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COMMENTS ON DEBT DYNAMICS

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The present paper deals with the accumulation of public debt based on different kinds of non-linear models. The same problem is analysed here in three different models. In the first model difference between growth rate and interest rate depends lineary on the debt/GDP ratio and the budget deficit. In the second model version this connection was non-linear, so two kinds of economic policy could be applied. In the third version the growth rate as well as the interest rate are in close connection with the debt/GDP ratio, in this case a stable equilibrium or a saddle path could be shown up.

Keywords: debt/GDP ratio, non-linear models, equilibrium, stability, dept trap

Jel classification index: E6, H 63, O42

1. INTRODUCTION

The studies, articles and textbooks dealing with the accumulation of public debt with time consider the equation describing a simple dynamic relationship among the four factors of initial debt/GDP ratio, real interest rate, real GDP growth rate and primary budget deficit as an analytical tool of predominant importance almost without exception.[1] In a mathematical form, the formula is as follows:

$$\frac{db}{dt} \equiv \dot{b}(t) = (r-g)\,b(t) + x \tag{1}$$

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[1] Foreign and home debt are not distinguished. This assumption is justified if uncovered interest parity holds.

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where $b=\frac{B}{Y}$, $x=\frac{D}{Y}$ is the proportion of the debt/GDP ratio and of budget deficit to GDP, r is the real interest rate and g is the real growth rate[2]. The differential equation (1) assumes that the size of the interest rate, the growth rate and the deficit is constant. Therefore, the accumulation of debt may be investigated only along the predefined parameters, and the nature of the dynamic process is the same over any time horizon. Due to all this, however a paradox it may seem, the differential equation (1) is of a very "static nature", or, to put it more precisely, it is too simplified and does not exploit the abundance of dynamic relationships.

The simple static nature represents a serious limitation especially from an economic perspective, because to investigate the debt problem is reasonable only in a longer time horizon, where it is a rather heroic assumption to stipulate the constancy of real interest rate, growth rate and primary deficit. That is why much broader investigation opportunities are offered by the further dynamisation of equation (1) and converting it into the following generic form:

$$\frac{db}{dt}=[r(t)-g(t)]b(t)+x(t) \tag{1'}$$

Although allowing for time-varying parameters renders the problem analytically more complicated, it also highlights the need for specifying the processes driving the development of the real interest rate, the growth rate, and the budget deficit. This article investigates a few possibilities, and it will try to answer some new questions arising in connection with debt dynamics. Accordingly, the structure of the study will be as follows.

Section 2 briefly reviews the traditional approach to study debt dynamics. Section 3 investigates debt dynamics when the real interest rate and the growth rate are allowed to change with the debt ratio and the primary budget deficit, with the latter treated as an exogenous variable. Section 4 considers a two-equation model for the debt ratio and the primary budget deficit. Jointly examining these two variables allows for analysing different economic policy strategies for debt management. Section 5 performs a similar model after endogenising the real interest rate, the growth rate and the primary budget deficit.

[2] This relationship was conceived by Domar (1944), who examined the development of the debt burdens and national income.

2. THE TRADITIONAL APPROACH

Differential equation (1) is very simple to solve and is easy to interpret in economic terms. Let us look at the exact mathematical solution first:

$$b(t) = [b(0) - b^*]e^{(r-g)t} + b^* \qquad \text{and} \qquad b^* = -\frac{x}{r-g} \tag{2}$$

The debt/GDP ratio will be in equilibrium if $\dot{b} = 0$. Using the condition we get the equilibrium value of $b^* = -\frac{x}{r-g}$. Accordingly, four scenarios may be distinguished.

Case 1: $r > g$ and *$x < 0$.*

Equilibrium will not be stable in this case and the value of the equilibrium debt/GDP ratio will be positive (see the line on the right in *Figure 1*). The real interest rate, which is higher than the growth rate takes the system increasingly further away from equilibrium. When the initial debt ratio is below its equilibrium level, the primary surplus more than offsets interest payments and the debt ratio falls. When the initial debt ratio is above its equilibrium level, the primary surplus is insufficient to offset interest payments and the debt continues to be accumulated. Reversing the chain of logic allows for deriving the debt-stabilisation strategy as well. For a given initial $b(0)$ and *$(r–g)$,* it is possible to determine a level of primary budget surplus for which $b^* > b(0)$. Running an appropriately large primary surplus allows reducing the debt ratio.

Case 2: $r > g$ and *$x > 0$.*

Equilibrium will not be stable and the equilibrium value will be negative. In economic terms, this means that the government has a positive net asset position (it is a lender, not a borrower), and uses the interest revenue to finance a primary deficit. If initial net assets are lower than their equilibrium level, the interest revenue is insufficient to fully finance the primary deficit and net assets are run down. If initial net assets are above b^*, interest income not only finances the primary deficit, but helps further accumulate assets.

Case 3: $r < g$ and $x < 0$.

In this case the equilibrium will be (asymptotically) stable because the growth rate exceeds the interest rate, and the equilibrium value of the debt ratio will be negative. "High" GDP growth decreases the relative importance of interest expenditure and the primary budget is in surplus. As a result, an indebted government is able to eliminate its debt and accumulate assets.

Case 4: $r < g$ and $x > 0$.

Equilibrium remains stable, but the equilibrium value will be positive, so the government will be in the position of a net debtor. In case of an initial debt/GDP ratio that is lower than equilibrium, budget deficit will be greater than the surplus arising out of growth in excess of interest expenditure, so b will increase. In the reverse case, in the event of b greater than the equilibrium, the growth rate will dominate over the deficit, and the economy will grow out of indebtedness.

The overview of these four cases clearly indicates the benefits and disadvantages of the traditional approach. On the one hand, the problem of debt dynamics is relatively simple to interpret, may be connected well to the basic categories and the relations between the scales of these problems, and it helps to formulate clear and easy-to-follow recommendations for economic policy. On the other hand, oversimplified treatment of problems may easily lead to wrong conclusions. Namely, conclusions based on the assumption that the relationship between r and g will not change in time, and that changing x has no impact on these variables, are not very realistic.

3. A SIMPLE NON-LINEAR MODEL

The general equation (1') formulated for introducing dynamics into the debt problem and solving this equation can take us closer to dynamic models that take interactions more into account. The general solution to (1') is as follows:

$$b(t) = e^{\int u dt}\left[c + \int w e^{\int u dt} dt\right] \qquad u(t) = r(t) - g(t), \qquad w(t) = x(t)$$

where $u(t)$ and $w(t)$ are functions that may be integrated, and c is a constant. To find a specific solution, the development $(r–g)$ and x over time has to be defined.

For the sake of simplicity we focus on, and assume that the value of u is influenced only by two factors, namely the budget deficit and the debt ratio, and that this relationship is linear:

$$u(t) = -\alpha x(t) + \beta b(t) \qquad \alpha, \beta > 0. \tag{3}$$

The first half of the definition stems from a "Keynesian inspiration" that a larger deficit promotes economic growth (cf. multiplier effect), but has no impact on interest rate (cf. liquidity trap); and, as a consequence, an increase in x reduces u.

Substituting definition (3) into the basic relationship (1'), we get a non-linear first-order differential equation in terms of b:

$$\frac{db}{dt} = \beta b^2(t) - \alpha x(t) b(t) + x(t) \cdot \tag{4}$$

Equation (4) is non-linear and is not separable[3]. To make the problem tractable, we assume that x is constant, and find the following equilibrium values after substituting $b = 0$:

$$b^*_{1,2} = \frac{\alpha x \pm \sqrt{\alpha^2 x^2 - 4\beta x}}{2\beta}. \tag{5}$$

In what follows, we assume that (5) has no complex roots. If , there will be one or two positive roots. In the latter case, the lower equilibrium will be locally stable, and the higher one unstable (*Figure 1*). A higher budget deficit will increase the distance between the two equilibrium values, and so does an increase in α: as higher deficit reduces the difference between interest rate and growth rate, it will lower the rate of debt accumulation (except for very small and very high b values). Increasing value β has the opposite essfect. When $x = \frac{4\beta}{\alpha^2}$, there is only one – positive – equilibrium value. This equilibrium situation is not stable (the trajectory approaches equilibrium from the left, but it does not do so from the right).

If $x = 0$, there is only the $b^* = 0$ equilibrium value, because the change in the debt/GDP ratio depends on the volume itself in a positive manner. In case $x < 0$, there will always be two equilibrium values, one of them negative and stable and the other positive and unstable (*Figure 2*).

In summary, assuming that the interest rate–growth rate differential depends on fiscal variables can give rise to multiple equilibria. Due to the existence of two equilibrium values, a simple monotone unidirectional movement is replaced by a movement that is more complex and closer to reality.

[3] The differential equations of this type are the so-called Ricatti equations, which have no general solution formula, see more about it in Abell – Braselton (2001), p. 151.

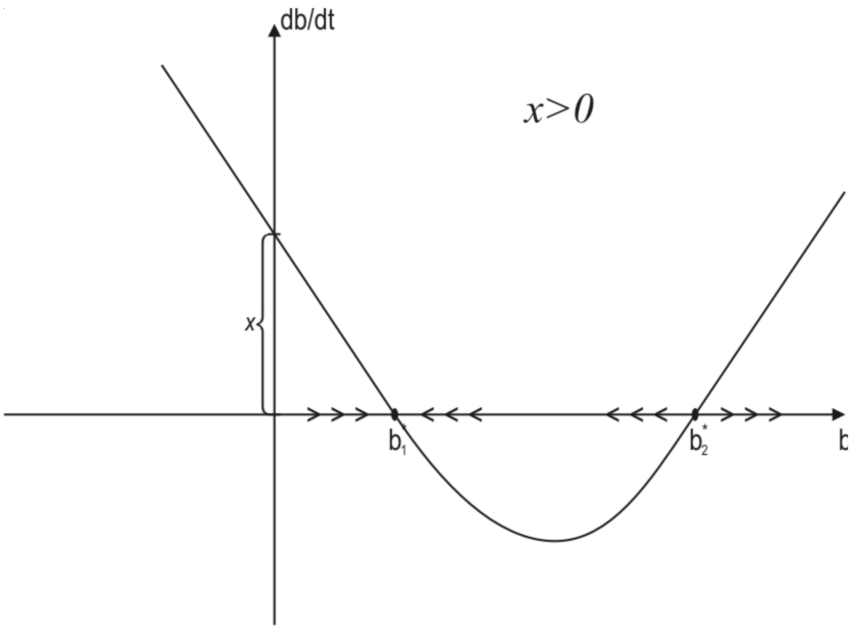

Figure 1. Equilibrium and stability in the simple non-linear model, x > 0 case

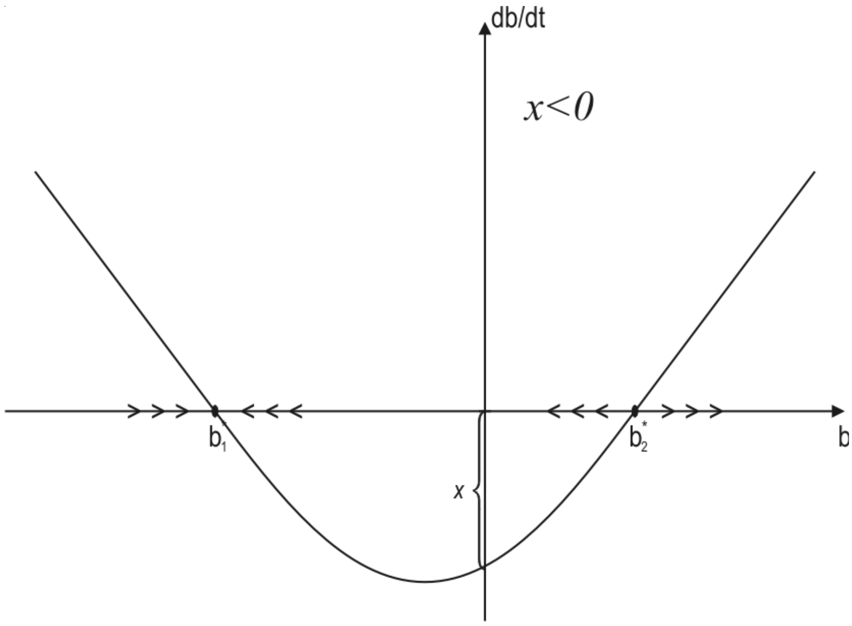

Figure 2. Equilibrium and stability in the simple non-linear model, x < 0 case

4. A TWO-EQUATION NON-LINEAR MODEL, WITH AN ECONOMIC POLICY RULE

In this section, we endogenise the budget deficit by means of a simple fiscal policy rule, and allow for a more complicated dependence of the interest rate–growth rate differential on fiscal variables.

Based on empirical facts it seems an obvious assumption that low values of the debt/GDP ratio have a negative impact on the difference of the interest rate and the growth rate (i.e. favourable on the growth rate and non-incremental on the interest rate).[4] However, the situation changes over a certain level of debt/GDP ratio, and the value of *u* will increase more and more with an increase in *b*. The following equation illustrates such a relationship:

$$u(t) = u^N + \alpha[b(t) - b^N]^3 \tag{6}$$

where u^N and b^N are the "normal" values of and the debt ratio.

In addition, u also depends on the budget deficit:

$$\begin{aligned} u(t) &= u^N - \beta x(t) \qquad \text{if } x_{\min} \le x(t) \le x_{\max} \\ u(t) &= u^N + \gamma x(t) \qquad \text{if } x(t) > x_{\max} \quad \text{or} \quad x(t) < x_{\min} \end{aligned} \tag{7}$$

where β and γ are positive parameters.

Taking into account relationships (6) and (7) and linking them to definition (1), the following *general* relationship may be provided for the changes in the debt/GDP ratio:

$$\begin{aligned} \dot{b} = F(b,x) \quad F_b > 0, \quad & F_x > 0 \quad \text{if } x > \overline{x} \quad \text{or} \quad x < \underline{x} \\ & F_x < 0 \quad \text{if } \underline{x} \le x \le \overline{x} \end{aligned} \tag{8}$$

In (8) both *b* and *x* influence the rate of debt accumulation directly and indirectly, via their effect on the interest rate–growth rate differential. For the debt ratio, the direct and indirect effects work in the same direction, while this is not necessarily the case for the budget deficit. The indirect effect may dominate in some range of *x*. Therefore, the defined impact applies within limits that are more narrow and differ from those specified in (7), i.e. $\overline{x} < x_{\max}$ and $\underline{x} > x_{\min}$.

4 In the paper of Missale – Blanchard (1994) instead of interest rate the expected inflation depends on the level of debt/GDP ratio.

If $\dot{b}$ = 0, the slope of function (8) will be positive in the interval $[\underline{x}, \overline{x}]$, because F_x < 0, and F_b > 0, but negative elsewhere (*Figure 3*).

$$\left.\frac{dx}{db}\right|_{\dot{b}=0} = -\frac{F_b}{F_x} > 0 \qquad \text{if} \quad F_x < 0.$$

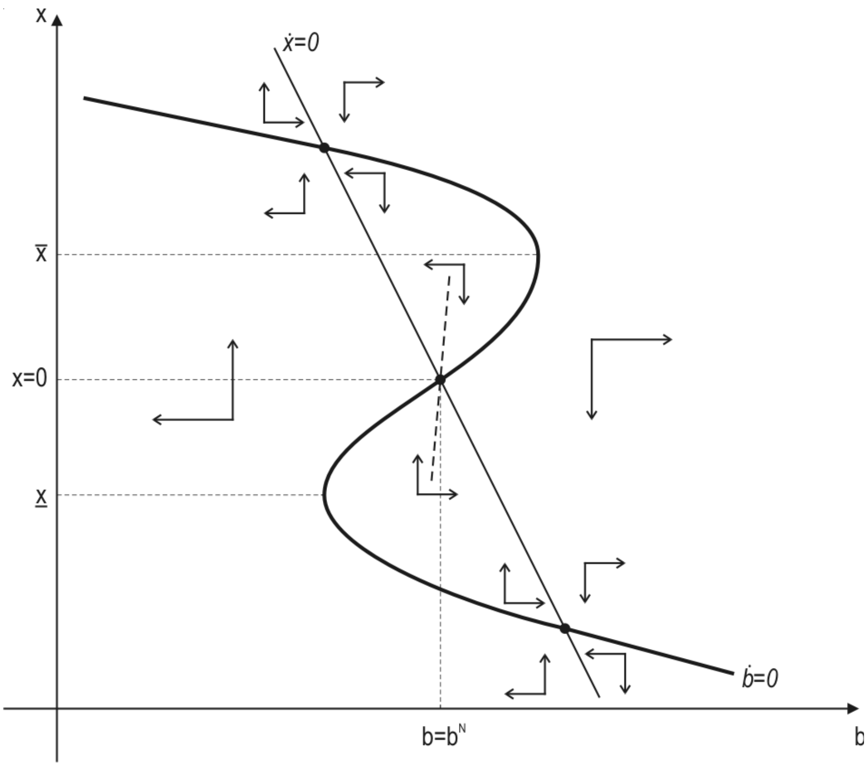

Figure 3. Equilibrium and stability in the two-equation non-linear model – striving for equilibrium (conservative) case

We still have to provide for the dynamisation of our other variable *x*. Let us assume that the government decides on changing the budget balance, knowing the size of *b* and *x*. Let us further assume that the government is committed to restoring financial equilibrium, so in response to an increase in the debt/GDP ratio it reduces the deficit, and seeing a high deficit it tries to improve the budgetary situation, that is:

$$\dot{x} = H(b,x) \qquad H_b < 0, \quad H_x < 0. \tag{9}$$

An alternative to this may be a policy that sets the objective of improving the real economic equilibrium situation, which is less sensitive to financial equilibrium and tries to get out of the indebtedness trap by "going forward", i.e, improving the growth rate through fiscal expansion. In this case, $H_x > 0$. In the default case $\dot{x} = 0$ loci will have a negative slope *(Figure 3)*, but in the alternative case it will have a positive slope *(Figure 4)*.

$$\left.\frac{dx}{db}\right|_{\dot{x}=0} = -\frac{H_b}{H_x} < 0 \qquad \text{if} \quad H_x < 0.$$

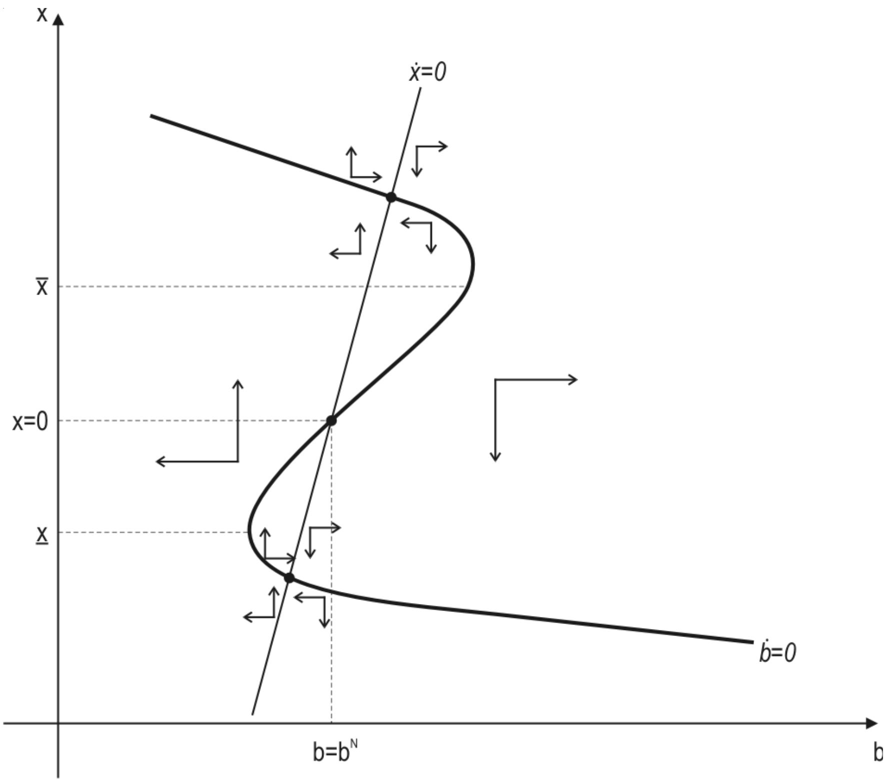

Figure 4. Equilibrium and stability in the two-equation non-linear model – growing out of the debt-trap case

In order to analyse the non-linear model defined by (8) and (9), let us consider the linearised version around the equilibrium values:[5]

$$\begin{bmatrix} \dot{b} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} F_b & F_x \\ H_b & H_x \end{bmatrix} \begin{bmatrix} b-b^* \\ x-x^* \end{bmatrix}. \tag{10}$$

Three equilibrium situations are possible (*Figures 3* and *4*). Let us first look at the medium equilibrium, which falls in the $[\underline{x}, \overline{x}]$ interval. Based on the initial conditions fixed

$$\mathrm{tr} J = F_b + H_x < 0, \qquad \text{if} \qquad |H_x| > F_b$$

$$\det J = F_b H_x - F_x H_b < 0$$

which means that the equilibrium is *not stable* and can only be approached along a saddle path. In economic terms, the policymaker has to be lucky to start in an initial (b, x) position on the saddle path, and then disciplined, following a path for x to keep the economy moving along the saddle path.

In the other two cases that are outside the specified interval, a stable equilibrium may exists if the Jacobi matrix's determinant is positive, i.e.,

$$|F_x H_b| > F_b H_x$$

and of course, tr $J < 0$ is continuous. The equilibrium trajectories will be stable spirals (as it is apparent from *Figure 3*) because $(\mathrm{tr}\, J)^2 < 4\det J$. In economic terms these two equilibrium situations may be interpreted so that the process of debt accumulation with a relatively high deficit may come to a resting point only with a low debt/GDP ratio, and a relatively high debt/GDP ratio may be maintained in a stable manner only with a considerable budget surplus.

In the alternative case when we assume an economic policy running forward, the medium equilibrium is not stable because tr J becomes positive due to $|H_x| > 0$. On the other hand, if $F_x H_b > F_b H_x$ (that is, if the slope of the $\dot{x} = 0$ curve is greater than that of the $\dot{b} = 0$ curve), det $J < 0$, as a result of which conversions to the saddle path is possible. The other two equilibrium situations will be unstable (or more precisely, the trajectories will be unstable spirals), because in this case both tr J and det J will be positive *(Figure 4)*.

[5] For the methodological background to solve non-linear differential equations see for example Gandolfo (1997) and Kaplan – Glass (1995).

5. A NON-LINEAR MODEL WITH ENDOGENOUS VARIABLES

Let us start again from the basic assumption (1') and try to give a definition to the three variables x, g and r, preferably within the given range of variables.[6]

$$\dot{b}(t) = [r(t) - g(t)]b(t) + x(t) \tag{1'}$$

$$x(t) = \gamma b(t) - \delta b^2(t) \tag{11}$$

$$g(t) = g^p - \omega b(t) \tag{12}$$

$$\begin{aligned} \dot{r}(t) &= \beta[b(t) - k(t)] \\ k(t) &= k^s + \alpha[r(t) - r^f] \end{aligned} \tag{13}$$

where k is the supply of loan capital (or capital inflow), $\alpha, \beta, \gamma, \delta, \omega > 0$ are parameters and g^p, k^s, $r^f > 0$ are constants. Equation (11) formulates that an increase in indebtedness sooner or later gives rise to budgetary restriction, and after a debt/GDP size over a certain level all economic policy courses will strive for surplus. Equation (12) states that the growth rate is negatively influenced by public debt. The interest-rate change is determined on the basis of demand for loan capital (b) and supply of loan capital (k), as it appears from the first equation of (13). The supply of loan capital changes on the basis of the relationship between domestic interest rate (r) and foreign interest rate (r^f) – beyond fixed basic supply (k^s) – according to the second equitation of (13), in which r^f was assumed to be a constant for simplicity's sake.

Taking equations (11)–(13) into account we get the following system of differential equations on (1'):

$$\dot{b}(t) = (\gamma - g^p)b(t) + (\omega - \delta)b^2(t) + r(t)b(t) \tag{14}$$

$$\dot{r}(t) = \beta b(t) - \alpha\beta r(t) + \alpha\beta r^f - \beta k^s.$$

Using the substitution $\dot{b} = \dot{r} = 0$ for determining the equilibrium situations it follows that

[6] It should be noted that the mainstream economic theory used a different way to endogenise these variables: it applied growth theory and the dynamic consumption optimisation, see for example Diamond (1965) and Blanchard (1985).

$$b(t)[(\gamma-g^{p})+(\omega-\delta)b(t)+r(t)]=0$$

$$b(t)-\alpha r(t)+\alpha r^{f}-k^{s}=0\,.$$

The equations of the equilibrium lines may be determined directly from this form and they are possible to chart in the [b, r] system of axis (*Figures 5* and *6*). Furthermore, it is possible to define the criteria for equilibrium and stability. Accordingly, we differentiate between two cases of parameter values:

Case 1:

$$g^{p}-\gamma>r^{f}-\frac{1}{\alpha}k^{s} \qquad \textit{and} \qquad \frac{1}{\alpha}>\delta-\omega$$

Case 2:

$$g^{p}-\gamma<r^{f}-\frac{1}{\alpha}k^{s} \qquad \textit{and} \qquad \frac{1}{\alpha}<\delta-\omega$$

In the first case the equilibrium is not stable, and only convergence along the saddle path is feasible *(Figure 5)*. In the second case the equilibrium will be stable *(Figure 6)*.

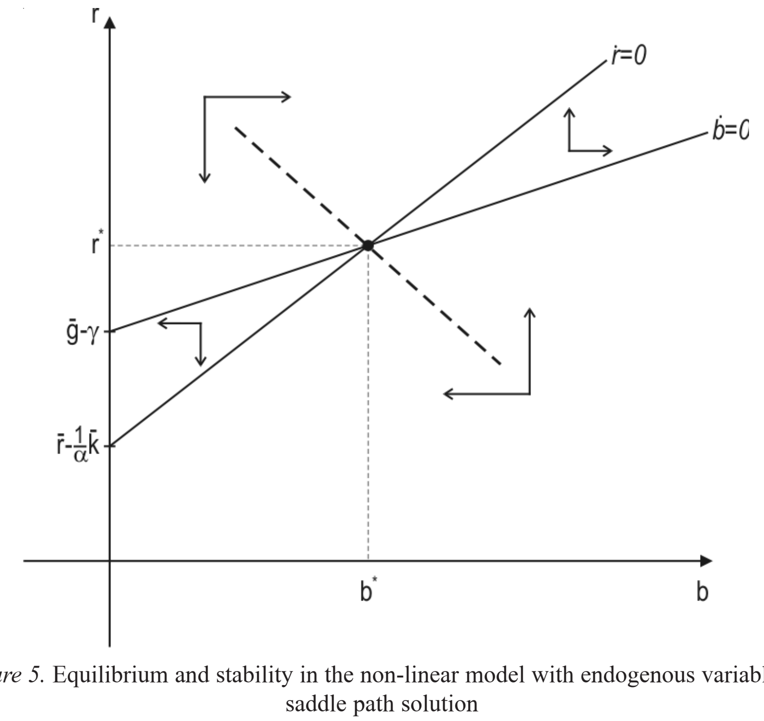

Figure 5. Equilibrium and stability in the non-linear model with endogenous variables – saddle path solution

The linearised version of the system of equations near the equilibrium values is:

$$\begin{bmatrix} \dot{b} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} (\omega-\delta)b^2 & b^* \\ \beta & -\alpha\beta \end{bmatrix} \begin{bmatrix} b-b^* \\ r-r^* \end{bmatrix}$$

The conditions for stability will be as follows:

$$\mathrm{tr}J = (\omega-\delta)b^* - \alpha\beta < 0$$

$$\det J = -\beta b^*[\alpha(\omega-\delta)+1] > 0.$$

If $\delta > \omega$, the first condition will be met, and if $1/\alpha < \delta - \omega$ also holds, the second condition will also be met (this is *Case 2* just mentioned, represented in *Figure 6*). Otherwise, only convergence along the saddle path is feasible (in line with *Case 1* and *Figure 5*).

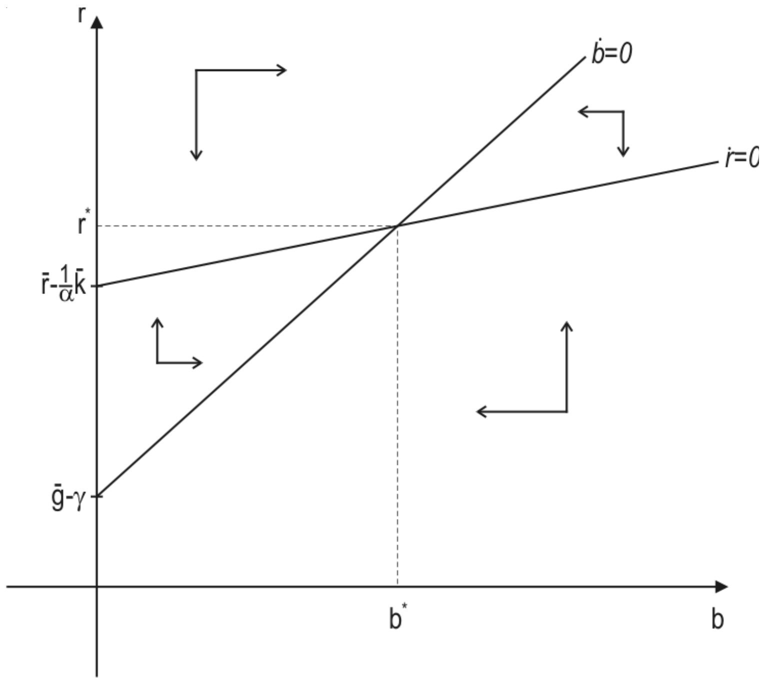

Figure 6. Equilibrium and stability in the non-linear model with endogenous variables – asymptotically stable equilibrium

The economic explanation of the two cases is relatively simple. The model has essentially three key parameters, α, β and ω. Parameter α shows the strength of capital inflow (or capital mobility). If capital mobility is low, it will have a low value, and accordingly its reciprocal will be large. And if $1/\alpha < \delta - \omega$, the system will be unstable as indicated by mathematical conditions. This situation occurs when there are not enough funds to finance debts, and therefore interest rate increases. An increase in interest rate further increases debts, which again generates a new wave of shortage of financing funds, and so on. If capital mobility is high, there will be enough capital inflow on the basis of the interest rate difference, which prevents further increase in the interest rate, and consequently the further increase of indebtedness.

If the value of δ is high, fiscal policy is conservative: the high debt ratio leads to a significant reduction in the budget deficit (or even triggers running surpluses). Parameter ω with the increase of b reduces the growth rate, so has a negative impact on stability. To ensure stability, a high value for $\delta - \omega$ is necessary, relative to $1/\alpha$.

6. SUMMARY

The classical debt dynamics relationship may be rendered more dynamics and thereby brought closer to practical application if we assume that the growth rate and the real interest rate change over time. In the first model version we considered the difference of $g - r = u$ to be a variable depending lineary on the value of the debt/GDP ratio and the budget deficit. This modification added subtlety to the dynamic behaviour of the system in a number of respects, even when we retained the exogenous nature of budget deficit x.

In the second version of the model, we allowed u to depend on the debt ratio and the budget deficit in a non-linear fashion. In addition, the budget deficit was endogenised by way of a fiscal policy rule. We considered two versions of economic policy as to their nature of dependence: the conservative version striving for equilibrium, and the "we will grow out of the debt trap" approach of fleeing forward. The dynamic operation of the system became much more complex and complicated than earlier. More equilibrium situations were generated, the characteristics of which were also different. Almost all cases that may theoretically arise were generated: stable and unstable equilibrium, convergence along the saddle path, spiral trajectory, etc.

In the third version of the model, both the growth rate and the interest rate were subject basically to the development of the debt/GDP ratio in addition to which we took into account exogenous factors: for growth the potential growth rate and for the real interest rate the external interest rate. We also dealt with capital-market conditions where demand for loan capital was determined by public debt and supply was determined by the internal and external interest rates. This way, however, our model contains two variables, because *r* also became an endogenous variable in addition to *b*. Given the set of conditions a stable equilibrium situation or a hogback line solution may emerge.

Naturally, there are many ways to further develop the basic relationship concerning debt dynamics, in line with the areas of interest of developer researchers and the practical problems that arise. Nevertheless, two directions of development appear to be rather evident on the basis of the findings of this study: on the one hand, it may be worthwhile to trace the operation of the system along various alternative scenarios retaining the exogenous character, but not the deterministic character of budget deficit. On the other hand, it might also be worthwhile to extend the model towards growth models, so that the current growth rate be an entirely endogenous variable. Accordingly, dynamic model with many variables would emerge, which would allow not only for studying the development of the debt/GDP ratio, but also the effect this has on the variables that have a fundamental influence on the operation of the real economy.

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