_n, b_n) \rightarrow 0$$

and similar observation holds for the generalized distance measure ρ as well. In fact, this follows rather easily from the continuity of h_1 and the property (e1). In particular, we obtain that "convergence" in any of the generalized distance measures d, ρ is equivalent to the convergence in the norm $\|\cdot\|$.

In order to show that the condition (d3) holds for close enough $a, b \in U_1$, assume on the contrary that we have sequences $a_n, b_n \in U_1$ and $x_n \in L_{a_n, b_n}$ such that $\|a_n - b_n\| \rightarrow 0$ and

$$d(x_n, b_n x_n^{-1} b_n) < (3/2)d(x_n, b_n)$$

holds for every positive integer n . This latter inequality means that

$$\|h_1((x_n b_n^{-1})^2)\| < (3/2)\|h_1(x_n b_n^{-1})\|$$

holds for all n . Since $d(a_n, x_n) = d(a_n, b_n) \rightarrow 0$, we have $a_n x_n^{-1}, a_n b_n^{-1} \rightarrow 1$ in norm which apparently implies that $x_n b_n^{-1} \rightarrow 1$ in norm. On the other hand, we know that $|h(z^2)| \geq (3/2)|h(z)|$ for all $z \in \mathbb{T}$ which are close enough to 1. Therefore, we obtain $\|h_1((x_n b_n^{-1})^2)\| \geq 3/2\|h_1(x_n b_n^{-1})\|$ for large enough n , a contradiction. This shows that the condition (d3) is satisfied for close enough $a, b \in U_1$. Applying Theorem 11 it follows that there is $\delta > 0$ such that for every $\|a - b\| < \delta$, $a, b \in U_1$ we have

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b).$$

Just as in the first section of the proof of Theorem 1 in [5] we then deduce that

$$\phi_0(ba^{-1}b) = \phi_0(b)\phi_0(a)^{-1}\phi_0(b)$$

holds not only locally, but also globally, i.e., for any $a, b \in U_1$. Putting $b = 1$ we get $\phi_0(a^{-1}) = \phi_0(a)^{-1}$ for every $a \in U_1$ and then obtain that

$$(10) \quad \phi_0(bab) = \phi_0(b)\phi_0(a)\phi_0(b), \quad a, b \in U_1.$$

By the equivalence of the convergence in d, ρ and in the norm we deduce that ϕ_0 is norm-continuous. Therefore, following the proof of Theorem 1 in [5] on page 160-161 employing one-parameter unitary groups, we infer that there is a bijective map $f : M_{1s} \rightarrow M_{2s}$ with $f(0) = 0$ for which

$$\phi_0(\exp(itx)) = \exp(itf(x)), \quad t \in \mathbb{R}, x \in M_{1s}.$$

Just as in the proof of Theorem 4 we claim that f is a scalar multiple of a norm-isometry. To verify this, we observe that one can prove similarly to (5) that

$$\frac{d(\exp(itx), \exp(ity))}{|t|} \rightarrow |h'_1(1)|\|x - y\|$$

holds for all $x, y \in M_{1s}$ as $t \rightarrow 0$. We omit the details. Similarly, we obtain that

$$\frac{\rho(\exp(itx), \exp(ity))}{|t|} \rightarrow |h'_2(1)|\|x - y\|$$

holds for all $x, y \in M_{2s}$ as $t \rightarrow 0$. Since ϕ_0 respects the pair d, ρ of generalized distance measures, i.e., satisfies (9), it follows that $|h'_1(1)|\|x - y\| = |h'_2(1)|\|f(x) - f(y)\|$, $x, y \in M_{1s}$. This implies that we have a positive scalar c such that $(1/c)f$ is a surjective isometry from M_{1s} onto M_{2s} . Just as in the proof of Theorem 4, since $f(0) = 0$, by Mazur-Ulam theorem we infer that f is linear and next apply Kadison's theorem [6, Thorem 2] to obtain that $(1/c)f(1)$ is a central symmetry in M_2 and there is a Jordan *-isomorphism J from M_1 onto M_2 such that

$$f(x) = f(1)J(x), \quad x \in M_{1s}.$$

Set $p = (1 + (1/c)f(1))/2$. Then p is a central projection in M_2 and

$$f(x) = c(pJ(x) - (1 - p)J(x)), \quad x \in M_{1s}.$$

Next an easy calculation yields that

$$(11) \quad \phi_0(\exp ix) = \exp(c(pJ(ix) - (1-p)J(ix))) \\ = pJ(\exp(icx)) + (1-p)J(\exp(-icx))$$

holds for every $x \in M_{1s}$. We assert that c is necessarily an integer. Indeed, since ϕ_0 is unital and satisfies (10), it follows that ϕ_0 sends symmetries to symmetries. Therefore, for any non-zero projection q in M_1 , the spectrum of

$$\phi_0(\exp i\pi q) = pJ(\exp(ic\pi q)) + (1-p)J(\exp(-ic\pi q))$$

is contained in $\{-1, 1\}$. Since J preserves the spectrum and p is central, it follows that at least one of the sets $\{1, \exp(ic\pi)\}$, $\{1, \exp(-ic\pi)\}$ (p might be trivial) is contained in $\{-1, 1\}$. This gives as that c is an integer and since it is assumed also to be positive, we obtain that c is a positive integer. Therefore, by (11) we have

$$(12) \quad \phi_0(a) = pJ(a^c) + (1-p)J(a^{-c}), \quad a \in U_1.$$

Now we prove that $c = 1$. Indeed, assuming that the central projection p above is non-trivial, inserting scalars $a = z1$, $z \in \mathbb{T}$ and $b = 1$ into (12), and using the generalized distance measure preserving property of ϕ_0 , we easily obtain

$$|h_1(z)| = \max\{|h_2(z^c)|, |h_2(z^{-c})|\}$$

for all $z \in \mathbb{T}$. Since h_1, h_2 has unique roots at $z = 1$, we infer that c must be 1. Similar argument works also in the case where p is trivial. This completes the proof of the implication (12.4) \Rightarrow (12.5).

Assume (12.5) holds. We compute

$$(13) \quad r(\phi(a)\phi(b)^{-1} - 1) = \|\phi(a)\phi(b)^{-1} - 1\| \\ = \|u_0(pJ(a)J(b)^{-1} + (1-p)J(a)^{-1}J(b))u_0^{-1} - 1\| \\ = \|pJ(a)J(b)^{-1} + (1-p)J(a)^{-1}J(b) - 1\| \\ = \max\{\|p(J(a)J(b)^{-1} - 1)\|, \|(1-p)(J(a)^{-1}J(b) - 1)\|\}.$$

Furthermore, by taking adjoints we have

$$(14) \quad \|(1-p)(J(a)^{-1}J(b) - 1)\| = \|(1-p)(J(b)^{-1}J(a) - 1)\| \\ = \|J(b)^{-1}(1-p)(J(a) - J(b))\| = \|(1-p)(J(a) - J(b))J(b)^{-1}\| \\ = \|(1-p)(J(a)J(b)^{-1} - 1)\|$$

since $1-p$ commutes with every element in M_2 . It follows from (13) and (14) that

$$r(\phi(a)\phi(b)^{-1} - 1) = \max\{\|p(J(a)J(b)^{-1} - 1)\|, \|(1-p)(J(a)J(b)^{-1} - 1)\|\} \\ = \|p(J(a)J(b)^{-1} - 1) + (1-p)(J(a)J(b)^{-1} - 1)\| = \|J(a)J(b)^{-1} - 1\| \\ = r(J(a)J(b)^{-1} - 1) = r(ab^{-1} - 1).$$

The last equality follows from the spectral multiplicativity of J . Thus we obtain (12.3).

Let us consider now the first part of the theorem. Assume (12.1) holds. It trivially implies (12.3) which implies (12.5). Consequently, there exists a Jordan *-isomorphism J from M_1 onto M_2 , an element $u_0 \in U_2$ and a central projection $p \in A_2$ such that

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \quad a \in U_1;$$

It is not hard to verify that the central projection p above must be the identity and hence we obtain (12.2). The implication (12.2) \Rightarrow (12.1) is trivial to check and hence the proof of the theorem is complete. □

Corollary 13. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ and ψ are surjections from U_1 onto U_2 . Then the following two conditions are equivalent:*

$$(13.1) \quad \sigma(ab) = \sigma(\phi(a)\psi(b)), \quad a, b \in U_1;$$

(13.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \psi(a) = J(a) u_0^{-1}, \quad a \in U_1.$$

Moreover the following three conditions are also equivalent:

$$(13.3) \quad r(ab - 1) = r(\phi(a)\psi(b) - 1), \quad a, b \in U_1;$$

(13.4) *there exists a pair $h_1, h_2 : \mathbb{T} \rightarrow \mathbb{C}$ of continuous functions which satisfy (e1)-(e2) and we have*

$$\|h_1(ab)\| = \|h_2(\phi(a)\psi(b))\|, \quad a, b \in U_1;$$

(13.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2 , a central projection $p \in M_2$, and $u_0 \in U_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \psi(a) = (pJ(a) + (1-p)J(a)^{-1})u_0^{-1}, \quad a \in U_1.$$

Proof. Setting $b = a^{-1}$, from both of (13.1) and (13.4) we obtain $\psi(a^{-1}) = \phi(a)^{-1}$. Easy application of Theorem 12 gives the implications (13.1) \Rightarrow (13.2) and (13.4) \Rightarrow (13.5). The rest of the proof can be shown by arguments already employed in the previous part of the paper. For example, the implication (13.5) \Rightarrow (13.3) can be proved by a reasoning similar to the one that has appeared in the proof of the implication (12.5) \Rightarrow (12.3). We omit the details. \square

Corollary 14. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$. Suppose that ϕ is a surjection from U_1 onto U_2 . Then the following conditions are equivalent:*

$$(14.1) \quad \sigma(ab) = \sigma(\phi(a)\phi(b)), \quad a, b \in U_1;$$

(14.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and a central symmetry $u_0 \in U_2$ such that*

$$\phi(a) = u_0 J(a), \quad a \in U_1.$$

Moreover the following three conditions are also equivalent:

$$(14.3) \quad r(ab - 1) = r(\phi(a)\phi(b) - 1), \quad a, b \in U_1;$$

(14.4) *there exists a pair $h_1, h_2 : \mathbb{T} \rightarrow \mathbb{C}$ of continuous functions which satisfy (e1)-(e2) and we have*

$$\|h_1(ab)\| = \|h_2(\phi(a)\phi(b))\|, \quad a, b \in U_1;$$

(14.5) *there exists a Jordan *-isomorphism J from M_1 onto M_2 , a central projection $p \in M_2$, and a central symmetry $u_0 \in U_2$ such that*

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \quad a \in U_1.$$

Proof. We apply Theorem 13 for $\psi = \phi$. The only implications that need closer look are (14.1) \Rightarrow (14.2) and (14.4) \Rightarrow (14.5). Assuming (14.1) we have a Jordan *-isomorphism $J : M_1 \rightarrow M_2$ and an element $u_0 \in U_2$ such that $\phi(a) = u_0 J(a) = J(a) u_0^{-1}$, $a \in U_1$. Since the unitary group linearly generate the whole algebra, it follows that $u_0 x = x u_0^{-1}$ holds for all $x \in M_2$ which readily implies that $u_0 = u_0^{-1}$ and then that u_0 is central. Similar argument applies for the implication (14.4) \Rightarrow (14.5), the rest of the proof is either easy or can be proved by already employed arguments. \square

In order to avoid overcomplications in the formulations of the remaining results, in what follows we shall omit conditions regarding the invariance properties of the transformations under considerations with respect to generalized distance measures. We are convinced that having read the paper carefully up to this point, it would be an easy task for the reader to complete the results with such additional equivalent conditions.

Theorem 15. *Let M_j be a von Neumann algebra with unitary group U_j for $j = 1, 2$ and F a non-empty set. Suppose that Φ_j and Ψ_j are surjections from F onto U_j for $j = 1, 2$. Then the following two conditions are equivalent*

$$(15.1) \quad \sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y)), \quad x, y \in F;$$

(15.2) *there exists a Jordan *-isomorphism J from M_1 onto M_2 and $u_0 \in U_2$ such that*

$$\Phi_2(x) = u_0 J(\Phi_1(x)), \Psi_2(x) = J(\Psi_1(x)) u_0^{-1}, \quad x \in F.$$

Moreover, the following two conditions are also equivalent:

$$(15.3) \quad r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_2(x)\Psi_2(y) - 1), \quad x, y \in F;$$

(15.4) *there exists a Jordan *-isomorphism J from M_1 onto M_2 , a central projection $p \in M_2$, and $u_0 \in U_2$ such that*

$$\Phi_2(x) = u_0(pJ(\Phi_1(x)) + (1-p)J(\Phi_1(x))^{-1}), \quad x \in F$$

and

$$\Psi_2(x) = (pJ(\Psi_1(x)) + (1-p)J(\Psi_1(x))^{-1})u_0^{-1}, \quad x \in F.$$

Proof. Suppose that (15.2) holds. We simply infer that

$$\sigma(\Phi_2(x)\Psi_2(y)) = \sigma(J(\Phi_1(x))J(\Psi_1(y))) = \sigma(\Phi_1(x)\Psi_1(y)), \quad x, y \in F.$$

Suppose now that (15.1) holds. We first observe that $\Phi_1(x) = \Phi_1(x')$ implies that $\Phi_2(x) = \Phi_2(x')$. Indeed, assume that $\Phi_1(x) = \Phi_1(x')$. Then we obtain

$$\sigma(\Phi_2(x)\Psi_2(y)) = \sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_1(x')\Psi_1(y)) = \sigma(\Phi_2(x')\Psi_2(y)), \quad y \in F.$$

Pick $y \in F$ with $\Psi_2(y) = \Phi_2(x)^{-1}$. Such an element $y \in F$ exists since $\Psi_2(F) = U_2$. With this y we have that

$$\{1\} = \sigma(\Phi_2(x)\Psi_2(y)) = \sigma(\Phi_2(x')\Psi_2(y)).$$

From this we infer that $1 = \Phi_2(x')\Psi_2(y)$, thus we have

$$\Phi_2(x') = \Psi_2(y)^{-1} = \Phi_2(x).$$

In the same way we see that $\Psi_1(x) = \Psi_1(x')$ implies that $\Psi_2(x) = \Psi_2(x')$. Define maps $\phi, \psi : U_1 \rightarrow U_2$ by $\phi(\Phi_1(x)) = \Phi_2(x)$ and $\psi(\Psi_1(x)) = \Psi_2(x)$, $x \in F$. Clearly, ϕ, ψ are well defined surjections from U_1 onto U_2 . Moreover, we have

$$\sigma(ab) = \sigma(\phi(a)\psi(b)), \quad a, b \in U_1.$$

By Theorem 13 there exists a Jordan *-isomorphism from M_1 onto M_2 and $u_0 \in U_2$ such that

$$\phi(a) = u_0J(a), \quad \psi(a) = J(a)u_0^{-1}, \quad a \in U_1$$

and we easily conclude that (15.2) holds.

The implication (15.4) \Rightarrow (15.3) can be proved by a reasoning similar to the one that has appeared in the proof of the implication (12.5) \Rightarrow (12.3).

Suppose now that (15.3) holds. We first observe that $\Phi_1(x) = \Phi_1(x')$ implies that $\Phi_2(x) = \Phi_2(x')$ for any $x, x' \in F$. Indeed, assume $\Phi_1(x) = \Phi_1(x')$. Then we have

$$\begin{aligned} r(\Phi_2(x)\Psi_2(y) - 1) &= r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_1(x')\Psi_1(y) - 1) \\ &= r(\Phi_2(x')\Psi_2(y) - 1), \quad y \in F. \end{aligned}$$

As $\Psi_2(F) = U_2$, there exists $y \in F$ with $\Psi_2(y) = \Phi_2(x)^{-1}$. With this y we obtain

$$0 = r(\Phi_2(x)\Psi_2(y) - 1) = r(\Phi_2(x')\Psi_2(y) - 1).$$

As $\Phi_2(x')\Psi_2(y)$ is unitary, we have

$$\|\Phi_2(x')\Psi_2(y) - 1\| = r(\Phi_2(x')\Psi_2(y) - 1) = 0$$

implying

$$\Phi_2(x') = \Psi_2(y)^{-1} = \Phi_2(x).$$

In a similar way we see that $\Psi_1(x) = \Psi_1(x')$ implies $\Psi_2(x) = \Psi_2(x')$. Once again, define maps $\phi, \psi : U_1 \rightarrow U_2$ by $\phi(\Phi_1(x)) = \Phi_2(x)$ and $\psi(\Psi_1(x)) = \Psi_2(x)$, $x \in F$ which turn to be well defined and surjective. Moreover, we infer that

$$r(ab - 1) = r(\phi(a)\psi(b) - 1), \quad a, b \in U_1.$$

Then by Theorem 13 there exists a Jordan *-isomorphism, a central projection $p \in M_2$ and $u_0 \in U_2$ such that

$$\phi(a) = u_0(pJ(a) + (1-p)J(a)^{-1}), \quad \psi(a) = (pJ(a) + (1-p)J(a)^{-1})u_0^{-1}, \quad a \in U_1.$$

This apparently gives us that (15.4) holds. \square

Finally, we present corollaries of the former results from which non-linear spectral multiplicativity type conditions can be deduced for maps between the self-adjoint parts of von Neumann algebras to be Jordan $*$ -isomorphisms.

Corollary 16. *Let M_j be a von Neumann algebra for $j = 1, 2$. Suppose that f and g are bijections from M_{1s} onto M_{2s} . Then the following two conditions are equivalent:*

$$(16.1) \quad \sigma(\exp(ix)\exp(iy)) = \sigma(\exp(if(x))\exp(ig(y))), \quad x, y \in M_{1s};$$

(16.2) *there exists a Jordan $*$ -isomorphism J from M_1 onto M_2 and $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0 \exp(iJ(x)), \quad \exp(ig(x)) = (\exp(iJ(x)))u_0^{-1}, \quad x \in M_{1s}.$$

In particular, if f and g are homogeneous, then we have $f = g = J$ and $u_0 = 1$.

Moreover, the following two conditions are also equivalent:

$$(16.3) \quad r(\exp(ix)\exp(iy) - 1) = r(\exp(if(x))\exp(ig(y)) - 1), \quad x, y \in M_{1s};$$

(16.4) *there exists a Jordan $*$ -isomorphism J from M_1 onto M_2 , a central projection $p \in M_2$ and $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0(p \exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1})$$

and

$$\exp(ig(x)) = (p \exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1})u_0^{-1}$$

for every $x \in M_{1s}$.

In particular, if f and g are homogeneous, then we have $f = g = (2p-1)J$ and $u_0 = 1$.

Proof. Suppose that (16.2) holds. We infer that

$$\begin{aligned} \sigma(\exp(if(x))\exp(ig(y))) &= \sigma(u_0 \exp(iJ(x)) \exp(iJ(y)) u_0^{-1}) \\ &= \sigma(J(\exp(ix))J(\exp(iy))) = \sigma(\exp(ix)\exp(iy)), \quad x, y \in M_{1s}. \end{aligned}$$

In particular, if f is homogeneous, then $f(0) = 0$. It follows that $u_0 = 1$ and we have

$$\exp(itf(x)) = \exp(if(tx)) = \exp(itJ(x)), \quad t \in \mathbb{R}, x \in M_{1s}.$$

Letting $t \rightarrow 0$, from

$$(\exp(itf(x)) - 1)/t = (\exp(itJ(x)) - 1)/t$$

we obtain $f(x) = J(x)$, $x \in M_{1s}$. In the same way we deduce $g(x) = J(x)$, $x \in M_{1s}$ if g is homogeneous.

Suppose that (16.1) holds. Set $F = M_{1s}$ and define $\Phi_1, \Psi_1 : M_{1s} \rightarrow U_1$ by $\Phi_1(x) = \Psi_1(x) = \exp(ix)$, $x \in M_{1s}$. Also define $\Phi_2, \Psi_2 : M_{1s} \rightarrow U_2$ by $\Phi_2(x) = \exp(if(x))$ and $\Psi_2(x) = \exp(ig(x))$, $x \in M_{1s}$. As $\exp iM_{js} = U_j$ for $j = 1, 2$, the maps Φ_j and Ψ_j are surjective for $j = 1, 2$. Apparently, we have

$$\sigma(\Phi_1(x)\Psi_1(y)) = \sigma(\Phi_2(x)\Psi_2(y)), \quad x, y \in F.$$

Then by Theorem 15 there exists a Jordan $*$ -isomorphism $J : M_1 \rightarrow M_2$ and $u_0 \in U_2$ such that

$$\exp(if(x)) = \Phi_2(x) = u_0 J(\Phi_1(x)) = u_0 J(\exp(ix)) = u_0 \exp(iJ(x))$$

and

$$\exp(ig(x)) = \Psi_2(x) = J(\Psi_1(x))u_0^{-1} = J(\exp(ix))u_0^{-1} = (\exp(iJ(x)))u_0^{-1}$$

for every $x \in M_{1s}$, and hence we obtain (16.2).

Now suppose that (16.4) holds. Then by a simple calculation we have that

$$\begin{aligned} \exp(if(x))\exp(ig(y)) \\ = u_0(pJ(\exp(ix))J(\exp(iy)) + (1-p)J(\exp(ix))^{-1}J(\exp(iy))^{-1})u_0^{-1}. \end{aligned}$$

Using a calculation similar to the one we have applied in the proof of the implication (12.5) \Rightarrow (12.3) in Theorem 12 we have that

$$r(\exp(ix)\exp(iy) - 1) = r(\exp(if(x))\exp(ig(y)) - 1), \quad x, y \in M_{1s}$$

and hence we obtain (16.3). In particular, if f is homogeneous, then $f(0) = 0$. Thus we have

$$1 = \exp(if(0)) = u_0(pJ(\exp(i0)) + (1-p)J(\exp(i0))^{-1}) = u_0.$$

It follows that

$$\begin{aligned} \exp(itf(x)) &= \exp(if(tx)) = p\exp(iJ(tx)) + (1-p)(\exp(iJ(tx)))^{-1} \\ &= p\exp(itJ(x)) + (1-p)\exp(-itJ(x)), \quad x \in M_{1s}. \end{aligned}$$

Letting $t \rightarrow 0$, from

$$(\exp(itf(x)) - 1)/it = p(\exp(itJ(x)) - 1)/it + (1-p)(\exp(-itJ(x)) - 1)/it,$$

we deduce

$$f(x) = (2p-1)J(x), \quad x \in M_{1s}.$$

In a similar manner we obtain $g(x) = (2p-1)J(x)$, $x \in M_{1s}$.

Suppose that (16.3) holds. Set $F = M_{1s}$ and once again define $\Phi_1, \Psi_1 : M_{1s} \rightarrow U_1$ by $\Phi_1(x) = \Psi_1(x) = \exp(ix)$ and $\Phi_2, \Psi_2 : M_{1s} \rightarrow U_2$ by $\Phi_2(x) = \exp(if(x))$, $\Psi_2(x) = \exp(ig(x))$, $x \in M_{1s}$. Then Φ_j and Ψ_j are both surjective maps for $j = 1, 2$. Apparently, we have

$$r(\Phi_1(x)\Psi_1(y) - 1) = r(\Phi_2(x)\Psi_2(y) - 1), \quad x, y \in M_{1s}.$$

By Theorem 15 there exists a Jordan $*$ -isomorphism J from M_1 onto M_2 , a central projection $p \in M_2$ and a unitary $u_0 \in U_2$ such that

$$\Phi_2(x) = u_0(pJ(\Phi_1(x)) + (1-p)J(\Phi_1(x))^{-1})$$

and

$$\Psi_2(x) = (pJ(\Psi_1(x)) + (1-p)J(\Psi_1(x))^{-1})u_0^{-1}$$

hold for every $x \in M_{1s}$. Then

$$\begin{aligned} \exp(if(x)) &= u_0(J(\exp(ix)) + (1-p)J(\exp(ix))^{-1}) \\ &= u_0(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}) \end{aligned}$$

and

$$\begin{aligned} \exp(ig(x)) &= (pJ(\exp(ix)) + (1-p)J(\exp(ix))^{-1})u_0^{-1} \\ &= (p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1})u_0^{-1} \end{aligned}$$

for every $x \in M_{1s}$. This completes the proof. \square

The following statement is an easy consequence of Corollary 16, one just needs to take $g = f$ (and have a short look at the argument in the proof of Corollary 14 concerning centrality).

Corollary 17. *Let M_j be a von Neumann algebra for $j = 1, 2$. Suppose that f is a bijection from M_{1s} onto M_{2s} . Then the following two conditions are equivalent:*

- (17.1) $\sigma(\exp(ix)\exp(iy)) = \sigma(\exp(if(x))\exp(if(y)))$, $x, y \in M_{1s}$;
- (17.2) *there exists a Jordan $*$ -isomorphism J from M_1 onto M_2 and a central symmetry $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0\exp(iJ(x)), \quad x \in M_{1s}.$$

In particular, if f is homogeneous, then we have $f = J$ and $u_0 = 1$.

The following two conditions are also equivalent:

- (17.3) $r(\exp(ix)\exp(iy) - 1) = r(\exp(if(x))\exp(if(y)) - 1)$, $x, y \in M_{1s}$;
- (17.4) *there exists a Jordan $*$ -isomorphism $J : M_1 \rightarrow M_2$, a central projection $p \in M_2$ and a central symmetry $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0(p\exp(iJ(x)) + (1-p)(\exp(iJ(x)))^{-1}), \quad x \in M_{1s}.$$

In particular, if f is homogeneous, then we have $f = (2p-1)J$ and $u_0 = 1$.

Our last statement is again a simple consequence of Corollary 16, one needs to consider the map $g(y) = -f(-y)$, $y \in M_{1s}$.

Corollary 18. *Let M_j be a von Neumann algebra for $j = 1, 2$. Suppose that f is a bijection from M_{1s} onto M_{2s} . Then the following two conditions are equivalent:*

- (18.1) $\sigma(\exp(ix)(\exp(iy))^{-1}) = \sigma(\exp(if(x))(\exp(if(y)))^{-1})$, $x, y \in M_{1s}$;
 (18.2) *there exists a Jordan $*$ -isomorphism J from M_1 onto M_2 and $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0 \exp(iJ(x)), \quad x \in M_{1s}.$$

In particular, if f is homogeneous, then we have $f = J$ and $u_0 = 1$.

The following two conditions are also equivalent:

- (18.3) $r(\exp(ix)(\exp(iy))^{-1} - 1) = r(\exp(if(x))(\exp(if(y)))^{-1} - 1)$, $x, y \in M_{1s}$;
 (18.4) *there exists a Jordan $*$ -isomorphism $J : M_1 \rightarrow M_2$, a central projection $p \in M_2$ and unitary $u_0 \in U_2$ such that*

$$\exp(if(x)) = u_0(p \exp(iJ(x)) + (1 - p)(\exp(iJ(x)))^{-1}), \quad x \in M_{1s}.$$

In particular, if f is homogeneous, then we have $f = (2p - 1)J$ and $u_0 = 1$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NIIGATA UNIVERSITY, NIIGATA 950-2181 JAPAN
E-mail address: hatori@math.sc.niigata-u.ac.jp

DEPARTMENT OF ANALYSIS, BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, H-6720 SZEGED, ARADI VÉRTANÚK TERE 1., HUNGARY AND MTA-DE "LENDÜLET" FUNCTIONAL ANALYSIS RESEARCH GROUP, INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, H-4010 DEBRECEN, P.O. BOX 12, HUNGARY
E-mail address: molnarl@math.u-szeged.hu
URL: <http://www.math.unideb.hu/~molnarl/>