**Title:** ASYMPTOTIC INFERENCE FOR NEARLY UNSTABLE AR($p$) PROCESSES

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**Abstract.** In this paper nearly unstable AR($p$) processes (in other words, models with characteristic roots near the unit circle) are studied. Our main aim is to describe the asymptotic behaviour of the least squares estimators of the coefficients. A convergence result is presented for the general complex-valued case. The limit distribution is given by the help of some continuous time AR processes. We apply the results for real-valued nearly unstable AR($p$) models. In this case the limit distribution can be identified with the maximum likelihood estimator of the coefficients of the corresponding continuous time AR processes.

## 1. INTRODUCTION

Consider the autoregressive AR($p$) model

$$\begin{cases} X_k = \beta_1 X_{k-1} + \ldots + \beta_p X_{k-p} + \varepsilon_k, & k = 1, 2, \ldots \\ X_0 = X_{-1} = \ldots = X_{1-p} = 0, \end{cases} \tag{1}$$

where $\varepsilon_k$ is the (unobservable) random disturbance (noise) at time $k$, and $\beta_1, \ldots, \beta_p$ are unknown parameters. The least-squares estimator (LSE) of the parameter

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$$

based on the observations $X_1, \ldots, X_n$ is given by

$$\widehat{\boldsymbol{\beta}}_n = \Big(\sum\nolimits_{k=1}^n \widetilde{X}_{k-1}\widetilde{X}_{k-1}'\Big)^{-1} \sum\nolimits_{k=1}^n X_k \widetilde{X}_{k-1}, \tag{2}$$

where

$$\widetilde{X}_k = (X_k, X_{k-1}, \ldots, X_{k-p+1})'.$$

The polynomial $\varphi$ defined by

$$\varphi(z) = 1 - \beta_1 z - \ldots - \beta_p z^p$$

is called the *characteristic polynomial* of the AR($p$) model (1).

When all roots of $\varphi$ are outside the unit circle, the model (1) is said to be *asymptotically stationary*. Under the assumption that the $\varepsilon_k$'s are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = \sigma^2$, the LSE of $\boldsymbol{\beta}$ is asymptotically normal:

$$\Big(\sum\nolimits_{k=1}^n \widetilde{X}_{k-1}\widetilde{X}_{k-1}'\Big)^{1/2} (\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, I), \quad \text{as } n \to \infty, \tag{3}$$

where $\stackrel{\mathcal{D}}{\longrightarrow}$ denotes convergence in distribution and $I$ is the unit matrix (see Mann and Wald [16] and Anderson [2]). By another normalization

$$\sqrt{n}\,(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \Sigma^{-1}), \quad \text{as } n \to \infty,$$

where the matrix $\Sigma$ can be expressed by the help of $\sigma^2$ and the covariance matrix of the stationary distribution.

When $\varphi$ has no roots inside the unit circle but has at least one root on the unit circle the model (1) is said to be *unstable*. It was shown by White [20] that in the case of the unstable AR(1) model $X_k = \beta X_{k-1} + \varepsilon_k$, $k \geq 1$, with $\beta = 1$, the variables $n(\widehat{\beta}_n - \beta)$ converge in law to a random variable:

$$n(\widehat{\beta}_n - \beta) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 W(t)\,dW(t)}{\int_0^1 W^2(t)\,dt}, \tag{4}$$

where $W(t)$, $t \geq 0$, is a standard Wiener process. In case of the unstable AR($p$) model Chan and Wei [7] proved that with suitable normalizing matrices $\delta_n$ the sequence $\delta_n^{-1}(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})$ converges in law and gave the representation of the limit distribution. This representation involves multiple stochastic integrals with respect to Wiener processes and has a very complicated form.

The result (4) led to the study of the following so-called nearly nonstationary (better to call it nearly unstable) AR(1) model:

$$\begin{cases} X_{n,k} = \beta_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \ldots, n \\ X_{n,0} = 0, \end{cases} \tag{5}$$

where $\beta_n = 1 + h/n$. It was shown by Chan and Wei [5], [6] that

$$\Big(\sum\nolimits_{k=1}^n X_{n,k-1}^2\Big)^{1/2} (\widehat{\beta}_n - \beta_n) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 Y(t)\,dW(t)}{\Big(\int_0^1 Y^2(t)\,dt\Big)^{1/2}}, \tag{6}$$

where $Y(t)$, $t \in [0, 1]$, is an Ornstein-Uhlenbeck process defined as the solution of the stochastic differential equation

$$dY(t) = hY(t)\,dt + dW(t), \qquad Y(0) = 0. \tag{7}$$

By another normalization

$$n\,(\widehat{\beta}_n - \beta_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t)\,dW(t)}{\int_0^1 Y^2(t)\,dt}, \tag{8}$$

see, for example, Phillips [18], Jeganathan [10], Dzhaparidze, Kormos, van der Meer and van Zuijlen [9]. (The above model is called also near integrated and is applied often in economic theory; see Phillips [18].)

Recently, Jeganathan [10] has considered nearly unstable AR($p$) models, i. e. AR($p$) models near to an unstable model:

$$\begin{cases} X_{n,k} = \beta_{1,n}X_{n,k-1} + \ldots + \beta_{p,n}X_{n,k-p} + \varepsilon_{n,k}, & k = 1, 2, \ldots, n \\ X_{n,0} = X_{n,-1} = \ldots = X_{n,1-p} = 0, \end{cases} \tag{9}$$

where the vector of parameters

$$\boldsymbol{\beta}_n = (\beta_{1,n}, \ldots, \beta_{p,n})'$$

is given by

$$\boldsymbol{\beta}_n = \boldsymbol{\beta} + \delta_n \mathbf{h}_n,$$

where

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$$

is a vector such that the polynomial

$$\varphi(z) = 1 - \beta_1 z - \ldots - \beta_p z^p$$

corresponds to an unstable AR($p$) model, $\{\delta_n\}$ are the same normalizing matrices obtained in Chan and Wei [7], and

$$\mathbf{h}_n = (h_{1,n}, \ldots, h_{p,n})'$$

is a sequence of vectors with $\mathbf{h}_n \to \mathbf{h}$. Jeganathan [10] proved that the sequence $\delta_n^{-1}\,(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n)$ converges in law and gave a very complicated representation for the limiting distribution in terms of multiple stochastic integrals with respect to Wiener processes.

One of the aims of the present paper is to find a simpler explanation for the asymptotic behaviour of the least-squares estimators in the nearly unstable AR($p$) model (9). The starting point of our investigation is the following equivalent formulation of (8). We consider $h$ instead of $\beta_n$ as a parameter. Then the LSE of $h$ is

$$\widehat{h}_n = n\,(\widehat{\beta}_n - \beta) = n\,(\widehat{\beta}_n - \beta_n) + h,$$

and we have

$$\widehat{h}_n \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t)\,dW(t)}{\int_0^1 Y^2(t)\,dt} + h = \frac{\int_0^1 Y(t)\,dY(t)}{\int_0^1 Y^2(t)\,dt}, \tag{10}$$

where the limit distribution in (10) turns out to be the maximum likelihood estimator (MLE) of the parameter $h$ in the model (7) (see, for example, Arató [3]). So if we use $h$ as a parameter then we do not have to normalize the LSE $\widehat{h}_n$. Remark that $h$ is connected with the rate of convergence in $\beta_n \to \beta$.

In the nearly unstable AR($p$) model (9) we suggest to use again parameters which are connected with the speed of approximation of roots of $\varphi$ (the characteristic polynomial of the limit unstable model). For the sake of simplicity we suppose that $\varphi$ has all its roots on the unit circle (purely unstable case). Then $\varphi$ can be written as

$$\varphi(z) = (1-z)^a(1+z)^b \prod_{j=1}^{\ell} \left((1 - e^{i\alpha_j}z)(1 - e^{-i\alpha_j}z)\right)^{m_j},$$

where $a, b, \ell, m_j$, $j = 1, \ldots, \ell$, are non-negative integers, $\alpha_j \in (0, \pi)$, $j = 1, \ldots, \ell$. We suggest to write $\varphi$ in the form

$$\varphi(z) = \prod_{j=1}^{q} (1 - a_j z)^{r_j},$$

where $q = 2 + 2\ell$, $a_j = e^{i\theta_j}$ and $\theta_1, \ldots, \theta_q \in (-\pi, \pi]$ are all different. We suppose that in the nearly unstable AR($p$) model (9) the characteristic polynomial $\varphi_n$ can be written as

$$\varphi_n(z) = \prod_{j=1}^{q} \prod_{k=1}^{r_j} (1 - a_{j,k,n} z),$$

where $a_{j,k,n} = e^{h_{j,k,n}/n + i\theta_j}$, $h_{j,k,n}$, $j = 1, \ldots, q$, $k = 1, \ldots, r_j$, $n \geq 1$, are complex numbers such that $h_{j,k,n} \to h_{j,k}$, as $n \to \infty$.

The main idea is to introduce another set of parameters, which are in one-to-one linear correspondence with the coefficients $\beta_{1,n}, \ldots, \beta_{p,n}$. For $j = 1, \ldots, q$, $k = 1, \ldots, r_j$, $n \geq 1$ let $c_{j,k,n} \in \mathbb{C}$ be defined (uniquely) by

$$\frac{\varphi_n(z)}{\varphi(z)} = \frac{\prod_{j=1}^q \prod_{k=1}^{r_j} (1 - a_{j,k,n} z)}{\prod_{j=1}^q (1 - a_j z)^{r_j}} = 1 - \sum_{j=1}^q \sum_{k=1}^{r_j} \frac{c_{j,k,n}}{n^k} \frac{a_j z}{(1 - a_j z)^k}, \qquad z \in \mathbb{C}. \tag{11}$$

In Theorem 5 a simple description of the limit distribution of the LSE $\widehat{c}_{j,k,n}$ will be given. Under some natural condition (C) on the $\varepsilon_{n,k}$'s, we will prove

$$\widehat{c}_{j,k,n} \xrightarrow{\mathcal{D}} \widehat{c}_{j,k}, \qquad \text{as} \quad n \to \infty,$$

jointly for $j = 1, \ldots, q$, $k = 1, \ldots, r_j$, where $\widehat{\mathbf{c}}_j = (\widehat{c}_{j,1}, \ldots, \widehat{c}_{j,r_j})'$ is given by

$$\widehat{\mathbf{c}}_j = S_j^{-1} \begin{pmatrix} \int_0^1 \overline{Y_j^{(r_j-1)}(t)} \, dY_j^{(r_j-1)}(t) \\ \vdots \\ \int_0^1 \overline{Y_j^{(0)}(t)} \, dY_j^{(r_j-1)}(t) \end{pmatrix},$$

where

$$S_j = \begin{pmatrix} \int_0^1 |Y_j^{(r_j-1)}(t)|^2 \, dt & \ldots & \int_0^1 \overline{Y_j^{(r_j-1)}(t)} Y_j^{(0)}(t) \, dt \\ \vdots & \ddots & \vdots \\ \int_0^1 \overline{Y_j^{(0)}(t)} Y_j^{(r_j-1)}(t) \, dt & \ldots & \int_0^1 |Y_j^{(0)}(t)|^2 \, dt \end{pmatrix},$$

and the processes $Y_j^{(k)}$, $j = 1, \ldots, q$, $k = 1, \ldots, r_j$, are given by the stochastic differential equation

$$\begin{cases} dY_j^{(r_j-1)}(t) = \big(c_{j,1} Y_j^{(r_j-1)}(t) + \ldots + c_{j,r_j} Y_j(t)\big) \, dt + dW_j(t), \\ Y_j(0) = Y_j^{(1)}(0) = \ldots = Y_j^{(r_j-1)}(0) = 0, \end{cases} \tag{12}$$

where $Y_j^{(1)}, \ldots, Y_j^{(r_j-1)}$ are the derivatives of $Y_j$, and $W_j(t)$, $t \in [0,1]$, $j = 1, \ldots, q$, are independent standard Wiener processes (which are real-valued for $\theta = 0$ or $\theta = \pi$, and complex-valued otherwise), and the characteristic polynomial of the model (12) is given by

$$1 - c_{j,1} z - \ldots - c_{j,r_j} z^{r_j} = \prod_{k=1}^{r_j} (1 - h_{j,k} z). \tag{13}$$

(For information on continuous time autoregressive processes cf. Arató [3].) Roughly speaking, the model (12) can be written as

$$\prod_{k=1}^{r_j} (d - h_{j,k}) Y_j(t) \, dt = dW_j(t), \tag{14}$$

where $d$ is the differential operator, and for the LSE of the parameters $h_{j,k,n}$ of the discrete time model (9) we prove the joint convergence for $j = 1, \ldots, q$, $k = 1, \ldots, r_j$

$$\widehat{h}_{j,k,n} \to \widehat{h}_{j,k} \quad \text{as} \quad n \to \infty,$$

where $\widehat{h}_{j,k}$ is the MLE of the parameter $h_{j,k}$ in the continuous time model (14).

In the present paper we clarify the relationship between general complex-valued discrete and continuous time AR($p$) models. As a consequence we are able to understand and to simplify the complicated expressions of Jeganathan [10] for the limit distribution of the least squares estimators in real-valued discrete settings. One of the advantages of our approach of studying complex-valued models is that we avoid complicated formulas with sines and cosines. In Section 7 we show how to use our results for real-valued AR($p$) models. Section 8 contains some examples demonstrating how to derive limit theorems for the least squares estimators of the coefficients of the discrete time models.

## 2. PRELIMINARIES, NOTATIONS

We shall use the notations

$$\mathbf{X}_n = (X_{n,1}, \dots, X_{n,n})', \qquad \boldsymbol{\varepsilon}_n = (\varepsilon_{n,1}, \dots, \varepsilon_{n,n})'.$$

Let $B$ denote the $n \times n$ backshift matrix, i. e. $B = (b_{jk})$, where

$$b_{jk} = \begin{cases} 1 & \text{if } j = k+1, \\ 0 & \text{else.} \end{cases}$$

The model (9) can be written in the short form

$$\varphi_n(B)\mathbf{X}_n = \boldsymbol{\varepsilon}_n.$$

For $\theta \in (-\pi, \pi]$ let $T_\theta$ be the $n \times n$ rotation matrix with angle $-\theta$, i. e. $T_\theta = (t_{jk})$, where

$$t_{jk} = \begin{cases} e^{-ik\theta} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

We have the simple commutation relation

$$T_\theta B = e^{-i\theta} B T_\theta. \tag{15}$$

For $n \times n$ matrices $A_1$ and $A_2$ with $A_1 A_2^{-1} = A_2^{-1} A_1$ we shall write sometimes $A_1/A_2$ instead of $A_1 A_2^{-1}$ or $A_2^{-1} A_1$.

For a complex number $z \in \mathbb{C}$ we denote by $\Re(z)$ and $\Im(z)$ the real and the imaginary part, respectively. We shall use the complex $d$-dimensional space $\mathbb{C}^d$ endowed with the inner product

$$\langle z, w \rangle = z_1 \overline{w}_1 + \dots + z_d \overline{w}_d$$

and with the norm

$$\|z\| = (|z_1|^2 + \dots + |z_d|^2)^{1/2}$$

for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, $w = (w_1, \dots, w_d) \in \mathbb{C}^d$. We denote by $C([0,1] \to \mathbb{R}^d)$ and $C([0,1] \to \mathbb{C}^d)$ the spaces of continuous functions with values in $\mathbb{R}^d$ and $\mathbb{C}^d$, respectively, endowed with the supremum norm. The supremum norm and the Skorokhod metric on the space $D([0,1] \to \mathbb{C}^d)$ will be denoted by $\|\cdot\|_\infty$ and $\rho$, respectively.

For measurable mappings $\Phi, \Phi_n : D([0,1] \to \mathbb{C}^k) \to D([0,1] \to \mathbb{C}^\ell)$, $n = 1, 2, \dots$ we shall write $\Phi_n \rightsquigarrow \Phi$ if $\|\Phi_n(x_n) - \Phi(x)\|_\infty \to 0$ for all $x_n \in D([0,1] \to \mathbb{C}^k)$, $x \in C([0,1] \to \mathbb{C}^k)$ with $\|x_n - x\|_\infty \to 0$. We shall need the following simple lemma, which is based on the continuous mapping theorem and the Skorokhod-construction.

LEMMA 1. *Let $\Phi, \Phi_n : D([0,1] \to \mathbb{C}^k) \to D([0,1] \to \mathbb{C}^\ell)$, $n = 1, 2, \dots$ be measurable mappings such that $\Phi_n \rightsquigarrow \Phi$. Let $Z, Z_n$, $n = 1, 2, \dots$ be stochastic processes with values in $D([0,1] \to \mathbb{C}^k)$ such that $Z_n \xrightarrow{\mathcal{D}} Z$ in $D([0,1] \to \mathbb{C}^k)$ and almost all trajectories of $Z$ are continuous. Then $\Phi_n(Z_n) \xrightarrow{\mathcal{D}} \Phi(Z)$ in $D([0,1] \to \mathbb{C}^\ell)$.*

PROOF. Due to the Skorokhod-construction we can find processes $\widetilde{Z}_n$ and a process $\widetilde{Z}$, such that $\widetilde{Z}_n \overset{\mathcal{D}}{=} Z_n$, $\widetilde{Z} \overset{\mathcal{D}}{=} Z$ and

$$\rho(\widetilde{Z}_n, \widetilde{Z}) \to 0 \quad \text{a. s.}$$

Using the fact that $\widetilde{Z}$ has continuous trajectories a. s., we conclude that

$$\|\widetilde{Z}_n - \widetilde{Z}\|_\infty \to 0 \quad \text{a. s.}$$

Thus we have

$$\|\Phi_n(\widetilde{Z}_n) - \Phi(\widetilde{Z})\|_\infty \to 0 \quad \text{a.s.}$$

and hence

$$\Phi_n(\widetilde{Z}_n) \xrightarrow{\mathcal{D}} \Phi(\widetilde{Z}) \quad \text{in } D([0,1] \to \mathbb{C}^\ell).$$

The last relation implies the desired result. ■

## 3. PARAMETRIZATIONS OF THE AR($p$) MODEL

In addition to the parameters $\beta_{1,n},\ldots,\beta_{p,n}$ and $h_{j,k,n}$, $j=1,\ldots,q$, $k=1,\ldots,r_j$ of the model (9) we introduce another two other systems of parameters, which both tend to the limit $c_{j,k}$, $j=1,\ldots,q$, $k=1,\ldots,r_j$ (given in (13)) as $n\to\infty$, and will be useful for the investigation of the LSE.

For $j=1,\ldots,q$, $n\geq 1$ let $d_{j,0,n},\ldots,d_{j,r_j,n}\in\mathbb{C}$ be defined (uniquely) by

$$\frac{\prod_{k=1}^{r_j}(1-a_{j,k,n}z)}{(1-a_jz)^{r_j}}=d_{j,0,n}-\sum\nolimits_{k=1}^{r_j}\frac{d_{j,k,n}}{n^k(1-a_jz)^k},\qquad z\in\mathbb{C}. \tag{16}$$

Lemma 2. *For* $j=1,\ldots,q$, $k=1,\ldots,r_j$ *we have*

$$\lim\nolimits_{n\to\infty}d_{j,k,n}=c_{j,k},$$

*where* $c_{j,k}$ *is defined by (13).*

Proof. Let $u\in\mathbb{C}$, $u\neq 0$. Then substituting

$$z=\frac{1}{a_j}\left(1-\frac{1}{nu}\right)$$

into (16) we obtain

$$\begin{aligned}d_{j,0,n}-\sum\nolimits_{k=1}^{r_j}d_{j,k,n}u^k &= \prod_{k=1}^{r_j}\left(\frac{a_{j,k,n}}{a_j}-n\left(\frac{a_{j,k,n}}{a_j}-1\right)u\right)\\ &\to \prod_{k=1}^{r_j}(1-h_{j,k}u)=1-c_{j,1}u-\ldots-c_{j,r_j}u^{r_j},\end{aligned}$$

since $a_{j,k,n}=e^{h_{j,k,n}/n+i\theta_j}\to e^{i\theta_j}=a_j$ and

$$n\left(\frac{a_{j,k,n}}{a_j}-1\right)=n\left(e^{h_{j,k,n}/n}-1\right)\to h_{j,k}. \qquad \blacksquare$$

For $j=1,\ldots,q$, $k=1,\ldots,r_j$, $n\geq 1$ let $c_{j,k,n}\in\mathbb{C}$ be defined (uniquely) by

$$\frac{\varphi_n(z)}{\varphi(z)}=\frac{\prod_{j=1}^q\prod_{k=1}^{r_j}(1-a_{j,k,n}z)}{\prod_{j=1}^q(1-a_jz)^{r_j}}=1-\sum\nolimits_{j=1}^q\sum\nolimits_{k=1}^{r_j}\frac{c_{j,k,n}}{n^k}\frac{a_jz}{(1-a_jz)^k},\qquad z\in\mathbb{C}. \tag{17}$$

Lemma 3. *For* $j=1,\ldots,q$, $k=1,\ldots,r_j$, $n\geq 1$ *we have*

$$c_{j,k,n}=\sum\nolimits_{\ell=0}^{r_j-k}\frac{\psi_{n,j}^{(\ell)}(a_j^{-1})}{n^\ell(-a_j)^\ell\ell!}d_{j,k+\ell,n}+\frac{c_{j,k+1,n}}{n},$$

*where* $c_{j,r_j+1,n}=0$ *and*

$$\psi_{n,j}(z)=\prod_{\substack{1\leq m\leq q\\ m\neq j}}\prod_{k=1}^{r_m}\left(\frac{1-a_{m,k,n}z}{1-a_jz}\right)\qquad z\in\mathbb{C}.$$

*For* $j=1,\ldots,q$, $k=1,\ldots,r_j$, *and sufficiently large* $n\geq 1$ *we have*

$$d_{j,k,n}=\sum\nolimits_{\ell=0}^{r_j-k}\frac{(z\tilde\psi_{n,j})^{(\ell)}(a_j^{-1})}{n^\ell(-a_j)^{\ell-1}\ell!}c_{j,k+\ell,n},$$

*where* $\tilde\psi_{n,j}(z)=1/\psi_{n,j}(z)$, $z\in\mathbb{C}$.

Proof. For $j=1,\ldots,q$ let $\Gamma_j$ be a closed curve around the point $a_j^{-1}$, not containing $a_\ell^{-1}$, $\ell\neq j$. Applying Cauchy's Integral Theorem we obtain for $k=1,2,\ldots,r_j$

$$\frac{1}{2\pi i}\int_{\Gamma_j}\frac{\varphi_n(z)}{\varphi(z)}n^{k-1}(1-a_jz)^{k-1}dz$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\Gamma_j}\left(1-\sum\nolimits_{m=1}^{q}\sum\nolimits_{\ell=1}^{r_m}\frac{c_{m,\ell,n}}{n^\ell}\frac{a_m z}{(1-a_m z)^\ell}\right)n^{k-1}(1-a_j z)^{k-1}dz\\
&= -\frac{1}{2\pi i}\int_{\Gamma_j}\left(\frac{c_{j,k,n}}{n}\frac{a_j z}{1-a_j z}+\frac{c_{j,k+1,n}}{n^2}\frac{a_j z}{(1-a_j z)^2}\right)dz\\
&= \frac{1}{a_j}\left(\frac{c_{j,k,n}}{n}-\frac{c_{j,k+1,n}}{n^2}\right),
\end{aligned}
$$

consequently

$$
c_{j,k,n} = \frac{a_j}{2\pi i}\int_{\Gamma_j} n^k(1-a_j z)^{k-1}\frac{\varphi_n(z)}{\varphi(z)}dz + \frac{c_{j,k+1,n}}{n}.
$$

On the other hand

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_j} n^k(1-a_j z)^{k-1}\frac{\varphi_n(z)}{\varphi(z)}dz\\
&\quad= \frac{1}{2\pi i}\int_{\Gamma_j} n^k(1-a_j z)^{k-1}\frac{\prod_{\ell=1}^{r_j}(1-a_{j,\ell,n}z)}{(1-a_j z)^{r_j}}\psi_{n,j}(z)dz\\
&\quad= \frac{1}{2\pi i}\int_{\Gamma_j} n^k(1-a_j z)^{k-1}\left(d_{j,0,n}-\sum\nolimits_{\ell=1}^{r_j}\frac{d_{j,\ell,n}}{n^\ell(1-a_j z)^\ell}\right)\psi_{n,j}(z)dz\\
&\quad= -\sum\nolimits_{\ell=k}^{r_j} d_{j,\ell,n}\frac{1}{2\pi i}\int_{\Gamma_j}\frac{\psi_{n,j}(z)}{n^{\ell-k}(1-a_j z)^{\ell-k+1}}dz\\
&\quad= -\sum\nolimits_{\ell=0}^{r_j-k}\frac{\psi_{n,j}^{(\ell)}(a_j^{-1})}{n^\ell(-a_j)^{\ell+1}\ell!}d_{j,k+\ell,n}.
\end{aligned}
$$

The second statement can be proved similarly. ■

Remark 1. For $j=1,\dots,q$ we obtain

$$
\lim_{n\to\infty}\psi_{n,j}(a_j^{-1}) = 1, \tag{18}
$$

since for $m \neq j$ we have $\lim_{n\to\infty}(1-a_{m,k,n}a_j^{-1})/(1-a_m a_j^{-1}) = 1$. Moreover, for $j=1,\dots,q$, $\ell=1,2,\dots$ $\lim_{n\to\infty}\psi_{n,j}^{(\ell)}(a_j^{-1})$ exists. Similarly, for $j=1,\dots,q$ we obtain

$$
\lim_{n\to\infty}\tilde{\psi}_{n,j}(a_j^{-1}) = 1, \tag{19}
$$

and for $j=1,\dots,q$, $\ell=1,2,\dots$ $\lim_{n\to\infty}(z\tilde{\psi}_{n,j})^{(\ell)}(a_j^{-1})$ exists. Consequently we have the following easy corollary.

Corollary 1. *For* $j=1,\dots,q$, $k=1,\dots,r_j$ *we have*

$$
\lim\nolimits_{n\to\infty} c_{j,k,n} = c_{j,k},
$$

*where* $c_{j,k}$ *is defined by (13).*

# 4. COMPLEX-VALUED AR(1) PROCESSES

In this section we collect some convergence results for nearly ubstable complex-valued AR(1) processes which will be used in the next sections.

For every $n=1,2,\dots$ consider the complex-valued AR(1) model

$$
\begin{cases} X_{n,k} = \beta_n X_{n,k-1} + \varepsilon_{n,k}, & k=1,2,\dots,n\\ X_{n,0}=0, \end{cases} \tag{20}
$$

where $\{\varepsilon_{n,k}\}$ is an array of complex random variables and $\beta_n = e^{h_n/n+i\theta}$, where $\{h_n\}$ is a sequence of complex numbers such that $h_n \to h$, and $\theta \in (-\pi,\pi]$. Then $\beta_n \to \beta = e^{i\theta}$.

For $\theta_1 \in (-\pi, \pi]$ the random step functions

$$\begin{aligned} Y_{n,\theta_1}(t) &= \frac{1}{\sqrt{n}} e^{-i[nt]\theta_1} X_{n,[nt]} = \frac{1}{\sqrt{n}} (T_{\theta_1} \mathbf{X}_n)_{[nt]}, \quad t \in [0,1] \\ M_{n,\theta_1}(t) &= \frac{1}{\sqrt{n}} \sum\nolimits_{k=1}^{[nt]} e^{-ik\theta_1} \varepsilon_{n,k} = \frac{1}{\sqrt{n}} \sum\nolimits_{k=1}^{[nt]} (T_{\theta_1} \boldsymbol{\varepsilon}_n)_k, \quad t \in [0,1] \end{aligned}$$

can be considered as random elements in the complex Skorokhod space $D([0,1] \to \mathbb{C})$.

First we investigate convergence of the sequence $Y_{n,\theta}$, $n \geq 1$, in $D([0,1] \to \mathbb{C})$.

THEOREM 1. *Suppose that* $M_{n,\theta} \xrightarrow{\mathcal{D}} M$ *in* $D([0,1] \to \mathbb{C})$, *where* $M(t)$, $t \in [0,1]$, *is a complex-valued continuous semimartingale.*

*Then there exist measurable mappings* $\Phi, \Phi_n : D([0,1] \to \mathbb{C}) \to D([0,1] \to \mathbb{C}^2)$, $n = 1, 2, \ldots$, *such that* $(M_{n,\theta}, Y_{n,\theta}) = \Phi_n(M_{n,\theta})$ *and* $\Phi_n \rightsquigarrow \Phi$, *where* $\Phi(M) = (M, Y)$, *and* $Y(t)$, $t \in [0,1]$, *is the complex-valued Ornstein-Uhlenbeck process defined as the solution of the stochastic differential equation*

$$dY(t) = hY(t)\,dt + dM(t), \qquad Y(0) = 0. \tag{21}$$

*Particularly,* $Y_{n,\theta} \xrightarrow{\mathcal{D}} Y$ *in* $D([0,1] \to \mathbb{C})$.

REMARK 2. The converse statement is also true and can be proved similarly. We also remark that convergence $M_{n,\theta_1} \xrightarrow{\mathcal{D}} M$ for some $\theta_1 \neq \theta$ does not imply convergence of the sequence $Y_{n,\theta_1}$, rotated by a 'wrong' angle.

PROOF. The model (20) can be written as

$$(I - e^{h_n/n + i\theta} B)\mathbf{X}_n = \boldsymbol{\varepsilon}_n,$$

consequently the commutation relation (15) implies that the *rotated* observations $\mathbf{Z}_n = T_\theta \mathbf{X}_n$ form again a nearly unstable complex AR(1) model

$$(I - e^{h_n/n} B)\mathbf{Z}_n = \boldsymbol{\zeta}_n, \tag{22}$$

where $\boldsymbol{\zeta}_n = T_\theta \boldsymbol{\varepsilon}_n$ contains the rotated random disturbances.

It is known that the process $Y(t)$, $t \in [0,1]$, can be expressed as

$$Y(t) = \int_0^t e^{h(t-s)}\,dM(s), \quad t \in [0,1].$$

Itô's formula gives also the representation

$$Y(t) = M(t) + h \int_0^t e^{h(t-s)} M(s)\,ds, \quad t \in [0,1].$$

A similar formula holds for the random step functions $Y_{n,\theta}(t)$, $t \in [0,1]$:

$$Y_{n,\theta}(t) = M_{n,\theta}(t) + h_n \int_0^{[nt]/n} e^{h_n([nt]/n - s)} M_{n,\theta}(s)\,ds, \tag{23}$$

since (22) implies

$$Y_{n,\theta}(t) = \frac{1}{\sqrt{n}} \sum\nolimits_{k=1}^{[nt]} e^{h_n([nt]-k)/n} \zeta_{n,k},$$

and the discrete analogue of the partial integration yields

$$\begin{aligned} h_n \int_0^{[nt]/n} e^{-h_n s} M_{n,\theta}(s)\,ds &= h_n \sum\nolimits_{j=1}^{[nt]} M_{n,\theta}\left((j-1)/n\right) \int_{(j-1)/n}^{j/n} e^{-h_n s}\,ds \\ &= \sum\nolimits_{j=1}^{[nt]} M_{n,\theta}\left((j-1)/n\right) \left(e^{-h_n(j-1)/n} - e^{-h_n j/n}\right) \\ &= \frac{1}{\sqrt{n}} \sum\nolimits_{j=1}^{[nt]} \sum\nolimits_{k=1}^{j-1} \zeta_{n,k} \left(e^{-h_n(j-1)/n} - e^{-h_n j/n}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \sum_{j=k+1}^{[nt]} \zeta_{n,k} \left( e^{-h_n(j-1)/n} - e^{-h_n j/n} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \zeta_{n,k} \left( e^{-h_n k/n} - e^{-h_n [nt]/n} \right) \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \zeta_{n,k} \left( e^{-h_n k/n} - e^{-h_n [nt]/n} \right) \\
&= e^{-h_n [nt]/n} (Y_{n,\theta}(t) - M_{n,\theta}(t)).
\end{aligned}
$$

Hence the processes $(M, Y)$ and $(M_{n,\theta}, Y_{n,\theta})$ can be expressed as

$$
(M, Y) = \Phi(M), \qquad (M_{n,\theta}, Y_{n,\theta}) = \Phi_n(M_{n,\theta}), \quad n = 1, 2, \dots,
$$

where the measurable mappings $\Phi, \Phi_n : D([0,1] \to \mathbb{C}) \to D([0,1] \to \mathbb{C}^2)$, $n = 1, 2, \dots$ are defined as follows

$$
\begin{aligned}
\Phi(x)(t) &= \left( x(t),\, x(t) + h \int_0^t e^{h(t-s)} x(s) ds \right), \\
\Phi_n(x)(t) &= \left( x(t),\, x(t) + h_n \int_0^{[nt]/n} e^{h_n([nt]/n - s)} x(s) ds \right).
\end{aligned}
$$

Applying Lemma 1 we obtain the last statement. ■

Consider now the random step functions

$$
U_{n,\theta_1}(t) = \frac{1}{n^{3/2}} \sum_{k=1}^{[nt]} e^{-i(k-1)\theta_1} X_{n,k-1},
$$

where $\theta_1 \in (-\pi, \pi]$.

Theorem 2. *Let $\theta_1 \in (-\pi, \pi]$. Let us suppose that $(M_{n,\theta}, M_{n,\theta_1}) \xrightarrow{\mathcal{D}} (M, M_1)$ in $D([0,1] \to \mathbb{C}^2)$, where $(M(t), M_1(t))$, $t \in [0,1]$, is a continuous semimartingale with values in $\mathbb{C}^2$.*

*Then there exist measurable mappings $\Phi, \Phi_n : D([0,1] \to \mathbb{C}^2) \to D([0,1] \to \mathbb{C}^3)$, $n = 1, 2, \dots$, such that $(M_{n,\theta}, M_{n,\theta_1}, U_{n,\theta_1}) = \Phi_n(M_{n,\theta}, M_{n,\theta_1})$ and $\Phi_n \rightsquigarrow \Phi$, where*

$$
\Phi(M, M_1)(t) = \begin{cases} (M(t), M(t), \int_0^t Y(s)\, ds), & \text{if } \theta_1 = \theta, \\ (M(t), M_1(t), 0), & \text{if } \theta_1 \neq \theta, \end{cases}
$$

*and $Y(t)$, $t \in [0,1]$, is defined by (21). Particularly,*

$$
\frac{1}{n^{3/2}} \sum_{k=1}^{[nt]} e^{-i(k-1)\theta_1} X_{n,k-1} \xrightarrow{\mathcal{D}} \begin{cases} \int_0^t Y(s)\, ds, & \text{if } \theta_1 = \theta, \\ 0, & \text{if } \theta_1 \neq \theta \end{cases}
$$

*in $D([0,1] \to \mathbb{C})$.*

Proof. If $\theta_1 = \theta$ then

$$
U_{n,\theta}(t) = \frac{1}{n^{3/2}} \sum_{k=1}^{[nt]} e^{-i(k-1)\theta} X_{n,k-1} = \frac{1}{n} \sum_{k=1}^{[nt]} Y_{n,\theta}\left((k-1)/n\right) = \int_0^{[nt]/n} Y_{n,\theta}(s)\, ds,
$$

and Theorem 1 gives the result.

If $\theta_1 \neq \theta$ then using

$$
X_{n,k} = \sum_{j=1}^{k} \beta_n^{k-j} \varepsilon_{n,j}
$$

we obtain

$$
\begin{aligned}
\sum_{k=1}^{[nt]} e^{-i(k-1)\theta_1} X_{n,k-1} &= \sum_{k=1}^{[nt]} e^{-i(k-1)\theta_1} \sum_{j=1}^{k-1} \beta_n^{k-j-1} \varepsilon_{n,j} \\
&= \sum_{j=1}^{[nt]-1} \sum_{k=j+1}^{[nt]} e^{-i(k-1)\theta_1} \beta_n^{k-j-1} \varepsilon_{n,j} \\
&= \sum_{j=1}^{[nt]-1} \frac{e^{-i[nt]\theta_1} \beta_n^{[nt]-j} - e^{-ij\theta_1}}{e^{-i\theta_1}\beta_n - 1} \varepsilon_{n,j},
\end{aligned}
$$

consequently

$$U_{n,\theta_1}(t) = \frac{1}{n(\beta_n - e^{i\theta_1})}\left(\beta_n e^{i([nt]-1)(\theta-\theta_1)} Y_{n,\theta}\left(t-n^{-1}\right) - e^{i\theta_1} M_{n,\theta_1}\left(t-n^{-1}\right)\right),$$

and Theorem 1 gives the result. ∎

For every $n = 1, 2, \ldots$ consider the complex ARMA(1,1) process defined by

$$\begin{cases} \tilde{\varepsilon}_{n,k} - \beta_n \tilde{\varepsilon}_{n,k-1} = \varepsilon_{n,k} - \beta \varepsilon_{n,k-1}, & k = 1, 2, \ldots, n \\ \tilde{\varepsilon}_{n,0} = 0, \end{cases} \tag{24}$$

and the random step functions

$$\widetilde{M}_{n,\theta_1}(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{-ik\theta_1} \tilde{\varepsilon}_{n,k}, \qquad t \in [0,1],$$

where $\theta_1 \in (-\pi, \pi]$.

Corollary 2. *Let $\theta_1 \in (-\pi, \pi]$. Let us suppose that $(M_{n,\theta}, M_{n,\theta_1}) \xrightarrow{\mathcal{D}} (M, M_1)$ in $D([0,1] \to \mathbb{C}^2)$, where $(M(t), M_1(t))$, $t \in [0,1]$, is a continuous semimartingale with values in $\mathbb{C}^2$.*

*Then there exist measurable mappings $\Phi, \Phi_n : D([0,1] \to \mathbb{C}^2) \to D([0,1] \to \mathbb{C}^3)$, $n = 1, 2, \ldots$, such that $\left(M_{n,\theta}, M_{n,\theta_1}, \widetilde{M}_{n,\theta_1}\right) = \Phi_n(M_{n,\theta}, M_{n,\theta_1})$ and $\Phi_n \rightsquigarrow \Phi$, where*

$$\Phi(M, M_1) = \begin{cases} (M, M, Y), & \text{if } \theta_1 = \theta, \\ (M, M_1, M_1), & \text{if } \theta_1 \neq \theta, \end{cases}$$

*and $Y(t)$, $t \in [0,1]$, is defined by (21). Particularly,*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{-ik\theta_1} \tilde{\varepsilon}_{n,k} \xrightarrow{\mathcal{D}} \begin{cases} Y(t), & \text{if } \theta_1 = \theta, \\ M_1(t), & \text{if } \theta_1 \neq \theta \end{cases}$$

*in $D([0,1] \to \mathbb{C})$.*

Remark 3. Using the notation

$$\tilde{\boldsymbol{\varepsilon}}_n = (\tilde{\varepsilon}_{n,1}, \ldots, \tilde{\varepsilon}_{n,n})'$$

we can write the ARMA(1,1) model (24) in the form

$$(I - \beta_n B)\tilde{\boldsymbol{\varepsilon}}_n = (I - \beta B)\boldsymbol{\varepsilon}_n,$$

that is,

$$\tilde{\boldsymbol{\varepsilon}}_n = \frac{I - \beta B}{I - \beta_n B} \boldsymbol{\varepsilon}_n = (I - \beta B)\mathbf{X}_n.$$

Corollary 2 can be interpreted in the following way: the above ARMA(1,1) process after a 'wrong' rotation have similar property as the original $\{\varepsilon_{n,k}\}$, $k = 1, \ldots, n$, sequence, while after the 'appropriate' rotation it is the solution of a stochastic differential equation governed by the process $M$.

Proof. If $\theta_1 = \theta$ then using the commutation relation (15) we obtain

$$\begin{aligned} \widetilde{M}_{n,\theta}(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (T_\theta \tilde{\boldsymbol{\varepsilon}}_n)_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (T_\theta (I - e^{i\theta} B)\mathbf{X}_n)_k \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} ((I - B) T_\theta \mathbf{X}_n)_k = \frac{1}{\sqrt{n}} T_\theta X_{n,[nt]} = Y_{n,\theta}(t), \end{aligned}$$

and Theorem 1 gives the result.

If $\theta_1 \neq \theta$ then using

$$\tilde{\varepsilon}_{n,k} = ((I - \beta B)\mathbf{X}_n)_k = X_{n,k} - e^{i\theta} X_{n,k-1} = \varepsilon_{n,k} + e^{i\theta}(e^{h_n/n} - 1) X_{n,k-1}$$

we get

$$\begin{aligned} \widetilde{M}_{n,\theta_1}(t) &= M_{n,\theta_1}(t) + \frac{1}{\sqrt{n}} e^{i\theta}(e^{h_n/n} - 1) \sum_{k=1}^{[nt]} e^{-ik\theta_1} X_{n,k-1} \\ &= M_{n,\theta_1}(t) + n(e^{h_n/n} - 1) e^{i(\theta-\theta_1)} U_{n,\theta_1}(t). \end{aligned}$$

Consequently, Theorem 1, Corollary 2 and Lemma 1 imply the statements. ∎

## 5. COMPLEX-VALUED AR($p$) PROCESSES

For every $n = 1, 2, \ldots$ consider the complex-valued AR($p$) model

$$\begin{cases} X_{n,k} = \beta_{1,n} X_{n,k-1} + \ldots + \beta_{p,n} X_{n,k-p} + \varepsilon_{n,k}, & k = 1, 2, \ldots, n \\ X_{n,0} = X_{n,-1} = \ldots = X_{n,1-p} = 0, \end{cases} \tag{25}$$

where $\{\varepsilon_{n,k}\}$ is an array of complex random variables and $\beta_{1,n}, \ldots, \beta_{p,n}$ are complex numbers. We suppose that the characteristic polynomial of the model has the form

$$\varphi_n(z) = 1 - \beta_{1,n} z - \ldots - \beta_{p,n} z^p = \prod_{j=1}^{q} \prod_{k=1}^{r_j} (1 - a_{j,k,n} z),$$

where $a_{j,k,n} = e^{h_{j,k,n}/n + i\theta_j}$, $h_{j,k,n} \in \mathbb{C}$, $j = 1, \ldots, q$, $k = 1, \ldots, r_j$, $n \geq 1$, such that $h_{j,k,n} \to h_{j,k}$, as $n \to \infty$, and $\theta_1, \ldots, \theta_q \in (-\pi, \pi]$ are different numbers. Clearly

$$\varphi_n(z) \to \varphi(z) = \prod_{j=1}^{q} (1 - a_j z)^{r_j},$$

where $a_j = e^{i\theta_j}$, $j = 1, \ldots, q$. Obviously $p = \sum_{j=1}^{q} r_j$.

For $j = 1, \ldots, q$, $k = 0, 1, \ldots, r_j - 1$ consider the random step functions

$$Y_{j,n}^{(k)}(t) = \frac{1}{n^{r_j - k - 1/2}} \left( T_{\theta_j} (I - a_j B)^{k - r_j} \varphi(B) \mathbf{X}_n \right)_{[nt]}, \qquad t \in [0, 1].$$

THEOREM 3. *Suppose that* $(M_{n,\theta_1}, \ldots, M_{n,\theta_q}) \xrightarrow{\mathcal{D}} (M_1, \ldots, M_q)$ *in* $D([0,1] \to \mathbb{C}^q)$, *where* $(M_1(t), \ldots, M_q(t))$, $t \in [0, 1]$, *is a continuous semimartingale with values in* $\mathbb{C}^q$.
*Then*

$$Y_{j,n}^{(k)} \xrightarrow{\mathcal{D}} Y_j^{(k)}, \qquad \text{as} \quad n \to \infty, \tag{26}$$

*jointly for* $j = 1, \ldots, q$, $k = 0, 1, \ldots, r_j - 1$, *in* $D([0,1] \to \mathbb{C}^p)$, *where*

$$(Y_j^{(k)}(t); j = 1, \ldots, q, \, k = 0, 1, \ldots, r_j - 1), \quad t \in [0, 1],$$

*is the Ornstein-Uhlenbeck process with values in* $\mathbb{C}^p$ *defined as the solution of the system of stochastic differential equations*

$$\begin{cases} dY_j^{(r_j - 1)}(t) = \left( c_{j,1} Y_j^{(r_j - 1)}(t) + \ldots + c_{j,r_j} Y_j^{(0)}(t) \right) dt + dM_j(t), \\ dY_j^{(k)}(t) = Y_j^{(k+1)}(t)\, dt, \qquad k = 0, 1, \ldots, r_j - 2 \\ Y_j^{(0)}(0) = Y_j^{(1)}(0) = \ldots = Y_j^{(r_j - 1)}(0) = 0, \end{cases} \tag{27}$$

*for* $j = 1, \ldots, q$, *where the complex numbers* $c_{j,1}, \ldots, c_{j,r_j}$, $j = 1, \ldots, q$ *are given by*

$$1 - c_{j,1} z - \ldots - c_{j,r_j} z^{r_j} = \prod_{k=1}^{r_j} (1 - h_{j,k} z), \qquad z \in \mathbb{C}.$$

PROOF. For $j = 1, \ldots, q$, $k = 0, 1, \ldots, r_j$, $n \geq 1$ consider the random step functions

$$U_{j,k,n}(t) = \frac{1}{\sqrt{n}} \sum\nolimits_{\ell=1}^{[nt]} \left( T_{\theta_j} \prod_{m=1}^{k} \left( \frac{I - a_j B}{I - a_{j,m,n} B} \right) \prod_{\substack{1 \leq u \leq q \\ u \neq j}} \prod_{v=1}^{r_u} \left( \frac{I - a_u B}{I - a_{u,v,n} B} \right) \boldsymbol{\varepsilon}_n \right)_\ell,$$

where

$$\prod_{m=1}^{0} \left( \frac{I - a_j B}{I - a_{j,m,n} B} \right) := I.$$

Repeatedly using Corollary 2 we conclude that there exist measurable mappings
$\Phi, \Phi_n : D([0,1] \to \mathbb{C}^q) \to D([0,1] \to \mathbb{C}^{2q+p})$, $n = 1, 2, \ldots$, such that

$$\left( M_{n,\theta_1}, \ldots, M_{n,\theta_q}, U_{1,0,n}, \ldots, U_{1,r_1,n}, \ldots, U_{q,0,n}, \ldots, U_{q,r_q,n} \right) = \Phi_n(M_{n,\theta_1}, \ldots, M_{n,\theta_q})$$

and $\Phi_n \rightsquigarrow \Phi$, where

$$\Phi_n(M_1,\ldots,M_q) = (M_1,\ldots,M_q,U_{1,0},\ldots,U_{1,r_1},\ldots,U_{q,0},\ldots,U_{q,r_q}),$$

and $U_{j,k}(t)$, $t \in [0,1]$, $j = 1,\ldots,q$, $k = 0,1,\ldots,r_j$ is given by

$$\begin{cases} dU_{j,k}(t) = h_{j,k}U_{j,k}(t)dt + dU_{j,k-1}(t), & j = 1,\ldots,q, \ \ k = 1,\ldots,r_j \\ U_{j,0}(t) = M_j(t), & j = 1,\ldots,q. \end{cases}$$

Particularly,

$$U_{j,k,n} \xrightarrow{\mathcal{D}} U_{j,k}, \qquad \text{as} \quad n \to \infty, \tag{28}$$

jointly for $j = 1,\ldots,q$, $k = 0,1,\ldots,r_j$, in $D([0,1] \to \mathbb{C}^{q+p})$. Using

$$U_{j,k-1}(t) = U_{j,k}(t) - h_{j,k}\int_0^t U_{j,k}(s)\,ds$$

for $j = 1,\ldots,q$, $k = 1,\ldots,r_j$, $t \in [0,1]$, we obtain

$$\begin{aligned} dU_{j,r_j}(t) \quad = \quad & \Big(c_{j,1}U_{j,r_j}(t) + c_{j,2}\int_0^t U_{j,r_j}(s)\,ds + \\ & + \ldots + c_{j,r_j}\int_0^t\int_0^{s_1}\ldots\int_0^{s_{r_j-2}} U_{j,r_j}(s_{r_j-1})\,ds_1\ldots ds_{r_j-1}\Big)\,dt + dM_j(t). \end{aligned}$$

On the other hand, by the help of the commutation relation (15) we obtain for $j = 1,\ldots,q$

$$Y_{j,n}^{(r_j-1)}(t) = U_{j,r_j,n}(t), \qquad t \in [0,1], \tag{29}$$

since

$$\begin{aligned} Y_{j,n}^{(r_j-1)}(t) &= \frac{1}{\sqrt{n}}\left(T_{\theta_j}(I - a_jB)^{-1}\varphi(B)\mathbf{X}_n\right)_{[nt]} \\ &= \frac{1}{\sqrt{n}}\sum_{\ell=1}^{[nt]}(I - B)\left(T_{\theta_j}(I - a_jB)^{-1}\varphi(B)\mathbf{X}_n\right)_\ell \\ &= \frac{1}{\sqrt{n}}\sum_{\ell=1}^{[nt]}\left(T_{\theta_j}\varphi(B)\mathbf{X}_n\right)_\ell = U_{j,r_j,n}(t). \end{aligned}$$

Moreover, we have for $j = 1,\ldots,q$, $k = 0,1,\ldots,r_j - 2$

$$Y_{j,n}^{(k)}(t) = \int_0^{[nt]/n} Y_{j,n}^{(k+1)}(s)\,ds, \tag{30}$$

since by induction

$$\begin{aligned} Y_{j,n}^{(k)}(t) &= \frac{1}{n^{r_j-k-1/2}}\left(T_{\theta_j}(I - a_jB)^{k-r_j}\varphi(B)\mathbf{X}_n\right)_{[nt]} \\ &= \frac{1}{n^{r_j-k-1/2}}\sum_{\ell=1}^{[nt]}\left((I - B)T_{\theta_j}(I - a_jB)^{k-r_j}\varphi(B)\mathbf{X}_n\right)_\ell \\ &= \frac{1}{n^{r_j-k-1/2}}\sum_{\ell=1}^{[nt]}\left(T_{\theta_j}(I - a_jB)^{k-r_j+1}\varphi(B)\mathbf{X}_n\right)_\ell = \int_0^{[nt]/n} Y_{j,n}^{(k+1)}(s)\,ds. \end{aligned}$$

The convergence (28) implies

$$(U_{1,r_1,n},\ldots,U_{q,r_q,n}) \xrightarrow{\mathcal{D}} (U_{1,r_1},\ldots,U_{q,r_q}), \qquad \text{as} \quad n \to \infty,$$

from which together with (29) and (30) we conclude (26). ∎

For $j, \ell \in \{1,\ldots,q\}$, $j \neq \ell$, $k = 1,\ldots,r_j - 1$ consider the random step functions

$$Y_{j,\ell,n}^{(k)}(t) = \frac{1}{n^{r_j-k+1/2}}\left(T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_jB)^{k-r_j}\varphi(B)\mathbf{X}_n\right)_{[nt]}, \qquad t \in [0,1].$$

Theorem 4. *Suppose that* $(M_{n,\theta_1},\ldots,M_{n,\theta_q}) \xrightarrow{\mathcal{D}} (M_1,\ldots,M_q)$ *in* $D([0,1] \to \mathbb{C}^q)$, *where* $(M_1(t),\ldots,M_q(t))$, $t \in [0,1]$, *is a continuous semimartingale with values in* $\mathbb{C}^q$.

*Then*
$$Y_{j,\ell,n}^{(k)} \xrightarrow{\mathcal{D}} 0, \qquad \text{as} \quad n \to \infty, \tag{31}$$
*jointly for* $j, \ell \in \{1, \ldots, q\}$, $j \neq \ell$, $k = 1, \ldots, r_j - 1$.

Proof. For $k = r_j - 1$ we have

$$\begin{aligned} Y_{j,\ell,n}^{(r_j-1)}(t) &= \frac{1}{n^{3/2}} \big(T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{-1}\varphi(B)\mathbf{X}_n\big)_{[nt]} \\ &= \frac{1}{n^{3/2}} \sum\nolimits_{m=1}^{[nt]} \big((I-B)T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{-1}\varphi(B)\mathbf{X}_n\big)_m \\ &= \frac{1}{n^{3/2}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_\ell}(I - a_j B)^{-1}\varphi(B)\mathbf{X}_n\big)_m , \end{aligned}$$

and we can apply Theorem 2 (using $\varphi(B)\mathbf{X}_n$ instead of $\boldsymbol{\varepsilon}_n$), since Theorem 3 implies

$$\frac{1}{\sqrt{n}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_j}\varphi(B)\mathbf{X}_n\big)_m = Y_{j,n}^{(r_j-1)}(t) \xrightarrow{\mathcal{D}} Y_j^{(r_j-1)}$$

and

$$\frac{1}{\sqrt{n}} \sum\nolimits_{m=1}^{[nt]} (T_{\theta_\ell}\varphi(B)\mathbf{X}_n)_m = Y_{\ell,n}^{(r_\ell-1)}(t) \xrightarrow{\mathcal{D}} Y_\ell^{(r_\ell-1)}.$$

For $k = r_j - 2, r_j - 3, \ldots, 1$ we have

$$\begin{aligned} Y_{j,\ell,n}^{(k)}(t) &= \frac{1}{n^{r_j-k+1/2}} \big(T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{k-r_j}\varphi(B)\mathbf{X}_n\big)_{[nt]} \\ &= \frac{1}{n^{r_j-k+1/2}} \sum\nolimits_{m=1}^{[nt]} \big((I-B)T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{k-r_j}\varphi(B)\mathbf{X}_n\big)_m \\ &= \frac{1}{n^{3/2}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_\ell}(I - a_j B)^{-1} n^{k-r_j+1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m , \end{aligned}$$

and we can apply again Theorem 2 (using $n^{k-r_j+1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n$ instead of $\boldsymbol{\varepsilon}_n$), since

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_j} n^{k-r_j+1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_j}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \sum\nolimits_{m=1}^{[nt]} \big((I-B)T_{\theta_j}(I - a_j B)^{k-r_j}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \big(T_{\theta_j}(I - a_j B)^{k-r_j}\varphi(B)\mathbf{X}_n\big)_{[nt]} = Y_{j,n}^{(r_j-k)} \xrightarrow{\mathcal{D}} Y_j^{(r_j-k)} \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_\ell} n^{k-r_j+1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \sum\nolimits_{m=1}^{[nt]} \big(T_{\theta_\ell}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \sum\nolimits_{m=1}^{[nt]} \big((I-B)T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_m \\ &\quad = \frac{1}{n^{r_j-k-1/2}} \big(T_{\theta_\ell}(I - a_\ell B)^{-1}(I - a_j B)^{k-r_j+1}\varphi(B)\mathbf{X}_n\big)_{[nt]} = Y_{j,\ell,n}^{(k+1)}, \end{aligned}$$

hence we can use induction for $k = r_j - 2, r_j - 3, \ldots, 1$. ■

# 6. CONVERGENCE OF LEAST-SQUARES ESTIMATORS

The least-squares estimator of the parameter $\boldsymbol{\beta}_n = (\beta_{1,n}, \ldots, \beta_{p,n})'$ of the model (25) can be obtained by minimizing the sum of squares

$$\sum_{k=1}^{n} |X_{nk} - \beta_{1,n}X_{n,k-1} - \ldots - \beta_{p,n}X_{n,k-p}|^2 = \|\mathbf{X}_n - \beta_{1,n}B\mathbf{X}_n - \ldots - \beta_{p,n}B^p\mathbf{X}_n\|^2.$$

It is known that the LSE $\widehat{\boldsymbol{\beta}}_n = (\widehat{\beta}_{1,n}, \dots, \widehat{\beta}_{p,n})'$ is the unique solution of the Yule-Walker equations

$$\langle \mathbf{X}_n - \widehat{\beta}_{1,n} B\mathbf{X}_n - \dots - \widehat{\beta}_{p,n} B^p \mathbf{X}_n, B^\ell \mathbf{X}_n \rangle = 0, \qquad \ell = 1, \dots, p, \tag{32}$$

and $\widehat{\boldsymbol{\beta}}_n$ can be written in the form (2).

LEMMA 4. *The least-squares estimators* $\widehat{c}_{j,k,n}$, $j = 1, \dots, q$, $k = 1, \dots, r_j$ *are the unique solutions of the system of linear equations*

$$\sum_{j=1}^{q} \sum_{k=1}^{r_j} \langle q_{j,k}(B)\mathbf{X}_n, q_{u,v}(B)\mathbf{X}_n \rangle \frac{\widehat{c}_{j,k,n} - c_{j,k,n}}{n^{k+v}} = \frac{1}{n^v} \langle \boldsymbol{\varepsilon}_n, q_{u,v}(B)\mathbf{X}_n \rangle,$$

*for* $u = 1, \dots, q$, $v = 1, \dots, r_u$, *where*

$$q_{j,k}(z) = a_j z (1 - a_j z)^{-k} \varphi(z), \qquad z \in \mathbb{C}.$$

PROOF. The existence of the unique LSE $\widehat{c}_{j,k,n}$ follows from the one-to-one correspondence between the parameters $c_{j,k,n}$, $j = 1, \dots, q$, $k = 1, \dots, r_j$ and $\beta_{n,\ell}$, $\ell = 1, \dots, p$.

Introduce the notation

$$\widehat{\varphi}_n(z) = 1 - \widehat{\beta}_{1,n} z - \dots - \widehat{\beta}_{p,n} z^p, \qquad z \in \mathbb{C}, \quad n \geq 1.$$

The model (25) can be written as $\varphi_n(B)\mathbf{X}_n = \boldsymbol{\varepsilon}_n$, while the Yule-Walker equations (32) has the form $\langle \widehat{\varphi}_n(B)\mathbf{X}_n, B^\ell \mathbf{X}_n \rangle = 0$, $\ell = 1, \dots, p$. Consequently, the LSE $\widehat{\boldsymbol{\beta}}_n$ can be obtained from the system of equations

$$\langle (\varphi_n(B) - \widehat{\varphi}_n(B))\mathbf{X}_n, B^\ell \mathbf{X}_n \rangle = \langle \boldsymbol{\varepsilon}_n, B^\ell \mathbf{X}_n \rangle, \qquad \ell = 1, \dots, p,$$

which is equivalent to

$$\langle (\varphi_n(B) - \widehat{\varphi}_n(B))\mathbf{X}_n, q_{u,v}(B)\mathbf{X}_n \rangle = \langle \boldsymbol{\varepsilon}_n, q_{u,v}(B)\mathbf{X}_n \rangle,$$

$$u = 1, \dots, q, \qquad v = 1, \dots, r_u,$$

since the polynomials $q_{u,v}$, $u = 1, \dots, q$, $v = 1, \dots, r_u$, are linearly independent. From the definition of the parameters $c_{j,k,n}$, $j = 1, \dots, q$, $k = 1, \dots, r_j$ it follows

$$\varphi(z) - \varphi_n(z) = \sum_{j=1}^{q} \sum_{k=1}^{r_j} \frac{c_{j,k,n}}{n^k} q_{j,k}(z), \qquad z \in \mathbb{C}. \tag{33}$$

Obviously we have

$$\varphi(z) - \widehat{\varphi}_n(z) = \sum_{j=1}^{q} \sum_{k=1}^{r_j} \frac{\widehat{c}_{j,k,n}}{n^k} q_{j,k}(z), \qquad z \in \mathbb{C}. \tag{34}$$

Consequently we conclude the statement. ■

From now on we shall put the following condition on the random disturbances $\{\varepsilon_{n,k}\}$.

(C) $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$ is a triangular array of real square integrable martingale differences with respect to the filtrations $(\mathcal{F}_{nk})_{k=0,1,\dots,n;n\geq 1}$ such that for all $t \in [0,1]$

$$\frac{1}{nt} \sum_{k=1}^{[nt]} \mathbb{E}(\varepsilon_{nk}^2 | \mathcal{F}_{n,k-1}) \xrightarrow{\mathbb{P}} 1, \qquad \text{as} \quad n \to \infty$$

and

$$\forall \alpha > 0 \qquad \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\varepsilon_{nk}^2 \chi_{\{|\varepsilon_{nk}| > \alpha\sqrt{n}\}} | \mathcal{F}_{n,k-1}) \xrightarrow{\mathbb{P}} 0, \qquad \text{as} \quad n \to \infty.$$

We shall say that $W(t)$, $t \in [0,1]$, is a *standard complex-valued Wiener process*, if $W(t) = W_1(t) + iW_2(t)$, where $(\sqrt{2}W_1(t), \sqrt{2}W_2(t))$, $t \in [0,1]$, is a standard Wiener process with values in $\mathbb{R}^2$.

LEMMA 5. *Suppose that the array* $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, *satisfies the condition (C). Then*

$$(M_{n,\theta_1}, \dots, M_{n,\theta_q}) \xrightarrow{\mathcal{D}} (W_1, \dots, W_q)$$

*in* $D([0,1]\to\mathbb{C}^q)$*, where* $W_j(t),\ t\in[0,1],\ j=1,\dots,q,$ *are standard Wiener processes, real-valued for* $\theta_j=0$ *or* $\theta_j=\pi$*, and complex-valued otherwise. Moreover,* $W_j$ *and* $W_k$ *are independent if* $\theta_j\neq-\theta_k$*, and* $W_j=\overline{W_k}$ *if* $\theta_j=-\theta_k$*.*

Proof. The statement follows from a version of the functional central limit theorem on the space $D([0,1]\to\mathbb{R}^p)$ (Theorem 7.11 in Liptser and Shiryayev (1989)), remarking the facts that for $\theta\neq0,\ \theta\neq\pi$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos^2 k\theta=\frac{1}{2},\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin^2 k\theta=\frac{1}{2},\qquad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin k\theta\cos k\theta=0,$$

and for $\theta,\alpha\in(-\pi,\pi],\ \theta\neq\pm\alpha$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos k\theta\cos k\alpha=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin k\theta\sin k\alpha=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin k\theta\cos k\alpha=0.$$ ■

Lemma 6. *Suppose that the array* $\varepsilon_{n,k},\ k=1,\dots,n,\ n\geq1,$ *satisfies the condition (C). Then*

$$\frac{1}{n^{k+v}}\langle q_{j,k}(B)\mathbf{X}_n,q_{u,v}(B)\mathbf{X}_n\rangle \xrightarrow{\mathcal{D}} \begin{cases}\int_0^1 Y_j^{(r_j-k)}(t)\overline{Y_j^{(r_j-v)}(t)}dt, & \textit{if } j=u\\ 0, & \textit{if } j\neq u\end{cases}$$

$$\frac{1}{n^{v}}\langle \boldsymbol{\varepsilon}_n,q_{u,v}(B)\mathbf{X}_n\rangle \xrightarrow{\mathcal{D}} \int_0^1\overline{Y_u^{(r_u-v)}(t)}\,dW_u(t),$$

*jointly for* $j=1,\dots,q,\ k=1,\dots,r_j,\ u=1,\dots,q,\ v=1,\dots,r_u$*.*

Proof. If $j=u$ then

$$\begin{aligned}\frac{1}{n^{k+v}}\langle q_{j,k}(B)\mathbf{X}_n,q_{j,v}(B)\mathbf{X}_n\rangle &= \frac{1}{n^{k+v}}\langle a_jB(I-a_jB)^{-k}\varphi(B)\mathbf{X}_n,a_jB(I-a_jB)^{-v}\varphi(B)\mathbf{X}_n\rangle\\ &= \frac{1}{n}\sum_{\ell=1}^{n}Y_{j,n}^{(r_j-k)}((\ell-1)/n)\overline{Y_{j,n}^{(r_j-v)}((\ell-1)/n)}\\ &= \int_0^1 Y_{j,n}^{(r_j-k)}(t)\overline{Y_{j,n}^{(r_j-v)}(t)}\,dt,\end{aligned}$$

and the convergence follows from Theorem 3.

If $j\neq u$ then using the decomposition

$$q_{u,v}(z)=(1-a_uz)^{-v}\varphi(z)-(1-a_uz)^{-v+1}\varphi(z)$$

we obtain

$$\begin{aligned}&\frac{1}{n^{k+v}}\langle q_{j,k}(B)\mathbf{X}_n,q_{u,v}(B)\mathbf{X}_n\rangle=\\ &=\frac{1}{n^{k+v}}\left\langle T_{\theta_j}\frac{a_jB\varphi(B)}{(I-a_jB)^k}\mathbf{X}_n,T_{\theta_j}\left(\frac{\varphi(B)}{(1-a_uz)^v}-\frac{\varphi(B)}{(1-a_jz)^{v-1}}\right)\mathbf{X}_n\right\rangle\\ &=\left\langle BT_{\theta_j}\frac{\varphi(B)}{n^{k-1/2}(I-a_jB)^k}\mathbf{X}_n,T_{\theta_j}\frac{\varphi(B)}{n^{v+1/2}(I-a_uB)^v}\mathbf{X}_n\right\rangle\\ &\quad-\frac{1}{n}\left\langle BT_{\theta_j}\frac{\varphi(B)}{n^{k-1/2}(I-a_jB)^k}\mathbf{X}_n,T_{\theta_j}\frac{\varphi(B)}{n^{v-1/2}(I-a_uB)^{v-1}}\mathbf{X}_n\right\rangle\\ &=\left\langle BT_{\theta_j}\frac{\varphi(B)}{n^{k-1/2}(I-a_jB)^k}\mathbf{X}_n,(I-B)T_{\theta_j}\frac{\varphi(B)}{n^{v+1/2}(I-a_jB)(I-a_uB)^v}\mathbf{X}_n\right\rangle\\ &\quad-\frac{1}{n}\left\langle BT_{\theta_j}\frac{\varphi(B)}{n^{k-1/2}(I-a_jB)^k}\mathbf{X}_n,(I-B)T_{\theta_j}\frac{\varphi(B)}{n^{v-1/2}(I-a_jB)(I-a_uB)^{v-1}}\mathbf{X}_n\right\rangle\\ &=\int_0^1 Y_{j,n}^{(r_j-k)}(t)\,d\overline{Y_{u,j,n}^{(r_u-v)}(t)}-\frac{1}{n}\int_0^1 Y_{j,n}^{(r_j-k)}(t)\,d\overline{Y_{u,j,n}^{(r_u-v+1)}(t)},\end{aligned}$$

and the convergence follows from Theorems 3 and 4 combined with Proposition 6 in Jeganathan (1991).

Finally,

$$\frac{1}{n^v}\langle\boldsymbol{\varepsilon}_n,q_{u,v}(B)\mathbf{X}_n\rangle = \frac{1}{n^v}\sum_{\ell=1}^{n}\varepsilon_{n\ell}\overline{(a_uB(I-a_uB)^{-v}\varphi(B)\mathbf{X}_n)_\ell}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{n} n^{-1/2}(T_{\theta_u}\boldsymbol{\varepsilon}_n)_\ell \overline{n^{-v+1/2}\,(BT_{\theta_u}(I-a_uB)^{-v}\varphi(B)\mathbf{X}_n)_\ell} \\
&= \sum_{\ell=1}^{n} \Big(M_n^{(\theta_u)}(\ell/n) - M_n^{(\theta_u)}((\ell-1)/n)\Big)\,\overline{Y_{u,n}^{(r_u-v)}((\ell-1)/n)} \\
&= \int_0^1 \overline{Y_{u,n}^{(r_u-v)}(t)}\,dM_n^{(\theta_u)}(t),
\end{aligned}
$$

and the convergence follows again from Theorem 3 combined with Proposition 6 in Jeganathan (1991). ∎

THEOREM 5. *Suppose that the array* $\varepsilon_{n,k}$, $k=1,\dots,n$, $n\ge 1$, *satisfies the condition (C). Then*

$$\widehat{c}_{j,k,n} \xrightarrow{\mathcal{D}} \widehat{c}_{j,k}, \qquad \text{as} \quad n\to\infty,$$

*jointly for* $j=1,\dots,q$, $k=1,\dots,r_j$, *where* $\widehat{\mathbf{c}}_j=(\widehat{c}_{j,1},\dots,\widehat{c}_{j,r_j})'$ *is given by*

$$\widehat{\mathbf{c}}_j = S_j^{-1}\begin{pmatrix} \int_0^1 \overline{Y_j^{(r_j-1)}(t)}\,dY_j^{(r_j-1)}(t) \\ \vdots \\ \int_0^1 \overline{Y_j^{(0)}(t)}\,dY_j^{(r_j-1)}(t)\end{pmatrix}, \tag{35}$$

*where*

$$S_j = \begin{pmatrix} \int_0^1 |Y_j^{(r_j-1)}(t)|^2\,dt & \dots & \int_0^1 \overline{Y_j^{(r_j-1)}(t)}Y_j^{(0)}(t)\,dt \\ \vdots & \ddots & \vdots \\ \int_0^1 \overline{Y_j^{(0)}(t)}Y_j^{(r_j-1)}(t)\,dt & \dots & \int_0^1 |Y_j^{(0)}(t)|^2\,dt \end{pmatrix},$$

*and the processes* $Y_j^{(k)}$, $j=1,\dots,q$, $k=1,\dots,r_j$, *are given by (27).*

PROOF. Using Itô's formula we can write

$$\widehat{\mathbf{c}}_j = \mathbf{c}_j + S_j^{-1}\begin{pmatrix} \int_0^1 \overline{Y_j^{(r_j-1)}(t)}\,dW_j(t) \\ \vdots \\ \int_0^1 \overline{Y_j^{(0)}(t)}\,dW_j(t)\end{pmatrix},$$

where $\mathbf{c}_j=(c_{j,1},\dots,c_{j,r_j})'$.

From Lemma 4, Corollary 1 and Lemma 6 we can derive

$$\widehat{c}_{j,k,n} \xrightarrow{\mathcal{D}} \widehat{c}_{j,k}, \qquad \text{as} \quad n\to\infty,$$

jointly for $j=1,\dots,q$, $k=1,\dots,r_j$. ∎

REMARK 5. It is known that $\widehat{\mathbf{c}}_j$ is the MLE of $\mathbf{c}_j=(c_{j,1},\dots,c_{j,r_j})'$ in the complex-valued continuous time $r_j$-order autoregressive model

$$\begin{cases} dY_j^{(r_j-1)}(t) = \big(c_{j,1}Y_j^{(r_j-1)}(t)+\dots+c_{j,r_j}Y_j^{(0)}(t)\big)\,dt + dW_j(t), \\ dY_j^{(k)}(t) = Y_j^{(k+1)}(t)\,dt, \qquad k=0,1,\dots,r_j-2 \\ Y_j^{(0)}(0)=Y_j^{(1)}(0)=\dots=Y_j^{(r_j-1)}(0)=0, \end{cases}$$

where $W_j(t)$, $t\in[0,1]$, is a standard complex-valued Wiener process (see, for example, Arató [3]).

# 7. APPLICATIONS FOR REAL-VALUED AR($p$) PROCESSES

Consider now for every $n=1,2,\dots$ the real-valued AR($p$) model

$$\begin{cases} X_{n,k} = \beta_{1,n}X_{n,k-1}+\dots+\beta_{p,n}X_{n,k-p}+\varepsilon_{n,k}, \quad k=1,2,\dots,n \\ X_{n,0}=X_{n,-1}=\dots=X_{n,1-p}=0, \end{cases}$$

where $\{\varepsilon_{n,k}\}$ is an array of real random variables and $\beta_{1,n}, \dots, \beta_{p,n}$ are real numbers. For the sake of simplicity we suppose that the characteristic polynomial of the limit unstable model has all roots on the unit circle. Then the characteristic polynomial has the form

$$\varphi_n(z) = \prod_{k=1}^{r_1}(1 - e^{h_{1,k,n}/n}z)\prod_{k=1}^{r_2}(1 + e^{h_{2,k,n}/n}z)\prod_{j=3}^{\ell}\prod_{k=1}^{r_j}\Big((1 - e^{h_{j,k,n}/n+i\theta_j}z)(1 - e^{\overline{h}_{j,k,n}/n-i\theta_j}z)\Big),$$

where $r_1, \dots, r_\ell$ are non-negative integers, $r_1 + r_2 + 2(r_3 + \ldots + r_\ell) = p$, $h_{j,k,n} \in \mathbb{C}$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$, $n \geq 1$, such that $h_{j,k,n} \to h_{j,k}$, as $n \to \infty$, and

$$\prod_{k=1}^{r_1}(1 - e^{h_{1,k,n}/n}z), \qquad \prod_{k=1}^{r_2}(1 + e^{h_{2,k,n}/n}z)$$

are polynomials with real coefficients, and $\theta_j \in (0, \pi)$, $j = 3, \dots, \ell$, are pairwise different. We remark that for the complex conjugate pairs of roots we had to put complex conjugate pairs of parameters $h_{j,k,n}$ and $\overline{h}_{j,k,n}$, $j = 3, \dots, \ell$, $k = 1, \dots, r_j$, in order to assure that the polynomial $\varphi$ has real coefficients. Obviously it implies that in the other two parametrizations we have again complex conjugate pairs $d_{j,k,n}$, $\overline{d}_{j,k,n}$ and $c_{j,k,n}$, $\overline{c}_{j,k,n}$, $j = 3, \dots, \ell$, $k = 1, \dots, r_j$. We shall write $\theta_1 = 0$, $\theta_2 = \pi$. Lemma 5 has the following obvious corollary.

Corollary 3. *Suppose that the array* $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, *satisfies the condition (C). Then*

$$(M_{n,\theta_1}, \dots, M_{n,\theta_\ell}) \xrightarrow{\mathcal{D}} (W_1, \dots, W_\ell)$$

*in* $D([0,1] \to \mathbb{C}^\ell)$, *where* $W_j(t)$, $t \in [0,1]$, $j = 1, \dots, \ell$, *are independent standard Wiener processes, real-valued for* $j = 1, 2$, *and complex-valued for* $j = 3, \dots, \ell$.

Theorem 5 has the following corollary.

Corollary 4. *Suppose that the array* $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, *satisfies the condition (C). Then*

$$\widehat{c}_{j,k,n} \xrightarrow{\mathcal{D}} \widehat{c}_{j,k}, \qquad \text{as} \quad n \to \infty,$$

*jointly for* $j = 1, \dots, \ell$, $k = 1, \dots, r_j$, *and for all* $z \in \mathbb{C}$

$$\prod_{k=1}^{r_j}(1 - \widehat{h}_{j,k,n}z) \xrightarrow{\mathcal{D}} 1 - \widehat{c}_{j,1}z - \ldots - \widehat{c}_{j,r_j}z^{r_j}, \qquad \text{as} \quad n \to \infty,$$

*jointly for* $j = 1, \dots, \ell$, *where* $\widehat{\mathbf{c}}_j = (\widehat{c}_{j,1}, \dots, \widehat{c}_{j,r_j})'$ *is given by (35). In other words,* $\widehat{c}_{j,k}$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$ *are the MLE of the parameters* $c_{j,k}$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$, *in the continuous time model (27). Loosely speaking,* $\widehat{h}_{j,k,n} \xrightarrow{\mathcal{D}} \widehat{h}_{j,k}$.

Remark 5. From Corollary 4 we can derive convergence theorem for the LSE of the coefficients $\beta_{1,n}, \dots, \beta_{p,n}$ (see Theorem 1 in Jeganathan [10]; the stable case, i. e. when $h_{j,k,n} = 0$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$, is treated in Chan and Wei [7]).

From (33) we have

$$\begin{aligned}
&(\beta_{1,n} - \beta_1)z + \ldots + (\beta_{p,n} - \beta_p)z^p \\
&\quad = \sum_{k=1}^{r_1}\frac{c_{1,k,n}}{n^k}\frac{z\varphi(z)}{(1-z)^k} - \sum_{k=1}^{r_2}\frac{c_{2,k,n}}{n^k}\frac{z\varphi(z)}{(1+z)^k} \\
&\qquad + \sum_{j=3}^{\ell}\sum_{k=1}^{r_j}\left(\frac{c_{j,k,n}}{n^k}\frac{e^{i\theta_j}z\varphi(z)}{(1-e^{i\theta_j}z)^k} + \frac{\overline{c}_{j,k,n}}{n^k}\frac{e^{-i\theta_j}z\varphi(z)}{(1-e^{-i\theta_j}z)^k}\right),
\end{aligned}$$

for $z \in \mathbb{C}$. Obviously for all $n \geq 1$ there is an invertible real $p \times p$ matrix $A_n$ such that

$$\boldsymbol{\beta}_n - \boldsymbol{\beta} = A_n \mathbf{C}_n,$$

where

$$\mathbf{C}_n = (\mathbf{c}_{1,n}', \dots, \mathbf{c}_{\ell,n}')',$$

and

$$\mathbf{c}_{j,n} = \begin{cases} (c_{j,1,n}, \dots, c_{j,r_j,n})' & \text{for } j = 1, 2 \\ (\Re(c_{j,1,n}), \Im(c_{j,1,n}), \dots, \Re(c_{j,r_j,n}), \Im(c_{j,r_j,n}))' & \text{for } j = 3, \dots, \ell. \end{cases}$$

Clearly we have also

$$\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta} = A_n \widehat{\mathbf{C}}_n,$$

where $\widehat{\mathbf{C}}_n$ contains the LSE. Using Corollary 1 and 4 we conclude

$$A_n^{-1}(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n) = \widehat{\mathbf{C}}_n - \mathbf{C}_n \stackrel{\mathcal{D}}{\longrightarrow} \widehat{\mathbf{C}} - \mathbf{C} = (\widehat{\mathbf{c}}_1' - \mathbf{c}_1', \dots, \widehat{\mathbf{c}}_\ell' - \mathbf{c}_\ell')',$$

where

$$\mathbf{c}_j = \begin{cases} (c_{j,1}, \dots, c_{j,r_j})' & \text{for } j = 1, 2 \\ (\Re(c_{j,1}), \Im(c_{j,1}), \dots, \Re(c_{j,r_j}), \Im(c_{j,r_j}))' & \text{for } j = 3, \dots, \ell, \end{cases}$$

$\widehat{\mathbf{c}}_j$ contains the MLE, which is given by (35). Corollary 3 implies that now the processes

$$\{(Y_j^{(0)}(t), Y_j^{(1)}(t), \dots, Y_j^{(r_j)}(t)),\, t \in [0,1]\}, \qquad j = 1, \dots, \ell$$

are mutually independent, and real-valued for $j = 1, 2$, complex-valued for $j = 3, \dots, \ell$. Furthermore, $\widehat{c}_{j,k}$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$ are the MLE of the parameters $c_{j,k}$, $j = 1, \dots, \ell$, $k = 1, \dots, r_j$, in the continuous time model (27). We remark that Jeganathan (1991) and Chan and Wei (1988b) used a slightly different normalization of $\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n$, but the results are obviously equivalent.

# 8. EXAMPLES

For illustration first we shall study real-valued AR(2) models near to an unstable model given by

$$\begin{cases} X_{n,k} = \beta_{1,n} X_{n,k-1} + \beta_{2,n} X_{n,k-2} + \varepsilon_{n,k}, \quad k = 1, 2, \dots, n \\ X_{n,0} = X_{n,-1} = 0, \end{cases} \tag{36}$$

where $\{\varepsilon_{n,k}\}$ is an array of real random variables satisfying the condition (C) and $\beta_{1,n}$, $\beta_{2,n}$ are real numbers.

First consider the case when the limit unstable model has complex roots, i.e., its characteristic polynomial is

$$\varphi(z) = (1 - e^{i\theta} z)(1 - e^{-i\theta} z) = 1 - 2z\cos\theta + z^2.$$

Then we have $\beta_1 = 2\cos\theta$ and $\beta_2 = -1$. The characteristic polynomial of the model (36) has the form

$$\varphi_n(z) = (1 - e^{h_n/n + i\theta} z)(1 - e^{\overline{h}_n/n - i\theta} z),$$

where $h_n \in \mathbb{C}$ such that $h_n \to h$, as $n \to \infty$ and $\theta \in (0, \pi)$. Remark, that (13) implies $c = h$. From (33) we have

$$(\beta_{1,n} - \beta_1) z + (\beta_{2,n} - \beta_2) z^2 = \frac{1}{n} \left( c_n e^{i\theta} z (1 - e^{-i\theta} z) + \overline{c}_n e^{-i\theta} z (1 - e^{i\theta} z) \right)$$

for $z \in \mathbb{C}$. Comparing the coefficients of $z$ and $z^2$ we obtain

$$\begin{aligned} n(\beta_{1,n} - \beta_1) &= 2(\Re(c_n)\cos\theta - \Im(c_n)\sin\theta) \\ n(\beta_{2,n} - \beta_2) &= -2\Re(c_n). \end{aligned}$$

Applying Theorem 5 we conclude

$$n\,(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) = \begin{pmatrix} n(\widehat{\beta}_{1,n} - \beta_1) \\ n(\widehat{\beta}_{2,n} - \beta_2) \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{pmatrix} 2(\Re(\widehat{c})\cos\theta - \Im(\widehat{c})\sin\theta) \\ -2\Re(\widehat{c}) \end{pmatrix},$$

where

$$\widehat{c} = \frac{\int_0^1 \overline{Y(t)}\, dY(t)}{\int_0^1 |Y(t)|^2\, dt},$$

and $Y(t)$, $t \in [0,1]$, is the continuous time complex-valued AR(1) process given by

$$dY(t) = hY(t)dt + dW(t), \qquad Y(0) = 0,$$

where $W(t)$, $t \in [0,1]$, is a standard complex-valued Wiener process. Moreover, $\widehat{c}$ can be interpreted as the MLE of the parameter $h$. We have also

$$n\,(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_n) = \begin{pmatrix} n(\widehat{\beta}_{1,n} - \beta_{1,n}) \\ n(\widehat{\beta}_{2,n} - \beta_{2,n}) \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{pmatrix} 2(\Re(\widehat{c}-c)\cos\theta - \Im(\widehat{c}-c)\sin\theta) \\ -2\Re(\widehat{c}-c) \end{pmatrix},$$

where by Itô's formula we can derive

$$\widehat{c} - c = \frac{\int_0^1 \overline{Y(t)}\,dW(t)}{\int_0^1 |Y(t)|^2\,dt}.$$

The above convergence statement can be reformulated as

$$n\,(\widehat{\boldsymbol{\beta}}_n - \boldsymbol{\beta_n}) = \begin{pmatrix} n(\widehat{\beta}_{1,n} - \beta_{1,n}) \\ n(\widehat{\beta}_{2,n} - \beta_{2,n}) \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \frac{2}{s_Y^2} \begin{pmatrix} r_{YW}^{+}\cos\theta - r_{YW}^{-}\sin\theta \\ -r_{YW}^{+} \end{pmatrix},$$

where

$$\begin{aligned} s_Y^2 &= \int_0^1 (Y_1^2(t) + Y_2^2(t))\,dt \\ r_{YW}^{+} &= \int_0^1 (Y_1(t)\,dW_1(t) + Y_2(t)\,dW_2(t)) \\ r_{YW}^{-} &= \int_0^1 (Y_1(t)\,dW_2(t) - Y_2(t)\,dW_1(t)), \end{aligned}$$

$W_1(t)$, $W_2(t)$, $t \in [0,1]$, are independent real-valued standard Wiener processes, and the process $(Y_1(t), Y_2(t))$, $t \in [0,1]$, is given by

$$\begin{pmatrix} dY_1(t) \\ dY_2(t) \end{pmatrix} = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} \begin{pmatrix} Y_1(t)\,dt \\ Y_2(t)\,dt \end{pmatrix} + \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix},$$

with initial values $Y_1(0) = Y_2(0) = 0$, where $\lambda = \Re(h)$ and $\omega = \Im(h)$. We remark that Corollary 3.3.8 in Chan and Wei (1988b) contains convergence of $n(\widehat{\beta}_{2,n} + 1)$ in the stable case, i.e., when $h_n \equiv 0$.

Now consider the case when the limit unstable model has double roots equal to 1, i.e., its characteristic polynomial is

$$\varphi(z) = (1-z)^2 = 1 - 2z + z^2$$

and we have $\beta_1 = 2$ and $\beta_2 = -1$. The characteristic polynomial of the model (36) has the form

$$\varphi_n(z) = (1 - e^{h_{1,n}/n} z)(1 - e^{h_{2,n}/n} z),$$

where $h_{k,n} \in \mathbb{C}$ such that $h_{k,n} \to h_k$, as $n \to \infty$ for $k = 1, 2$, and the polynomial $\varphi_n$ has real coefficients. This implies that $h_{1,n}$ and $h_{2,n}$ are real numbers or conjugated complex numbers. The same is valid for $h_1$ and $h_2$. Remark, that (13) now has the form

$$1 - c_1 z - c_2 z^2 = (1 - h_1 z)(1 - h_2 z), \qquad z \in \mathbb{C},$$

hence $c_1 = h_1 + h_2$ and $c_2 = -h_1 h_2$. From (33) we have

$$(\beta_{1,n} - \beta_1)z + (\beta_{2,n} - \beta_2)z^2 = \frac{1}{n} c_{1,n} z(1-z) + \frac{1}{n^2} c_{2,n} z$$

for $z \in \mathbb{C}$. Comparing the coefficients of $z$ and $z^2$ we obtain

$$\begin{aligned} c_{1,n} &= -n(\beta_{2,n} - \beta_2) \\ c_{2,n} &= n^2(\beta_{1,n} - \beta_1) + n^2(\beta_{2,n} - \beta_2). \end{aligned}$$

Applying Corollary 4 as in Remark 4 we conclude

$$\begin{pmatrix} 0 & -n \\ n^2 & n^2 \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_1 \\ \widehat{\beta}_{2,n} - \beta_2 \end{pmatrix} = \begin{pmatrix} \widehat{c}_{1,n} \\ \widehat{c}_{2,n} \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix},$$

where

$$\begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix} = S^{-1} \begin{pmatrix} \int_0^1 \dot{Y}(t)\, d\dot{Y}(t) \\ \int_0^1 Y(t)\, d\dot{Y}(t) \end{pmatrix}, \qquad S = \begin{pmatrix} \int_0^1 (\dot{Y}(t))^2\, dt & \int_0^1 \dot{Y}(t) Y(t)\, dt \\ \int_0^1 Y(t)\dot{Y}(t)\, dt & \int_0^1 (Y(t))^2\, dt \end{pmatrix},$$

and $(Y(t), \dot{Y}(t))$, $t \in [0,1]$, is the continuous time real-valued AR(2) process

$$\begin{cases} d\dot{Y}(t) = \big((h_1 + h_2)\dot{Y}(t) - h_1 h_2 Y(t)\big)\, dt + dW(t), \\ dY(t) = \dot{Y}(t)\, dt, \\ Y(0) = \dot{Y}(0) = 0, \end{cases} \tag{37}$$

where $W(t)$, $t \in [0,1]$, is a standard real-valued Wiener process. Moreover, $\widehat{c}_1$, $\widehat{c}_2$ can be interpreted as the MLE of $c_1 = h_1 + h_2$ and $c_2 = -h_1 h_2$. By Itô's formula we can also derive

$$\begin{pmatrix} 0 & -n \\ n^2 & n^2 \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_{1,n} \\ \widehat{\beta}_{2,n} - \beta_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} S^{-1} \begin{pmatrix} \int_0^1 \dot{Y}(t)\, dW(t) \\ \int_0^1 Y(t)\, dW(t) \end{pmatrix}.$$

The case when the limit unstable model has double roots equal to $-1$, i.e., its characteristic polynomial is

$$\varphi(z) = (1+z)^2 = 1 + 2z + z^2$$

can be handled similarly, and we obtain

$$\begin{pmatrix} 0 & -n \\ -n^2 & n^2 \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_{1,n} \\ \widehat{\beta}_{2,n} - \beta_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} S^{-1} \begin{pmatrix} \int_0^1 \dot{Y}(t)\, dW(t) \\ \int_0^1 Y(t)\, dW(t) \end{pmatrix},$$

where $(Y(t), \dot{Y}(t))$, $t \in [0,1]$, is the continuous time real-valued AR(2) process given by (37).

Now consider the case when the limit unstable model has the roots 1 and $-1$, i.e., its characteristic polynomial is

$$\varphi(z) = (1-z)(1+z) = 1 - z^2$$

and we have $\beta_1 = 0$ and $\beta_2 = 1$. The characteristic polynomial of the model (36) has the form

$$\varphi_n(z) = (1 - e^{h_{1,n}/n} z)(1 + e^{h_{2,n}/n} z),$$

where $h_{k,n} \in \mathbb{R}$ such that $h_{k,n} \to h_k$, as $n \to \infty$ for $k = 1,2$. Remark, that (13) implies $c_k = h_k$, $k = 1,2$. From (33) we have

$$(\beta_{1,n} - \beta_1)z + (\beta_{2,n} - \beta_2)z^2 = \frac{1}{n}(c_{1,n} z(1+z) - c_{2,n} z(1-z))$$

for $z \in \mathbb{C}$. Comparing the coefficients of $z$ and $z^2$ we obtain

$$\begin{aligned} 2c_{1,n} &= n((\beta_{1,n} - \beta_1) + (\beta_{2,n} - \beta_2)) \\ 2c_{2,n} &= n(-(\beta_{1,n} - \beta_1) + (\beta_{2,n} - \beta_2)). \end{aligned}$$

Applying Corollary 4 as in Remark 4 we conclude

$$\begin{pmatrix} n & n \\ -n & n \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_1 \\ \widehat{\beta}_{2,n} - \beta_2 \end{pmatrix} = \begin{pmatrix} 2\,\widehat{c}_{1,n} \\ 2\,\widehat{c}_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} 2\,\widehat{c}_1 \\ 2\,\widehat{c}_2 \end{pmatrix},$$

where

$$\widehat{c}_k = \frac{\int_0^1 Y_k(t)\, dY_k(t)}{\int_0^1 Y_k^2(t)\, dt},$$

and $Y_k(t)$, $t \in [0,1]$, $k = 1,2$ are the (independent) continuous time real-valued AR(1) processes given by

$$dY_k(t) = h_k Y_k(t) dt + dW_k(t), \quad Y_k(0) = 0, \qquad k = 1,2,$$

where $W_k(t)$, $t \in [0,1]$, $k=1,2$, are independent standard real-valued Wiener processes. Moreover, $\widehat{c}_k$, $k=1,2$ can be interpreted as the MLE of the parameters $h_k$, $k=1,2$. We have also

$$\begin{pmatrix} n & n \\ -n & n \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_{1,n} \\ \widehat{\beta}_{2,n} - \beta_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} 2\int_0^1 Y_1(t)\, dW_1(t) / \int_0^1 Y_1^2(t)\, dt \\ 2\int_0^1 Y_2(t)\, dW_2(t) / \int_0^1 Y_2^2(t)\, dt \end{pmatrix}.$$

Next we investigate complex-valued AR(2) models near to an unstable model given by

$$\begin{cases} X_{n,k} = \beta_{1,n} X_{n,k-1} + \beta_{2,n} X_{n,k-2} + \varepsilon_{n,k}, & k = 1,2,\dots,n \\ X_{n,0} = X_{n,-1} = 0, \end{cases} \tag{38}$$

where $\beta_{1,n}$, $\beta_{2,n}$ are complex numbers and $\{\varepsilon_{n,k}\}$ is an array of real-valued random variables satisfying the condition (C).

First consider the case where the limit unstable model has different (complex) roots, i.e., its characteristic polynomial is

$$\varphi(z) = (1 - e^{i\theta_1} z)(1 - e^{-i\theta_2} z),$$

where $\theta_1, \theta_2 \in (-\pi, \pi]$, $\theta_1 \neq \theta_2$. The characteristic polynomial of the model (38) has the form

$$\varphi_n(z) = (1 - e^{h_{1,n}/n + i\theta_1} z)(1 + e^{h_{2,n}/n + i\theta_2} z),$$

where $h_{k,n} \in \mathbb{C}$ such that $h_{k,n} \to h_k$, as $n \to \infty$ for $k=1,2$. Remark, that (13) implies $c_k = h_k$, $k=1,2$. From (33) we have

$$(\beta_{1,n} - \beta_1) z + (\beta_{2,n} - \beta_2) z^2 = \frac{1}{n} (c_{1,n} e^{i\theta_1} z (1 - e^{i\theta_2} z) + c_{2,n} e^{i\theta_2} z (1 - e^{i\theta_1} z))$$

for $z \in \mathbb{C}$. Comparing the coefficients of $z$ and $z^2$ we obtain

$$\begin{aligned} (e^{i\theta_1} - e^{i\theta_2}) c_{1,n} &= n((\beta_{1,n} - \beta_1) + (\beta_{2,n} - \beta_2) e^{-i\theta_1}) \\ (e^{i\theta_1} - e^{i\theta_2}) c_{2,n} &= n(-(\beta_{1,n} - \beta_1) - (\beta_{2,n} - \beta_2) e^{-i\theta_2}). \end{aligned}$$

Applying Theorem 5 we conclude

$$\begin{pmatrix} n & ne^{-i\theta_1} \\ -n & -ne^{-i\theta_2} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_1 \\ \widehat{\beta}_{2,n} - \beta_2 \end{pmatrix} = (e^{i\theta_1} - e^{i\theta_2}) \begin{pmatrix} \widehat{c}_{1,n} \\ \widehat{c}_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} (e^{i\theta_1} - e^{i\theta_2}) \begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix},$$

where

$$\widehat{c}_k = \frac{\int_0^1 \overline{Y_k(t)}\, dY_k(t)}{\int_0^1 |Y_k(t)|^2\, dt},$$

and $Y_k(t)$, $t \in [0,1]$, $k=1,2$ are continuous time AR(1) processes given by

$$dY_k(t) = h_k Y_k(t) dt + dW_k(t), \quad Y_k(0) = 0, \qquad k = 1,2,$$

where $W_k(t)$, $t \in [0,1]$, $k=1,2$, are standard Wiener processes, real-valued if $\theta_k = 0$ or $\theta_k = \pi$, and complex-valued otherwise. Further, $W_1$ and $W_2$ are independent if $\theta_1 \neq -\theta_2$, and $W_1 = \overline{W}_2$ if $\theta_1 = -\theta_2$.

Moreover, $\widehat{c}_k$, $k=1,2$ can be interpreted as the MLE of the parameters $h_k$, $k=1,2$. We also have

$$\begin{pmatrix} n & ne^{-i\theta_1} \\ -n & -ne^{-i\theta_2} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_{1,n} \\ \widehat{\beta}_{2,n} - \beta_{2,n} \end{pmatrix} \xrightarrow{\mathcal{D}} (e^{i\theta_1} - e^{i\theta_2}) \begin{pmatrix} \int_0^1 \overline{Y_1(t)}\, dW_1(t) / \int_0^1 |Y_1(t)|^2\, dt \\ \int_0^1 \overline{Y_2(t)}\, dW_2(t) / \int_0^1 |Y_2(t)|^2\, dt \end{pmatrix}.$$

Now consider the case when the limit unstable model has double (complex) roots equal to $e^{i\theta}$, i.e., its characteristic polynomial is

$$\varphi(z) = (1 - e^{i\theta} z)^2,$$

where $\theta \in (-\pi, \pi]$. The characteristic polynomial of the model (38) has the form

$$\varphi_n(z) = (1 - e^{h_{1,n}/n + i\theta} z)(1 - e^{h_{2,n}/n + i\theta} z),$$

where $h_{k,n} \in \mathbb{C}$ such that $h_{k,n} \to h_k$, as $n \to \infty$ for $k = 1, 2$. Remark, that (13) implies $c_1 = h_1 + h_2$ and $c_2 = -h_1 h_2$. From (33) we have

$$(\beta_{1,n} - \beta_1)z + (\beta_{2,n} - \beta_2)z^2 = \frac{1}{n} c_{1,n} e^{i\theta} z(1 - e^{i\theta} z) + \frac{1}{n^2} c_{2,n} e^{i\theta} z$$

for $z \in \mathbb{C}$. Comparing the coefficients of $z$ and $z^2$ we obtain

$$\begin{aligned} c_{1,n} &= -n(\beta_{2,n} - \beta_2)e^{-2i\theta} \\ c_{2,n} &= n^2(\beta_{1,n} - \beta_1)e^{-i\theta} + n^2(\beta_{2,n} - \beta_2)e^{-2i\theta}. \end{aligned}$$

Applying Theorem 5 we conclude

$$\begin{pmatrix} 0 & -ne^{-2i\theta} \\ n^2 e^{-i\theta} & n^2 e^{-2i\theta} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_1 \\ \widehat{\beta}_{2,n} - \beta_2 \end{pmatrix} = \begin{pmatrix} \widehat{c}_{1,n} \\ \widehat{c}_{2,n} \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix},$$

where

$$\begin{pmatrix} \widehat{c}_1 \\ \widehat{c}_2 \end{pmatrix} = S^{-1} \begin{pmatrix} \int_0^1 \overline{\dot{Y}(t)}\, d\dot{Y}(t) \\ \int_0^1 \overline{Y(t)}\, d\dot{Y}(t) \end{pmatrix}, \qquad S = \begin{pmatrix} \int_0^1 |\dot{Y}(t)|^2\, dt & \int_0^1 \overline{\dot{Y}(t)} Y(t)\, dt \\ \int_0^1 \overline{Y(t)} \dot{Y}(t)\, dt & \int_0^1 |Y(t)|^2\, dt \end{pmatrix},$$

and $(Y(t), \dot{Y}(t))$, $t \in [0, 1]$, is the continuous time AR(2) process

$$\begin{cases} d\dot{Y}(t) = \big((h_1 + h_2)\dot{Y}(t) - h_1 h_2 Y(t)\big)\, dt + dW(t), \\ dY(t) = \dot{Y}(t)\, dt, \\ Y(0) = \dot{Y}(0) = 0, \end{cases}$$

where $W(t)$, $t \in [0, 1]$, is a standard Wiener process, real-valued if $\theta = 0$ or $\theta = \pi$, and complex-valued otherwise.

Moreover, $\widehat{c}_1$, $\widehat{c}_2$ can be interpreted as the MLE of $c_1 = h_1 + h_2$ and $c_2 = -h_1 h_2$. By Itô's formula we can also have

$$\begin{pmatrix} 0 & -ne^{-2i\theta} \\ n^2 e^{-i\theta} & n^2 e^{-2i\theta} \end{pmatrix} \begin{pmatrix} \widehat{\beta}_{1,n} - \beta_{1,n} \\ \widehat{\beta}_{2,n} - \beta_{2,n} \end{pmatrix} \stackrel{\mathcal{D}}{\longrightarrow} S^{-1} \begin{pmatrix} \int_0^1 \overline{\dot{Y}(t)}\, dW(t) \\ \int_0^1 \overline{Y(t)}\, dW(t) \end{pmatrix}.$$

Comparing the complex-valued AR(2) models with the real-valued AR(2) models we observe that convergence of least squares estimators in the real-valued models can be deived from the complex-valued case by taking into account of the extra requirement, that the coefficients should be real numbers. However, the formulations in the context of complex-valued models are remarkably simpler.

As we have seen, a multiple root in the model implies a higher order autoregressive component in the corresponding continuous time model. Different but not conjugated roots imply components driven by independent Wiener processes in the continuous time model. In case the roots are conjugated pairs, then the components are driven by conjugated complex-valued Wiener processes. A real root is connected to a real-valued Wiener process, and a complex root is connected to a complex-valued Wiener process, even if the model has real coefficients!

We finally note that convergence of least squares estimators in models with complex-valued disturbances $\{\varepsilon_{n,k}\}$ can be handled similarly, see the AR(1) case in Kormos, van der Meer, Pap and van Zuijlen [12].

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