Chapter 1
Turbulence modeling using fractional derivatives

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Abstract We propose a new turbulence model in this work. The main idea of the model is that the shear stresses are considered to be random variables and we assume that their differences with respect to time are Lévy-type distributions. This is a generalization of the classical Newton's law of viscosity. We tested the model on the classical backward facing step benchmark problem. The simulation results are in a good accordance with real measurements.

1.1 Introduction

Turbulence is a velocity fluctuation of the mean flow in fluid dynamics. For this phenomenon there is no any exact definition, we can hardly quantify it and its numerical simulation is also challenging. Its study has a long history, it is enough to refer to the famous wish of Albert Einstein: "After I die, I hope God will explain turbulence to me."
Our study is based on the Navier–Stokes equations as a widely accepted model for fluid dynamics. Starting from this point there are many variant ways to modeling this phenomenon, for example the direct numerical simulation, the large eddy simulation and modeling with the Reynolds averaged equations. We propose here a new model and a new way for modeling turbulence. We consider the quantity obtained from the Newton's law of viscosity as a special expected value for the shear stresses. According to our approach, in the simulation we should take into account not only the actual velocity field but also the history of the velocity field to calculate this expected value.

We generalize the Navier–Stokes equation, using this hypothesis and get a probabilistic–deterministic model.

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1.2 Preliminaries

The Navier-Stokes equations for incompressible fluids can be given as

$$\begin{aligned} \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= \frac{1}{\rho}\Big(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\Big) \\ \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} &= \frac{1}{\rho}\Big(\frac{\partial \sigma_y}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\Big) \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0. \end{aligned} \tag{1.1}$$

Here terms σ_x, σ_y denote the tensile stresses, τ_{xy}, τ_{yx} denote the shear stresses, ρ the fluid density and $\mathbf{v} = (v_x, v_y)$ the velocity vector. Acording to the Newton's law of viscosity we additionally have

$$\tau_{ij} = \mu\Big(\frac{\partial v_j}{\partial i} + \frac{\partial v_i}{\partial j}\Big), \quad i, j \in \Big\{x, y\Big\}. \tag{1.2}$$

The tensile stresses are given as

$$\sigma_i = -P + \mu\tau_{ii} = -P + 2\mu\frac{\partial v_i}{\partial i}, \quad i \in \Big\{x, y\Big\}. \tag{1.3}$$

Let us introduce the notations $p := \frac{P}{\rho}$ for the pressure and $\nu := \frac{\mu}{\rho}$ for the kinematic viscosity. Using (1.2), (1.3) in (1.1) we can make it explicit, to obtain the classical Navier–Stokes equations

$$\begin{aligned} \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} &= -\frac{\partial p}{\partial x} + \nu \Delta v_x \\ \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} &= -\frac{\partial p}{\partial y} + \nu \Delta v_y \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned} \tag{1.4}$$

1.3 Results

1.3.1 The fractional Newton's law of viscosity

To work with fractional order differentiation we need the following definition (see, *e.g.*, [1]).

Definition 1. For each $q \in [0, 1)$ and $a \in \mathbb{R}$ we say that f is q-times differentiable if the following limit exists:

$$\frac{1}{\Gamma(1-q)}\frac{\partial}{\partial t}\int_a^t \frac{f(s)}{(t-s)^q}\,\mathrm{ds} =:_a D^q f(t). \tag{1.5}$$

In [1] the authors investigated the accuracy of the approximation of (1.5):

$$_aD^q f(t) \approx [_aD_h^q]f(t) := \Big(\frac{t-a}{N}\Big)^{-q}\sum_{k=0}^{N-1}\binom{q}{k}(-1)^k f\big(t-k\frac{t-a}{N}\big). \tag{1.6}$$

Using the fractional order derivatives in (1.5) we introduce the following generalization of (1.2):

$$\tau_{ij}(t,\cdot) = \nu\Big[_{t-T}D^q\Big(\frac{\partial v_j}{\partial i}(t,\cdot)+\frac{\partial v_i}{\partial j}(t,\cdot)\Big)\Big],\quad 0\le q<1,\quad i,j\in\Big\{x,y\Big\}. \tag{1.7}$$

We modify the equations for the tensile stresses accordingly to obtain

$$\sigma_i(t,\cdot) = -p(t,\cdot)+\tau_{ii}(t,\cdot),\quad i\in\Big\{x,y\Big\}. \tag{1.8}$$

Using the notation in (1.5) and substituting (1.7) and (1.8) into (1.4) we arrive at the fractional Navier–Stokes equations

$$\begin{aligned}\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y} &= -\frac{\partial p}{\partial x} +_{t-T} D^q\Big(\nu\Delta v_x\Big)\\ \frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y} &= -\frac{\partial p}{\partial y} +_{t-T} D^q\Big(\nu\Delta v_y\Big)\\ \operatorname{div}\mathbf{v} &= 0.\end{aligned} \tag{1.9}$$

We need the following two theorems, the second one is discussed in [5] and we proved the first one in the Appendix. Before we recall that for $\alpha\in\mathbb{R}$ and $j\in\mathbb{N}$ we define $\binom{\alpha}{j}=\frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!}$.

Theorem 1. *For each $\alpha\in\mathbb{C}$ and $h=1/N$ the following is true:*

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}\binom{\alpha}{j}\frac{(-1)^j}{h^\alpha} = \frac{1}{\Gamma(1-\alpha)}. \tag{1.10}$$

Theorem 2. *For any $\alpha\in[0,1)$ and $j\in\mathbb{Z}^+$ we have $\binom{\alpha}{j}(-1)^j<0$ and the following equality holds:*

$$\sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j = 0. \tag{1.11}$$

Let $f:\mathbb{R}\to\mathbb{R}$ α-times differentiable by means of Definiton 1 and it is approximated using (1.6). For simplicity we assume that $T:=1$, then the following estimations are valid

$$
\begin{aligned}
{}_{t-1}D^{\alpha} f(t) &\approx \Big(\frac{1}{N}\Big)^{-\alpha} \sum_{k=0}^{N-1} \binom{\alpha}{k} (-1)^k f\big(t - \frac{k}{N}\big) \\
&\approx \Big(\frac{1}{N}\Big)^{-\alpha} \sum_{k=1}^{N-1} \binom{\alpha}{k} (-1)^{k+1} \Big[f(t) - f\big(t - \frac{k}{N}\big)\Big] \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{N-1} \Gamma(1-\alpha) \Big(\frac{1}{N}\Big)^{-\alpha} \binom{\alpha}{k} (-1)^{k+1} \Big[f(t) - f\big(t - \frac{k}{N}\big)\Big].
\end{aligned}
\tag{1.12}
$$

Let

$$
p_k = \Gamma(1-\alpha)\Big(\frac{1}{N}\Big)^{-\alpha} \binom{\alpha}{k} (-1)^{k+1}. \tag{1.13}
$$

According to Theorem 2 we have $p_k > 0$, and Theorem 1 gives that

$$
\lim_{N\to\infty} \sum_{k=1}^{N-1} p_k = 1. \tag{1.14}
$$

Consequently, the values $\{p_k\}_{k\in\mathbb{N}}$ define a probability density function, and the limit distribution is Lévy type. We can also conclude that the generalization (1.7) of the Newton's law can be considered as the expected value of the variation of the shear stresses, where the distribution function is defined by the values p_k. This serves as a motivation for our model.

Note that the standard Newton's law corresponds to the case $q = 0$ in (1.7), which can be interpreted as the distribution of the variation is Gaussian, and then the shear stresses are independent from the earlier stress values.

1.3.2 The algorithm

To discretize (1.9) we use the method of the work [3], which is a finite difference approximation on a staggered grid. The semidiscretization results then in the following ODE:

$$
\begin{aligned}
\mathbf{u}_t + L_h(\mathbf{u})\mathbf{u} + \mathrm{grad}_h p &= \mathbf{0} \\
\mathrm{div}_h \mathbf{u} &= 0,
\end{aligned}
\tag{1.15}
$$

where $L_h(\mathbf{u}) = D_h(\mathbf{u}) - \nu[{}_{t-T}D_h^{\alpha}]\Delta$, $D_h(\mathbf{u})\mathbf{u}$ is the approximation of the nonlinear terms, div_h is the discrete divergence , grad_h is the discrete gradient operator, ν is the viscosity parameter and $[{}_{t-T}D_h^{\alpha}]$ defined in (1.6).

We solve then equations (1.15) using a simple predictor-corrector algorithm. We start from an initial velocity field $\mathbf{u}^0$ and an initial value for the pressure and apply the time step τ. The main steps of the algorithm are the following.

1. Solve the first equation in (1.15) for $\mathbf{w}$:

$$\frac{\mathbf{w}-\mathbf{u}^n}{\tau}+L_h(\mathbf{u}^n)\mathbf{w}+\mathrm{grad}_h p^n=0. \tag{1.16}$$

2. Solve the following equation for q:

$$\mathrm{div}_h\mathrm{grad}_h q=\frac{1}{\tau}\mathbf{w}. \tag{1.17}$$

3. Compute the pressure values $p^{n+1}=q+p^n$.
4. Compute the velocity vector $\mathbf{u}^{n+1}=\mathbf{w}-\tau\mathrm{grad}_h q$.

1.3.3 The test problem

To test our simulation we use the real measurements of the work [2] and we also compare our results with other numerical predictions. We choose a classical benchmark problem, a backward facing step. The geometric setup of this problem is shown in Fig. 1.1. We set the fluid memory T = 2.5 s, and the time step $\tau = 0.005\,$s. It is sufficient to assume this fluid memory because for N = 500 and $\alpha = 0.2$ we have $1-\sum_{k=1}^{N} p_k = 2\cdot 10^{-4}$.

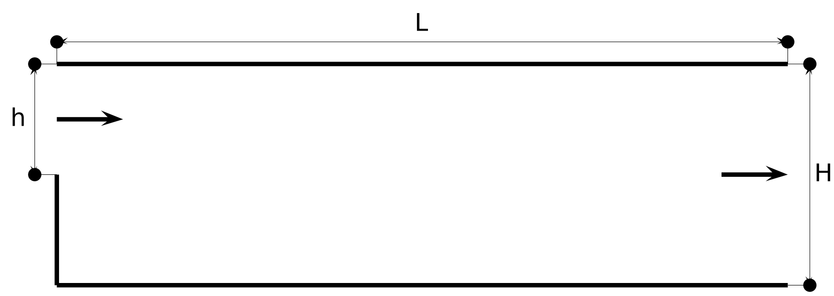

Fig. 1.1 The backward facing step problem.

The fluid flows into the channel on the upper part of the left hand side of the channel and it flows out at the right hand side. We set the geometry parameters to H = 1 cm, L = 10 cm, h = 0.5 cm and $\nu = \frac{2}{3}\cdot 10^{-5}\,\frac{\mathrm{m}^2}{s}$ and use the the Reynold's number $\mathrm{Re}=\frac{4hv_{\max}}{3\nu}$. With these the exact boundary conditions are the following:

- $x=0, y\in[H-h,h]$ (inflow section): $v_y=0$ and $v_x=-\frac{4(H-y)(H-h-y)}{h^2}v_{\max}$,

- $x = L$ (outflow section): $\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial x} = 0$ and $p = 0$,

- on the remaining part of the boundary: $v_x = v_y = 0$.

We notice that one can take also a channel before the inlet stage, because it has some effect on the velocity field [4]. Focusing to the simplest version of the problem we do not use this inlet channel. Whenever the problem seems to be easy, many recent calculations underpredict certain well-measurable quantities, the location of the so-called reattachment lengths r, s and rs. The corresponding error rate is about $5 - 15\%$. For a visualization of the reattachment lengths we refer to figure 1.2.

An important advance of our model is that we can predict this quantity very precisely for different Reynold's number by choosing the parameter α properly. We made some comparison with other predictions and summarized the results in Tables 1.1-1.3.

For the computations we divided the domain into 300×30 elementary cells. Implementing then the algorithm in Section 1.3.2 we found that the numerical method converges to a stationary solution. The simulated time was 30 s using a number of 6000 time steps both for the equations (1.4) and (1.9).

We also tested our model on a similar benchmark problem using a different parameter set $H = 1$ cm, $L = 12$ cm, $h = 0.6$ cm and $\nu = 8 \cdot 10^{-6}\ \frac{\mathrm{m}^2}{\mathrm{s}}$, with the Reynold's number Re $= 2425$. We made some comparison with other predictions with different turbulence models and measurements [9] and summarized the results in Table 1.4.

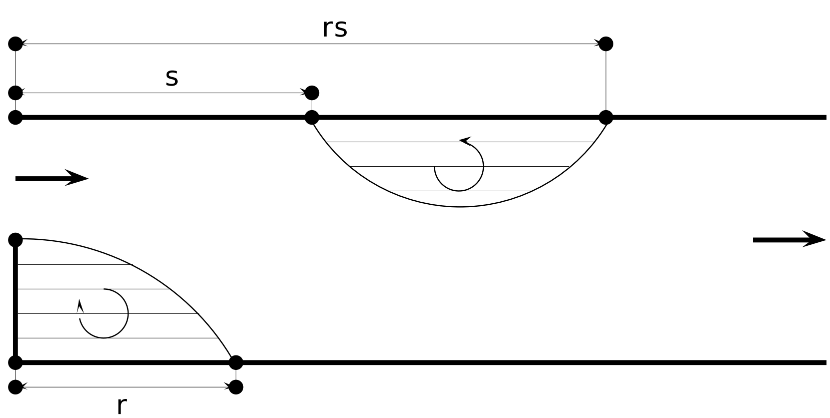

Fig. 1.2 Reattachment lengths r, s and rs. The subdomains of the computational domain with $v_x < 0$ are shaded.

Table 1.1 Summarized results for the reattachment lengths with Re = 800. FNS=fractional Navier–Stokes, NNS=Classical Navier–Stokes

	Experimental results		Computed Results					
	Length on	Exp. [2]	Lee, Mateescu [2]	Gartling [6]	Kim, Moin [7]	Sohn [8]	Present study, NNS	Present Study, FNS, ($\alpha = 0.06$)
Lower wall	r/H	6.45	6.0	6.1	6.0	5.8	6.11	6.43
Upper wall	s/H	5.15	4.80	4.85	–	–	5.08	5.33

Table 1.2 Numerical results for the reattachment lengths with Re = 1000. FNS=fractional Navier–Stokes, NNS=Classical Navier–Stokes

	Experimental results		Computed Results	
	Length on	Exp. [2]	Present study, NNS	Present Study, FNS, ($\alpha = 0.17$)
Lower wall	r/H	7.5	6.68	7.46
Upper wall	s/H	6.5	5.51	6.16

Table 1.3 Numerical results for the reattachment lengths with Re = 1200. FNS=fractional Navier–Stokes, NNS=Classical Navier–Stokes

	Experimental results		Computed Results	
	Length on	Exp. [2]	Present study, NNS	Present Study, FNS, ($\alpha = 0.24$)
Lower wall	r/H	8.5	7.16	8.50
Upper wall	s/H	7.5	5.93	7.06

Table 1.4 Summarized results for the reattachment lengths with Re = 2425. FNS=fractional Navier–Stokes, NNS=Classical Navier–Stokes

	Reattachment Length ratio r/H					
Reynolds number	Exp. [9]	$k-\epsilon$ [9]	RNG $k-\epsilon$ [9]	SA [9]	SST [9]	Present study, FNS, $\alpha = 0.4$
2425	9.2	6.3	6.93	8.54	9.4	9.06

1.4 Conclusion

We introduced a new turbulence model in this work by assuming that the variations of shear stresses are random variables and their distributions are Lévy-type. In this way we use two new parameter for the governing equations:

the fluid memory and a stability parameter. The most important task in the practical computations was to choose correctly the stability parameter, while the length of the memory is not so important in numerical calculations. We could predict well the reattachment lengths in a classical benchmark problem by a proper setting of the stability parameter.

We observed that for small Reynold's numbers the choice of parameter $\alpha = 0$, which corresponds to the classical Navier–Stokes equations, gives good accordance with the real measurements. If only the Reynold's number is increased and consequently, the flow becomes turbulent, the parameter α has to be also increased. For example, if $\mathrm{Re} = 800$, we found that the choice $\alpha = 0.06$ is optimal for the simulation. This corresponds to the fact, that turbulent flows can be described rather statistically than explicitly, and in the long run we can consider the present model also a statistical one.

Our future aim is to find experimentally the values α corresponding to the Reynold's number. It would also be important to compare this result with numerical experiments on further test problems.

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Appendix

Proof. (Theorem 1) Let $\alpha \in \mathbb{C}$ be any fixed complex number. Let x be a real or complex number such that $|x| < 1$, then

$$(1-x)^{\alpha} = \sum_{N=0}^{\infty} (-1)^N \binom{\alpha}{N} x^N. \tag{1.18}$$

It is easy to see that

$$(1-x)^{\alpha-1} = \frac{(1-x)^{\alpha-1}}{1-x} = \sum_{N=0}^{\infty} \left(\sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} \right) x^N. \tag{1.19}$$

On the other hand

$$(1-x)^{\alpha-1} = \sum_{N=0}^{\infty} (-1)^N \binom{\alpha-1}{N} x^N, \tag{1.20}$$

whence equating the coefficients of x^{N-1}, we obtain

$$
\begin{aligned}
\sum_{k=0}^{N-1}(-1)^k\binom{\alpha}{k} &= (-1)^{N-1}\binom{\alpha-1}{N-1}\\
&= \binom{N-\alpha-1}{N-1} = \frac{\Gamma(N-\alpha)}{\Gamma(N)\Gamma(1-\alpha)}.
\end{aligned}
\tag{1.21}
$$

Thus

$$
\begin{aligned}
\lim_{N\to\infty} N^\alpha \sum_{k=0}^{N-1}(-1)^k\binom{\alpha}{k} &= \lim_{N\to\infty}\frac{N^\alpha\Gamma(N-\alpha)}{\Gamma(N)\Gamma(1-\alpha)}\\
&= \frac{1}{\Gamma(1-\alpha)}\lim_{N\to\infty}\frac{N^\alpha\Gamma(N-\alpha)}{\Gamma(N)} = \frac{1}{\Gamma(1-\alpha)},
\end{aligned}
\tag{1.22}
$$

where we have used Stirling's formula (or the known asymptotics for gamma function ratios) in the last step. □

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