SUBCONVEXITY FOR TWISTED L-FUNCTIONS OVER NUMBER FIELDS VIA SHIFTED CONVOLUTION SUMS

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ABSTRACT. Assume that π is a cuspidal automorphic GL_2 representation over a number field F. Then for any Hecke character χ of conductor $\mathfrak{q}$, the subconvex bound

$$L(1/2,\pi\otimes\chi) \ll_{F,\pi,\chi_\infty,\varepsilon} \mathcal{N}(\mathfrak{q})^{3/8+\theta/4+\varepsilon}$$

holds for any $\varepsilon > 0$, where θ is any constant towards the Ramanujan-Petersson conjecture ($\theta = 7/64$ is admissible). In these notes, we derive this bound from the spectral decomposition of shifted convolution sums worked out by the author in [Mag].

1. INTRODUCTION

The subconvexity problem is concerned with the size of an L-function. Given a family of automorphic forms, we look for a bound of the form

$$L(1/2,\pi) \ll_\varepsilon \mathrm{cond}(\pi)^{\delta+\varepsilon},$$

which holds for each member of the family. Here, $\mathrm{cond}(\pi)$ denotes the analytic conductor of π, as described in [IS00, Section 2]. With $\delta = 1/4$, this is known as the convexity bound as it follows from the Phragmén-Lindelöf convexity principle combined with the functional equation and some bound on the half-plane $\Re s > 1$ (for the latter see [Mol02]). However, for most applications, one needs a stronger estimate. Any improvement in the exponent (i.e. any $\delta < 1/4$) is called a subconvex bound. The generalized Lindelöf hypothesis predicts that even $\delta = 0$ is admissible, and this would follow from the generalized Riemann hypothesis. On the other hand, several unconditional results are known. For example, for GL_1 L-functions over $\mathbf{Q}$ (i.e. Dirichlet L-functions), the famous Burgess bound [Bur63] is the above with $\delta = 3/16$. For automorphic GL_2 L-functions over number fields, the subconvexity problem was solved by Michel and Venkatesh [MV10] with an unspecified δ.

Recently, Blomer and Harcos [BH10] proved a Burgess type subconvex bound for twisted automorphic GL_2 L-functions over totally real number fields. The method is based on the generalization of their earlier work [BH08], a spectral decomposition of shifted convolution sums.

In this paper, we extend the subconvex bound of [BH10] to all number fields. Let π be an irreducible cuspidal representation of $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})$ with unitary central character, and let χ be a Hecke character of conductor $\mathfrak{q}$. Moreover, let θ be a constant towards the Ramanujan-Petersson conjecture (by [BB11], $\theta = 7/64$ is admissible).

Theorem 1. *For any $\varepsilon > 0$, we have the Burgess like subconvex bound*

$$L(1/2,\pi\otimes\chi) \ll_{F,\pi,\chi_\infty,\varepsilon} \mathcal{N}(\mathfrak{q})^{3/8+\theta/4+\varepsilon},$$

where $\mathcal{N}(\mathfrak{q})$ stands for the norm of $\mathfrak{q}$.

We remark that the same exponent was simultaneously achieved by Wu [Wu14], using a method built on [MV10].

The family of twisted L-functions considered in Theorem 1 was the first instance of the automorphic subconvexity problem to be studied systematically (see for example the works [Iwa87], [Duk88], [DFI93], [Byk96], [CI00], [CPSS], [BHM07], [Ven10]). Via Waldspurger type formulae, critical values of twisted

L-functions are connected to Fourier coefficients of modular forms of half-integral weight. The whole area has been highly motivated by Hilbert's eleventh problem: which integers are integrally represented by a given quadratic form over a number field, and subconvex bounds for twisted L-functions give rise to the asymptotic of representation numbers of quadratic forms (see [DSP90]). They also appear in the solution of other arithmetic equidistribution problems (consult [Coh05], [Zha05], [Ven10]). On the other hand, such subconvexity estimates are often used as ingredients in higher-rank subconvexity problems.

A note to the reader. This paper is the continuation of [Mag], therefore it is cited many times below. The results presented here (and also those of [Mag]) were part of the author's PhD thesis [Mag13b] at Central European University, Budapest.

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2. Notations

As in the introduction, let F be a number field (a finite algebraic extension of $\mathbf{Q}$) with ring of integers $\mathfrak{o}$ and adele ring $\mathbf{A}$. Assume that the degree of F over $\mathbf{Q}$ is $r+2s$, where F has r real and s complex places, and set $F_1,\dots,F_r$ for the pairwise inequivalent real and $F_{r+1},\dots,F_{r+s}$ for the pairwise inequivalent complex completions of F. Let then

$$F_\infty=\bigoplus_{j=1}^{r+s}F_j$$

be the Minkowski space of F. Set further

$$F_{\infty,+}^\times=\{(a_1,\dots,a_{r+s})\in F_\infty^\times : a_1,\dots,a_r>0\}$$

for the multiplicative group of totally positive elements (those that are positive at each real place). An important subset of this is the diagonal subgroup

$$F_{\infty,+}^{\mathrm{diag}}=\{(a_1,\dots,a_{r+s})\in F_{\infty,+}^\times : a_1=\ldots=a_{r+s}\}.$$

For any $a=(a_1,\dots,a_{r+s})\in F_\infty$, we define its infinity norm

$$|a|_\infty=\prod_{j=1}^{r}|a_j|\prod_{j=r+1}^{r+s}|a_j|^2.$$

If π is an irreducible cuspidal automorphic GL_2 representation with Hecke eigenvalues $\lambda_\pi(\mathfrak{a})$, then on the domain $\Re s>1$, the corresponding L-function is defined as

$$L(s,\pi)=\sum_{0\neq\mathfrak{a}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}.$$

Fixing the representation π, we may twist it with various GL_1 representations (i.e. Hecke characters) χ, and we get the L-function

$$L(s,\pi\otimes\chi)=\sum_{0\neq\mathfrak{a}\subseteq\mathfrak{o}}\frac{\lambda_{\pi\otimes\chi}(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})^s}.$$

Now $L(s,\pi)$ and also $L(s,\pi\otimes\chi)$ (for any Hecke character χ) continue holomorphically to the complex plane.

Throughout the paper, we use Vinogradov's notation: $A\ll_* B$ means that for some constant $c\in\mathbf{R}$ depending only on $*$, $|A|\le cB$. Also, by $A\asymp_* B$ we mean that both $A\ll_* B$ and $B\ll_* A$ hold.

3. Preliminaries: Kuznetsov formula and the density of the automorphic spectrum

3.1. A semi-adelic Kuznetsov formula over number fields. First of all, we introduce some more notations. Given an ideal $\mathfrak{c}$, let

$$\mathscr{C}_1(\mathfrak{c}) = \{\pi \in \mathscr{C}_1 \mid \mathfrak{c} \subseteq \mathfrak{c}_\pi\}, \qquad \mathscr{E}_1(\mathfrak{c}) = \{\chi \in \mathscr{E}_1 \mid \mathfrak{c} \subseteq \mathfrak{c}_{\chi,\chi^{-1}}\},$$

where $\mathscr{C}_1$, $\mathscr{E}_1$ stand for the cuspidal spectrum and the Eisenstein spectrum of trivial central character, respectively; while $\mathfrak{c}_\pi$, $\mathfrak{c}_{\chi,\chi^{-1}}$ are the analytic conductors, consult [Mag, Section 2].

Now we briefly discuss a variant of the Kuznetsov formula (for details, see [Mag13a, Theorem 1] or [Mag13b, Theorem 3], an extension of Venkatesh's version [Ven04]) that we will use later, the central character is still assumed to be trivial. In our notation, for a weight function h of the form described below and an ideal $\mathfrak{c} \subseteq \mathfrak{o}$,

$$\begin{aligned}
&[K(\mathfrak{o}):K(\mathfrak{c})]^{-1} \sum_{\pi \in \mathscr{C}_1(\mathfrak{c})} C_\pi^{-1} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_\pi^{-1}} h(\mathbf{r}_\pi) \lambda_\pi^{\mathfrak{t}}(\alpha \mathfrak{a}^{-1}) \overline{\lambda_\pi^{\mathfrak{t}}(\alpha' \mathfrak{a}'^{-1})} + CSC = \\
&\qquad \mathrm{const.}_F \Delta(\alpha\mathfrak{a}^{-1}, \alpha'\mathfrak{a}'^{-1}) \int h(\mathbf{r}) d\mu + \\
(1)\qquad &\qquad \mathrm{const.}_F \sum_{\mathfrak{m} \in C} \sum_{c \in \mathfrak{a}\mathfrak{m}\mathfrak{c}} \sum_{\varepsilon \in \mathfrak{o}_+^\times / \mathfrak{o}^{2\times}} \frac{KS(\varepsilon\alpha, \mathfrak{a}^{-1}\mathfrak{d}^{-1}; \alpha'\gamma_\mathfrak{m}, \mathfrak{a}'^{-1}\mathfrak{d}^{-1}; c, \mathfrak{a}^{-1}\mathfrak{m}^{-1}\mathfrak{d}^{-1})}{\mathscr{N}(c\mathfrak{a}^{-1}\mathfrak{m}^{-1})} \\
&\qquad\qquad \cdot \int \mathscr{B}h_{(\mathbf{r})}\left(4\pi \frac{(\alpha\alpha'\gamma_\mathfrak{m}\varepsilon)^{\frac{1}{2}}}{c}\right) d\mu,
\end{aligned}$$

where KS is a Kloosterman sum, $\mathscr{B}$ is a certain transform, and $d\mu$ is a certain measure of the space of the archimedean spectral parameters (and for convenience, we introduce its norm)

$$\mathbf{r} = (\nu_1, \dots, \nu_r, (\nu_{r+1}, p_{r+1}), \dots, (\nu_{r+s}, p_{r+s})), \qquad \mathscr{N}(\mathbf{r}) = \prod_{j=1}^{r} (1+|\nu_j|) \prod_{j=r+1}^{r+s} (1+|\nu_j|+|p_j|)^2.$$

(see [Mag, Section 2]). We explain the notation and the conditions: $\mathfrak{d}$ is the different; $\mathfrak{a}^{-1}$ and $\mathfrak{a}'^{-1}$ are nonzero fractional ideals; $\alpha \in \mathfrak{a}, \alpha \in \mathfrak{a}'$ such that $\alpha\alpha'$ is totally positive; C is a fixed set of narrow ideal class representatives $\mathfrak{m}$, for which $\mathfrak{m}^2\mathfrak{a}\mathfrak{a}'^{-1}$ is a principal ideal generated by a totally positive element $\gamma_\mathfrak{m}$; $\Delta(\alpha\mathfrak{a}^{-1}, \alpha'\mathfrak{a}'^{-1})$ is 1 if $\alpha\mathfrak{a}^{-1} = \alpha'\mathfrak{a}'^{-1}$, otherwise it is 0; CSC is an analogous integral over the Eisenstein spectrum. Further, the factors C_π depend only on the representations π in the summation: for its definition, see for example [Mag, Section 2], and for its estimates, consult [BH10, Sections 2.8-9] and [Mag13b, Chapter 3].

The weight function h we will use is of the form $h = \prod_j h_j$ (a product over the archimedean places), where h_j's are defined as follows. Let $a_j, b_j > 1, a'_j \in \mathbf{R}$ be given. Then at real places

$$h_j(\nu_j) = \begin{cases} e^{(\nu_j^2 - \frac{1}{4})/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, \\ 1, & \text{if } \nu_j \in \frac{1}{2} + \mathbf{Z}, \frac{3}{2} \le |\nu_j| \le b_j, \\ 0 & \text{otherwise,} \end{cases}$$

while at complex places

$$h_j(\nu_j, p_j) = \begin{cases} e^{(\nu_j^2 + a'_j p_j^2 - 1)/a_j}, & \text{if } |\Re\nu_j| < \frac{2}{3}, p_j \in \mathbf{Z}, |p_j| \le b_j, \\ 0 & \text{otherwise.} \end{cases}$$

For the purpose of this paper, we will choose our parameters as follows. At each place, $a_j > 1$ is arbitrary, then set $b_j = \sqrt{a_j}$. Furthermore, at complex places, we use $a'_j = -1$. In this setup, we have the bounds

$$\begin{aligned} &\int h_j(\nu_j) d\mu_j \ll a_j, \qquad \int (\mathscr{B}_j h_j)_{\nu_j}(t) d\mu_j \ll a_j \min(1, |t|^{1/2}); \\ &\int h_j(\nu_j, p_j) d\mu_j \ll a_j^2, \qquad \int (\mathscr{B}_j h_j)_{(\nu_j, p_j)}(t) d\mu_j \ll a_j \min(1, |t|), \end{aligned} \tag{2}$$

at real and complex places, respectively (see [BMP01, pp. 124-126], [BM03, Section 10] and [Mag13b, Lemma 5.3]).

3.2. **The density of the spectrum.** In this section, we estimate the density of the Eisenstein and the cuspidal spectrum in terms of the spectral parameters. These are the extensions of [BH10, Lemma 2 and Lemma 6]. After the suitable modifications, the proofs given there apply in the more general situation. In this section, we still assume that the central character is trivial, since this is the only case we shall use later.

3.2.1. *Density of the Eisenstein spectrum.*

Lemma 1. *Let $\mathfrak{c}_1^2\mathfrak{c}_2 = \mathfrak{c} \subseteq \mathfrak{o}$, where $\mathfrak{c}_2$ is squarefree. Then for $1 \le X \in \mathbf{R}$, $1 \le P \in \mathbf{Z}$,*

$$\int_{\substack{\varpi \in \mathscr{E}_1(\mathfrak{c}) \\ |\nu_{\varpi,j}| \le X \\ |p_{\varpi,j}| \le P}} 1 d\varpi \ll_F X^{r+s} P^s \mathscr{N}(\mathfrak{c}_1).$$

Proof. Any Hecke character χ can be factorized as $\chi = \chi_\infty \chi_{\mathrm{fin}}$. Here, $\chi_{\mathrm{fin}}|_{\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times}$ is a character of $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$. By [BH10, Lemma 1] or [Mag, Proposition 2.1], $\mathfrak{c}_\chi | \mathfrak{c}_1$, so there are at most $\varphi(\mathfrak{c}_1)$ possibilities for this restriction. Given a character ξ of $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$, we estimate the measure of the set S of those Hecke characters χ for which $\chi_{\mathrm{fin}}|_{\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times} = \xi$. If $S = \emptyset$, the measure is 0. If $S \neq \emptyset$, fix some $\chi_0 \in S$. Then to any χ in S, associate $\chi' = \chi\chi_0^{-1}$. From the non-archimedan part, we see χ' is trivial on $\prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times$. From the archimedan part, we see that for $a \in F_{\infty,+}^\times$, $\chi'(a_j) = |a_j|^{t_j}$, if $j \le r$, and $\chi'(a_j) = |a_j|^{t_j}(a_j/|a_j|)^{p_j}$, if $j > r$, where $t_j \in i[-2X, 2X]$, and $p_j \in [-2P, 2P] \cap \mathbf{Z}$. Fix the vector $(p_j)_{j>r} \in [-2P, 2P]^s \cap \mathbf{Z}^s$.

Now χ'_∞ is trivial on the group U^+ of totally positive units embedded in $F_{\infty,+}^\times$. Fix a generating set $\{u_1, \dots, u_{r+s-1}\}$ for the torsion-free part of U^+. Then by the notation of [BH10], take

$$M = \begin{pmatrix} \deg[F_1 : \mathbf{R}] & \dots & \deg[F_{r+s} : \mathbf{R}] \\ \deg[F_1 : \mathbf{R}] \log|u_{1,1}| & \dots & \deg[F_{r+s} : \mathbf{R}] \log|u_{1,r+s}| \\ \vdots & & \vdots \\ \deg[F_1 : \mathbf{R}] \log|u_{r+s-1,1}| & \dots & \deg[F_{r+s} : \mathbf{R}] \log|u_{r+s-1,r+s}| \end{pmatrix} \in \mathbf{R}^{(r+s)\times(r+s)}.$$

Then the column vector $t = (t_j)_j \in i[-2X, 2X]^{r+s}$ with $iT = \sum_j \deg[F_j : \mathbf{R}] t_j$ satisfies $Mt \in \{iT\} \times (2\pi i \mathbf{Z})^{r+s-1}$. Using that M is invertible and depends only on F, we see

$$\int_{-2(r+2s)X}^{2(r+2s)X} \#((\{T\} \times (2\pi i \mathbf{Z})^{r+s-1}) \cap Mi[-2X, 2X]^{r+s}) dT \ll_F X^{r+s},$$

since the integrand is $O_F(X^{r+s-1})$. Taking into account the finiteness of the torsion subgroup of U^+ and of $F^\times F_{\infty,+}^\times \prod_\mathfrak{p} \mathfrak{o}_\mathfrak{p}^\times \backslash \mathbf{A}^\times$, finally summing over $(p_j)_{j>r} \in [-2P, 2P]^s$, we obtain the statement. ☐

Corollary 2. *Let $\mathfrak{c}_1^2\mathfrak{c}_2 = \mathfrak{c} \subseteq \mathfrak{o}$, where $\mathfrak{c}_2$ is squarefree. Then for $1 \le X \in \mathbf{R}$,*

$$\int_{\substack{\varpi \in \mathscr{E}_1(\mathfrak{c}) \\ j \le r : |\nu_{\varpi,j}| \le X \\ j > r : |\nu_{\varpi,j}^2 - p_{\varpi,j}^2| \le X^2}} 1 d\varpi \ll_F X^{r+2s} \mathscr{N}(\mathfrak{c}_1).$$

3.2.2. *Density of the cuspidal spectrum.* Using the Kuznetsov formula, we may estimate the density of the cuspidal spectrum as follows.

Lemma 3. *Let $\mathfrak{c} \subseteq \mathfrak{o}$ be an ideal. Then for $1 \leq X_j \in \mathbf{R}^{r+s}$,*

$$\sum_{\substack{\varpi \in \mathscr{C}_1(\mathfrak{c}) \\ j \leq r: |\nu_{\varpi,j}| \leq X_j \\ j > r: |\nu_{\varpi,j}^2 - p_{\varpi,j}^2| \leq X_j^2}} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m})|^2 \ll_{F,\varepsilon}$$

$$\left(\prod_{j \leq r} X_j^{2+\varepsilon}\right)\left(\prod_{j > r} X_j^{4+\varepsilon}\right)\mathscr{N}(\mathfrak{c})^{1+\varepsilon} + \left(\prod_{j} X_j^{2+\varepsilon}\right)(\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c})))^{1/2}\mathscr{N}(\mathfrak{m})^{1/2+\varepsilon}.$$

Proof. This is the generalization of [BH10, Lemma 6], and we can repeat its proof. Choose a narrow class representative $\mathfrak{n}$ equivalent to $\mathfrak{m}^{-1}$ from a fixed set of narrow class representatives. Then for some $\alpha \in F^{\times}$, $\mathfrak{m} = \alpha\mathfrak{n}^{-1}$, and $1 \ll_F \mathscr{N}((\alpha))/\mathscr{N}(\mathfrak{m}) \ll_F 1$. We apply the Kuznetsov formula (1) with $\alpha = \alpha'$, $\mathfrak{a} = \mathfrak{a}' = \mathfrak{n}$, and the weight function is the one described above, setting $a_j = X_j^2$, $b_j = X_j$ at each archimedean place. On the spectral side of the Kuznetsov formula, we obtain an upper bound on the left-hand side of the statement, since the contribution of the Eisenstein spectrum is nonnegative. For $\varpi \in \mathscr{C}_1(\mathfrak{c})$, one obtains

$$[K(\mathfrak{o}):K(\mathfrak{c})]C_{\varpi} \ll_{F,\varepsilon} (\prod_j X_j)^{\varepsilon}\mathscr{N}(\mathfrak{c})^{1+\varepsilon},$$

consult [BH10, Section 2.9] and [Mag13b, Chapter 3] for the estimate of C_{ϖ}. Then by (2), the delta term gives

$$\ll_{F,\varepsilon} (\prod_{j \leq r} X_j^{2+\varepsilon})(\prod_{j > r} X_j^{4+\varepsilon})\mathscr{N}(\mathfrak{c})^{1+\varepsilon}.$$

As for the Kloosterman term, we use Weil's bound [Ven04, (13)] together with (2) to see it is

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathscr{N}(\mathfrak{c})^{1+\varepsilon} \max_{\mathfrak{a} \in C} \sum_{0 \neq c \in \mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathscr{N}((\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1})))^{1/2}}{\mathscr{N}(c\mathfrak{n}^{-1}\mathfrak{a}^{-1})^{1/2-\varepsilon}} \cdot \prod_{j \leq r} \min(1, |\alpha_j/c_j|^{1/2}) \prod_{j > r} \min(1, |\alpha_j/c_j|), \tag{3}$$

where C is a fixed set of narrow class representatives (depending only on F) such that $\mathfrak{a}^2$ is a totally positive ideal for each $\mathfrak{a} \in C$. Then sum over the elements c can be rewritten as a sum over the principal ideals (c), while the sum over the units is estimated in [BM98, Lemma 8.1]. Then the above display becomes

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathscr{N}(\mathfrak{c})^{1+\varepsilon} \max_{\mathfrak{a} \in C} \sum_{0 \neq (c) \subseteq \mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathscr{N}(\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1}))^{1/2}}{\mathscr{N}(c\mathfrak{n}^{-1}\mathfrak{a}^{-1})^{1/2-\varepsilon}} \cdot (1 + |\log(\mathscr{N}((\alpha/c)))|^{r+s-1})\min(1, \mathscr{N}((\alpha/c))),$$

which is obviously

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right)\mathscr{N}(\mathfrak{c})^{1+\varepsilon}\mathscr{N}(\mathfrak{m})^{1/2+2\varepsilon} \max_{\mathfrak{a} \in C} \sum_{0 \neq (c) \subseteq \mathfrak{n}\mathfrak{a}\mathfrak{c}} \frac{\mathscr{N}((\gcd(\mathfrak{m}, c\mathfrak{n}^{-1}\mathfrak{a}^{-1})))^{1/2}}{\mathscr{N}((c))^{1/2+\varepsilon}}.$$

We estimate now the sum. First we extend it to all nonzero ideals contained in $\mathfrak{n}\mathfrak{a}\mathfrak{c}$ (parametrized as $\mathfrak{b}\mathfrak{n}\mathfrak{a}\mathfrak{c}$, where $0 \neq \mathfrak{b} \subseteq \mathfrak{o}$), then we factorize out $\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}$. In this way we obtain

$$\frac{1}{\mathscr{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}} \sum_{\mathfrak{b} \subseteq \mathfrak{o}} \frac{\mathscr{N}(\gcd(\mathfrak{m}, \mathfrak{c}\mathfrak{b}))^{1/2}}{\mathscr{N}(\mathfrak{b})^{1+\varepsilon}} \ll_{F,\varepsilon} \frac{\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2}}{\mathscr{N}(\mathfrak{n}\mathfrak{a}\mathfrak{c})^{1+\varepsilon}}\mathscr{N}(\mathfrak{m})^{\varepsilon}.$$

Altogether, the contribution of the Kloosterman term is

$$\ll_{F,\varepsilon} \left(\prod_j X_j^{2+\varepsilon}\right) \mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))^{1/2} \mathscr{N}(\mathfrak{m})^{1/2+\varepsilon}.$$

Finally, recall that the contribution of the Eisenstein spectrum is nonnegative, hence we are done. □

Corollary 4. *Let $\mathfrak{c} \subseteq \mathfrak{o}$ be an ideal. Then for $1 \le X_j \in \mathbf{R}^{r+s}$,*

$$\sum_{\substack{\varpi \in \mathscr{C}_1(\mathfrak{c}) \\ j \le r: |\nu_{\varpi,j}| \le X_j \\ j > r: |\nu_{\varpi,j}^2 - p_{\varpi,j}^2| \le X_j^2}} \sum_{\mathfrak{t} \mid \mathfrak{c}\mathfrak{c}_\pi^{-1}} 1 \ll_{F,\varepsilon} \left(\prod_{j \le r} X_j^{2+\varepsilon}\right) \left(\prod_{j > r} X_j^{4+\varepsilon}\right) \mathscr{N}(\mathfrak{c})^{1+\varepsilon}.$$

4. Proof of Theorem 1

Assume that π is an irreducible cuspidal automorphic representation. Let $\mathfrak{q} \subseteq \mathfrak{o}$ be an ideal, χ a Hecke character of conductor $\mathfrak{q}$. We may also think of χ as a character on the group of fractional ideals coprime to $\mathfrak{q}$, extended to be 0 on other ideals. There exist characters χ_{fin} of $(\mathfrak{o}/\mathfrak{q})^\times$ and χ_∞ of $F_\infty^\times$ satisfying $\chi((t)) = \chi_{\text{fin}}(t)\chi_\infty(t)$ for all $t \in \mathfrak{o}$ coprime to $\mathfrak{q}$. The transition from one meaning to another of Hecke characters can be found at several places (see [Bum97, Sections 1.7 and 3.1], for example). Our goal is to estimate $L(1/2, \pi \otimes \chi)$ in terms of $\mathscr{N}(\mathfrak{q})$. Fix any $\varepsilon > 0$. From now on, the implicit constants in $\ll$ are always meant to depend on $F, \varepsilon, \pi, \chi_\infty$, even if it is not emphasized in the subscript like $\ll_{F,\varepsilon,\pi,\chi_\infty}$. Fix an ideal $\mathfrak{n}$ coprime to $\mathfrak{q}$ satisfying

$$\mathscr{N}(\mathfrak{n}) \ll \mathscr{N}(\mathfrak{q})^\varepsilon \tag{4}$$

and note that in every narrow ideal class, there is a representative $\mathfrak{n}$ with these properties.

First we introduce the following notation. For given positive real numbers $a < b$,

$$[[a,b]] = \{x \in F_{\infty,+}^\times : a \le |x_j| \le b\}. \tag{5}$$

Let G_0 be a smooth and compactly supported function on $F_{\infty,+}^1 = \{x \in F_{\infty,+}^\times : |x|_\infty = 1\}$ satisfying $\sum_{u \in \mathfrak{o}_+^\times} G_0(ux) = 1$ for all $x \in F_{\infty,+}^1$ (where $\mathfrak{o}_+^\times$ stands for the group of totally positive units). We extend this function to $F_{\infty,+}^\times$ as $G(x) = G_0(x/|x|_\infty)$, then $\sum_{u \in \mathfrak{o}_+^\times} G(ux) = 1$ for all $x \in F_{\infty,+}^\times$. Assume that G_0 is supported on $[[c_1, c_2]]$, then G is supported on $F_{\infty,+}^{\text{diag}}[[c_1,c_2]]$, where c_1, c_2 are constants depending only on F (recall (5)). Fix moreover a compact fundamental domain $\mathscr{G}_0$ for the action of $\mathfrak{o}_+^\times$ on $F_{\infty,+}^1$ and let $\mathscr{G} = F_{\infty,+}^{\text{diag}}\mathscr{G}_0$ be its extension to $F_{\infty,+}^\times$.

4.1. The amplification method.

Let ξ be a character of $(\mathfrak{o}/\mathfrak{q})^\times$. Parametrized by $\nu = (\nu_1, \dots, \nu_{r+s}) \in (i\mathbf{R})^{r+s}$, $p = (p_{r+1}, \dots, p_{r+s}) \in \mathbf{Z}^s$, assume that $W_{\nu,p}$ are functions on $F_{\infty,+}^\times$ satisfying the following properties:

(i) $W_{\nu,p}$ is smooth and supported on $[[c_3, c_4]]$ for some $c_3 < c_1$ and $c_4 > c_2$ depending only on F;
(ii) for any differential operator $\mathscr{D}$ of the form

$$\mathscr{D} = \left(\left(\frac{\partial}{\partial y_j}\right)^{\mu_j}_{j \le r} \left(\frac{\partial}{\partial y_j}\right)^{\mu_{j,1}}_{j > r} \left(\frac{\partial}{\partial \overline{y_j}}\right)^{\mu_{j,2}}_{j > r}\right),$$

with nonnegative integers $\mu_{j,*}$, we have

$$\mathscr{D}W_{\nu,p}(y) \ll_{\mathscr{D}} \prod_{j=1}^{r} (1+|\nu_j|)^{\mu_j} \prod_{j=r+1}^{r+s} (1+|\nu_j|+|p_j|)^{\mu_{j,1}+\mu_{j,2}}.$$

For convenience, introduce

$$\mathcal{N}(v,p) = \prod_{j=1}^{r}(1+|v_j|) \prod_{j=r+1}^{r+s} (1+|v_j|+|p_j|)^2. \tag{6}$$

Then set

$$\mathcal{L}_\xi(v,p) = \sum_{0<<t\in\mathfrak{n}} \frac{\lambda_\pi(t\mathfrak{n}^{-1})\xi(t)}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}} W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right), \tag{7}$$

where $0 << t$ means that we sum over the totally positive elements. The only assumption on the positive real number Y is that

$$Y \ll \mathcal{N}(\mathfrak{q})^{1+\varepsilon}. \tag{8}$$

Introduce $\mathcal{K} = \mathfrak{n}\cap F_{\infty,+}^{\mathrm{diag}}[[c_3,c_4]]$. We see that the numbers t that give a positive contribution are all in the set $\mathfrak{n}\cap\mathcal{K}$ and also satisfy $t\in[[c_3,c_4]]Y^{1/(r+2s)}$, this latter implies $|t|_\infty \asymp_F Y$.

Assume that L (the amplification length) is a further parameter satisfying

$$\log L \asymp \log \mathcal{N}(\mathfrak{q}). \tag{9}$$

Lemma 5. *Denote by $\Pi_{\mathfrak{q},+}(L,2L)$ the set of totally positive, principal prime ideals $\mathfrak{l}\subseteq\mathfrak{o}$ satisfying $\mathcal{N}(\mathfrak{l})\in[L,2L]$ and $\mathfrak{l}\nmid\mathfrak{q}$. Set $\pi_{\mathfrak{q},+}(L,2L) = \#\Pi_{\mathfrak{q},+}(L,2L)$. Then*

$$\pi_{\mathfrak{q},+}(L,2L) \gg L\mathcal{N}(\mathfrak{q})^{-\varepsilon}.$$

Proof. This follows immediately from the results [Nar74, Corollary 6 of Proposition 7.8 and Proposition 7.9(ii)] about the natural density of prime ideals in narrow ideal classes. (See also [Neu99, Chapter VII, §13] for analogous statements about the Dirichlet density.) □

Therefore,

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 = \frac{1}{\pi_{\mathfrak{q},+}(L,2L)^2}\left|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}} 1\right|^2$$

$$\ll \frac{\mathcal{N}(\mathfrak{q})^\varepsilon}{L^2}\sum_{\xi\in\widehat{(\mathfrak{o}/\mathfrak{q})^\times}}\left|\mathcal{L}_\xi(v,p)\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}}\xi(l)\overline{\chi_{\mathrm{fin}}(l)}\right|^2.$$

Observe that the ξ-sum is the square integral of the Fourier transform of the function

$$(\mathfrak{o}/\mathfrak{q})^\times \ni x \mapsto \sum_{t\in\mathfrak{n}\cap\mathcal{K}}\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)\\ lt\equiv x \pmod{\mathfrak{q}}}}\overline{\chi_{\mathrm{fin}}(l)}\frac{\lambda_\pi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right),$$

so Plancherel gives

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll \frac{\varphi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^\varepsilon}{L^2} \cdot \sum_{x\in(\mathfrak{o}/\mathfrak{q})^\times}\left|\sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G}\\(l)\in\Pi_{\mathfrak{q},+}(L,2L)}}\overline{\chi_{\mathrm{fin}}(l)}\sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K}\\ lt\equiv x \pmod{\mathfrak{q}}}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right)\right|^2.$$

This can be further majorized using $\varphi(\mathfrak{q}) \le \mathcal{N}(\mathfrak{q})$ and $(\mathfrak{o}/\mathfrak{q})^\times \subset \mathfrak{o}/\mathfrak{q}$, giving

$$|\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{l_1,l_2 \in \mathfrak{o}\cap\mathcal{G} \\ (l_1),(l_2)\in\Pi_{\mathfrak{q},+}(L,2L)}} \overline{\chi_{\mathrm{fin}}(l_1)}\chi_{\mathrm{fin}}(l_2) \tag{10}$$
$$\sum_{\substack{t_1,t_2\in\mathfrak{n}\cap\mathcal{K} \\ l_1t_1-l_2t_2\in\mathfrak{q}}} \frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_{v,p}\left(\frac{t_1}{Y^{1/(r+2s)}}\right)\overline{W_{v,p}\left(\frac{t_2}{Y^{1/(r+2s)}}\right)}.$$

In (10), the contribution of $l_1t_1 - l_2t_2 = 0$ will be referred as the diagonal contribution DC, and that of $l_1t_1 - l_2t_2 \neq 0$ as the off-diagonal contribution ODC. We will estimate them separately, then optimize the choice of the parameter L (keeping (9) in mind), which will give rise to an estimate of $\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)$. Using Mellin inversion, this bound on $\mathcal{L}_{\chi_{\mathrm{fin}}}(v,p)$ (with implicit parameters satisfying (4) and (8)) will give rise to a Burgess type subconvex bound on $L(1/2,\pi\otimes\chi)$.

4.2. **The diagonal contribution.** First we focus on DC. Then, by Cauchy-Schwarz,

$$DC \ll \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L^2} \sum_{\substack{l\in\mathfrak{o}\cap\mathcal{G} \\ (l)\in\Pi_{\mathfrak{q},+}(L,2L)}} \sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K} \\ |t|_\infty \asymp_F Y}} \frac{|\lambda_\pi(t\mathfrak{n}^{-1})|^2}{\mathcal{N}(t\mathfrak{n}^{-1})} |\{(l',t')\in(\mathfrak{o}\cap\mathcal{G})\times(\mathfrak{n}\cap\mathcal{K}) : l't' = lt\}|.$$

Here, $|\{(l',t')\in(\mathfrak{o}\cap\mathcal{G})\times(\mathfrak{n}\cap\mathcal{K}) : l't' = lt\}|$ is at most the number of divisors of (lt), which is $\ll \mathcal{N}((lt))^\varepsilon \ll (LY)^\varepsilon$. Using partial summation in [BH10, (77)], via (4) and (8), we see

$$\sum_{\substack{t\in\mathfrak{n}\cap\mathcal{K} \\ |t|_\infty \asymp_F Y}} \frac{|\lambda_\pi(t\mathfrak{n}^{-1})|^2}{\mathcal{N}(t\mathfrak{n}^{-1})} \ll \mathcal{N}(\mathfrak{q})^\varepsilon,$$

and estimate the number of prime ideals (l) trivially by $\ll L$. Altogether,

$$DC \ll \frac{\mathcal{N}(\mathfrak{q})^{1+\varepsilon}}{L}. \tag{11}$$

4.3. **Off-diagonal contribution: spectral decomposition and Eisenstein part.**

4.3.1. *Spectral decomposition.* The estimate of the off-diagonal contribution requires much more work. Assume $\mathcal{G}_0$ is supported on $[[c_5,c_6]]$ for some constants c_5,c_6 depending only on F. Then only $l_1,l_2 \in [[c_5L^{1/(r+2s)}, c_6L^{1/(r+2s)}]]$ and $t_1,t_2 \in [[c_3Y^{1/(r+2s)}, c_4Y^{1/(r+2s)}]]$ have nonzero contribution. If l_1,l_2,t_1,t_2 satisfy these constraints, then

$$l_1t_1 - l_2t_2 \in \mathcal{B} = \{x\in F_\infty : |x_j| \le c_7(LY)^{1/(r+2s)}\}$$

with $c_7 = 2c_4c_6$. Now a term in ODC corresponding to some fixed l_1,l_2 can be written as

$$\sum_{\substack{q\in\mathfrak{q}\mathfrak{n}\cap\mathcal{B} \\ q\neq 0}} \sum_{\substack{l_1t_1-l_2t_2=q \\ 0\neq t_1,t_2\in\mathfrak{n}}} \frac{\lambda_\pi(t_1\mathfrak{n}^{-1})\overline{\lambda_\pi(t_2\mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1t_2\mathfrak{n}^{-2})}} W_1\left(\frac{l_1t_1}{(LY)^{1/(r+2s)}};v,p\right)\overline{W_2\left(\frac{l_2t_2}{(LY)^{1/(r+2s)}};v,p\right)}, \tag{12}$$

where W_1,W_2 are smooth functions on $F_{\infty,+}^\times$ defined as

$$W_1(y;v,p) = W_{v,p}(yL^{1/(r+2s)}/l_1), \qquad W_2(y;v,p) = W_{v,p}(yL^{1/(r+2s)}/l_2).$$

Now by the assumptions made on $W_{v,p}$ and l_1,l_2, we have that W_1,W_2 are smooth of compact support $[[c_8,c_9]]$ (where c_8,c_9 depend on F) and for any differential operator $\mathcal{D}$ of the form

$$\mathcal{D} = \left(\left(\frac{\partial}{\partial y_j}\right)^{\mu_j}_{j\le r}\left(\frac{\partial}{\partial y_j}\right)^{\mu_{j,1}}_{j>r}\left(\frac{\partial}{\partial \overline{y_j}}\right)^{\mu_{j,2}}_{j>r}\right),$$

with nonnegative integers $\mu_{j,*}$, we have

(13) $$\mathscr{D}W_{1,2}(y;v,p) \ll_{\mathscr{D}} \mathscr{N}(v,p)^{\mu},$$

where $\mu = \max_j(\mu_{j,*})$ (recall (6)).

Now by [Mag, Theorem 1], (12) can be decomposed spectrally over the automorphic spectrum:

(14) $$\sum_{0\neq q\in q\mathfrak{n}\cap\mathscr{B}} \int_{(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \frac{\lambda_{\varpi}^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}} W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/(r+2s)}};v,p\right) d\varpi,$$

where $\mathfrak{c} = \mathfrak{c}_{\pi}\mathrm{lcm}((l_1),(l_2))$. Here, the integral over $(\mathfrak{c})$ means that only those automorphic representations ϖ might have nonzero contribution whose conductor $\mathfrak{c}_{\varpi}$ contains $\mathfrak{c}$. The superscript $\mathfrak{t}$ is due to the orthogonalization process of oldforms (consult [Mag, (2.13-14), (2.26), (2.28-29)]).

4.3.2. *Eisenstein spectrum.* First we estimate the contribution of the Eisenstein spectrum to (14). We derive a simple consequence of [Mag, Theorem 1].

Lemma 6. *Conditions as in* [Mag, Theorem 1]. *Assume $\mathscr{D}$ is a differential operator as in* [Mag, Proposition 3.6]. *Then for any $0<\varepsilon<1/4$ and nonnegative integers b,c', we have, for all $y\in F_{\infty}^{\times}$,*

$$\int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} (\mathscr{N}(\mathbf{r}_{\varpi}))^{c'} |\mathscr{D}W_{\varpi,\mathfrak{t}}(y)| d\varpi \ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c',P} \mathscr{N}(\mathfrak{l})^{1/4}\mathscr{N}((l_1l_2))^{\varepsilon}||W_1||_{S_{\alpha'}}||W_2||_{S_{\alpha'}}$$
$$\cdot\prod_{j=1}^{r}(|y_j|^{1/2-\varepsilon}+|y_j|^{1/2-\theta-\varepsilon})(\min(1,|y_j|^{-b}))\prod_{j=r+1}^{r+s}(|y_j|^{3/4}+|y_j|)(\min(1,|y_j|^{-b}))$$

with $\alpha' = 2(3r+4s+2)+(r+s)(a+b+2c'+4(r+2s))+2(7r+18s)$, where $\mathfrak{l}$ stands for the largest square divisor of $\mathrm{lcm}((l_1),(l_2))$.

Proof. Set $c = c'+2(r+2s)$. Apply first Cauchy-Schwarz,

$$\left(\int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} (\mathscr{N}(\mathbf{r}_{\varpi}))^{c'} |\mathscr{D}W_{\varpi,\mathfrak{t}}(y)| d\varpi\right)^2 \ll_F \int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} (\mathscr{N}(\mathbf{r}_{\varpi}))^{2c} |\mathscr{D}W_{\varpi,\mathfrak{t}}(y)|^2 d\varpi$$
$$\cdot\int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} (\mathscr{N}(\mathbf{r}_{\varpi}))^{-4(r+2s)} d\varpi.$$

Now the first integral is estimated in [Mag, Theorem 1], while by Corollary 2, the second integral is $\ll_F (\mathscr{N}(\mathfrak{l}))^{1/2}$. We are done by taking square-roots. □

Now we apply this with $\mathscr{D}=1, a=c'=0, b=2$. The largest square divisor of $\mathrm{lcm}((l_1),(l_2))$ is $\mathfrak{o}$, hence

$$\int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y;v,p)| \ll \mathscr{N}((l_1l_2))^{\varepsilon}||W_1||_{S_{\alpha_1}}||W_2||_{S_{\alpha_1}}$$

with some positive integer α_1 depending only on F, uniformly in y,v,p, where S_* stands for Sobolev norms (see [Mag, Section 2.6]). Moreover, by [Ven10, Lemma 8.4] and (13), for any positive α,

(15) $$||W_{1,2}||_{S_{\alpha}} \ll_{\alpha} \mathscr{N}(v,p)^{2\alpha}$$

giving

$$\int_{\mathscr{E}(\mathfrak{c})} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y;v,p)| \ll \mathscr{N}((l_1l_2))^{\varepsilon}\mathscr{N}(v,p)^{4\alpha_1}.$$

Taking into account [Mag, display between (2.29) and (2.30)], (4), (8) and (9), we see that the contribution of the Eisenstein spectrum to (14) is

$$\ll \mathcal{N}(v,p)^{4\alpha_1}\mathcal{N}(\mathfrak{q})^{\varepsilon}\sum_{0\neq q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}}\frac{\mathcal{N}(\gcd(\mathfrak{c},(q)))}{\sqrt{\mathcal{N}((q))}}.$$

In the sum, each ideal (q) appears with multiplicity $\ll \mathcal{N}(\mathfrak{q})^{\varepsilon}$. Indeed, each ideal $(q)\subseteq\mathfrak{o}$ has a generator q satisfying $|q_j|\geq c_5$ at each archimedean place. Hence the possible units ε for which $q\varepsilon\in\mathscr{B}$ all satisfy $|\varepsilon_j|\leq c_{10}(LY)^{1/(r+2s)}$ at each place, for some constant c_{10} depending only on F. The number of such units is $\ll \log(\mathcal{N}(\mathfrak{q}))^{r+s-1}$ by (8) and (9). Then the above display is

$$\ll \mathcal{N}(v,p)^{4\alpha_1}\mathcal{N}(\mathfrak{q})^{2\varepsilon}\sum_{\substack{0\neq(q)\subseteq\mathfrak{q}\mathfrak{n}\\ \mathcal{N}((q))\ll LY}}\frac{\mathcal{N}(\gcd(\mathfrak{c},(q)))}{\sqrt{\mathcal{N}((q))}}.$$

Here, the sum is $\ll \mathcal{N}(\mathfrak{q})^{-1+\varepsilon}(LY)^{1/2}$, since $\gcd(\mathfrak{c},(q))=\gcd(\mathfrak{c}_\pi,(q))$, which has norm $O_{F,\pi}(1)$.

Altogether, using again (8), in (14), the Eisenstein spectrum has contribution

$$\ll \mathcal{N}(v,p)^{4\alpha_1}\mathcal{N}(\mathfrak{q})^{-1/2+\varepsilon}L^{1/2}, \tag{16}$$

which is analogous to [BH10, (116)].

4.4. **Off-diagonal contribution: cuspidal spectrum.** Now we turn to the cuspidal part of (14), and since the relevant set is countable, we will write a sum in place of the integral there.

Set

$$\mathscr{C}(\mathfrak{c},\varepsilon)=\{\varpi\in\mathscr{C}(\mathfrak{c}):\mathcal{N}(\mathbf{r}_\varpi)\leq\mathcal{N}(\mathfrak{q})^{\varepsilon}\}.$$

Later we will prove that the contribution of representations outside $\mathscr{C}(\mathfrak{c},\varepsilon)$ is small. So restrict to $\mathscr{C}(\mathfrak{c},\varepsilon)$, and fix also the sign of q as follows. For any sign $\xi\in\{\pm1\}^r$, set

$$\mathscr{B}(\xi)=\{y\in\mathscr{B}:\operatorname{sign}(y)=\xi\}.$$

Then focus on the quantity

$$\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)}\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}}W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/(r+2s)}};v,p\right). \tag{17}$$

We follow again [BH10]. Consider the Mellin transform

$$\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)=\int_{F^\times_{\infty,+}}W_{\varpi,\mathfrak{t}}(\xi y;v,p)\prod_{j=1}^{r+s}|y_j|^{v'_j}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{p'_j}d^\times_\infty y. \tag{18}$$

We would like to invert this. As for p', observe that $W_{\varpi,\mathfrak{t}}(y;v,p)$ is continuous on the set where each $|y_j|$ is fixed (which is the product of s circles), so the standard Fourier analysis of the circle group is applicable.

Lemma 7. *Conditions as in* [Mag, Theorem 1]. *Assume $\mathscr{D}$ is a differential operator as* [Mag, Proposition 3.6]. *Then for any $0<\varepsilon<1/4$ and nonnegative integers b,c', we have, for all $y\in F_\infty^\times$,*

$$\sum_{\varpi\in\mathscr{C}(\mathfrak{c})}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}(\mathcal{N}(\mathbf{r}_\varpi))^{c'}|\mathscr{D}W_{\varpi,\mathfrak{t}}(y)|\ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c',P}\mathcal{N}((l_1l_2))^{1/2+\varepsilon}||W_1||_{S_{\alpha'}}||W_2||_{S_{\alpha'}}$$
$$\cdot\prod_{j=1}^{r}(|y_j|^{1/2-\varepsilon}+|y_j|^{1/2-\theta-\varepsilon})(\min(1,|y_j|^{-b}))\prod_{j=r+1}^{r+s}(|y_j|^{3/4}+|y_j|)(\min(1,|y_j|^{-b}))$$

with $\alpha'=2(3r+4s+2)+(r+s)(a+b+2c'+4(r+2s))+2(7r+18s)$.

Proof. The same as the proof of Lemma 6 above, the only difference is that we use Corollary 4 instead of Corollary 2. ∎

From this, it is clear that the set $(i\mathbf{R})^{r+s}$ (which is the product of $r+s$ lines) can be used for Mellin inversion (see [GR07, 17.41]). Therefore, (17) is

$$\ll \sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}}(LY)^{(v'_1+\ldots+v'_{r+s})/(r+2s)}$$
$$\cdot\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left(\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)}\frac{\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v'_j}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p'_j}\right)dv'_\infty.$$

By Cauchy-Schwarz, this is

$$\ll \sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}}\left(\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\right|^2\right)^{1/2}$$
$$(19)\qquad \cdot\left(\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)}\frac{\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v'_j}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p'_j}\right|^2\right)^{1/2}|dv'_\infty|.$$

In what follows, we estimate the Mellin part

$$(20)\qquad \left(\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\widehat{W}^{\xi}_{\varpi,\mathfrak{t}}(v',p';v,p)\right|^2\right)^{1/2}$$

and the arithmetic part

$$(21)\qquad \left(\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}}\left|\sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)}\frac{\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}}\prod_{j=1}^{r+s}|q_j|^{-v'_j}\prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p'_j}\right|^2\right)^{1/2}$$

separately.

4.4.1. *Estimate of the Mellin part.* Recall the definition (18) of the Mellin transform. Our plan is to insert differentiations (using that W's are highly differentiable) to show that the Mellin part decays fast in terms of $\mathscr{N}(v',p')$.

At real places ($j\le r$), for $v'_j\neq 0$,

$$\int_{\mathbf{R}^\times_+}W(y_j)y_j^{v'_j}d^\times_{\mathbf{R}}y_j=-\frac{1}{v'_j}\int_{\mathbf{R}^\times_+}y_j\frac{\partial}{\partial y_j}W(y_j)y_j^{v'_j}d^\times_{\mathbf{R}}y_j,$$

so at those real places, where $|v'_j|\ge 1$, we can gain a factor $|v'_j|^{-1}$ using the differential operator $y_j(\partial/\partial y_j)$. The complex places ($j>r$) can be handled similarly. For $v'_j\neq 0$,

$$\int_{\mathbf{C}^\times}W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j}d^\times_{\mathbf{C}}y_j=-\frac{1}{v'_j}\int_{\mathbf{C}^\times}|y_j|\frac{\partial}{\partial|y_j|}W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j}d^\times_{\mathbf{C}}y_j,$$

while for $p'_j\neq 0$,

$$\int_{\mathbf{C}^\times}W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j}d^\times_{\mathbf{C}}y_j=-\frac{1}{ip'_j}\int_{\mathbf{C}^\times}\frac{\partial}{\partial(y_j/|y_j|)}W(y_j)|y_j|^{v'_j}\left(\frac{y_j}{|y_j|}\right)^{p'_j}d^\times_{\mathbf{C}}y_j.$$

This means that at those complex places, where $|v'_j|\ge 1$ (or $|p'_j|\ge 1$, respectively), we can gain a factor $|v'_j|^{-1}$ (or $|p'_j|^{-1}$, respectively), by inserting the differential operator $y(\partial/\partial y)$ (or $\partial/\partial(y/|y|)$, respectively).

A simple calculation shows that for any real-differentiable complex function $f(z)$ with $z=re^{i\theta}$ ($r>0$, $\theta\in[0,2\pi]$), both $r\partial f/\partial r$ and $\partial f/\partial\theta$ are $\ll|z\partial f/\partial z|+|\bar{z}\partial f/\partial\bar{z}|$.

Therefore, set the differential operators

$$\mathscr{D}_{(e,f,g)} = \left(\left(\left(y_j \frac{\partial}{\partial y_j} \right)^{e_j} \right)_{j \le r}, \left(\left(y_j \frac{\partial}{\partial y_j} \right)^{f_j} \right)_{j > r}, \left(\left(\overline{y_j} \frac{\partial}{\partial \overline{y_j}} \right)^{g_j} \right)_{j > r} \right),$$

where $0 \le e_j \le 3$ $(j \le r)$, $0 \le f_j \le 6$, $0 \le g_j \le 6$ $(j > r)$. Then the above argument, together with (18) and Cauchy-Schwarz, implies that (20) is

$$\ll (\mathscr{N}(v',p'))^{-3/2} \sum_{(e,f,g)} \left(\int_{F_{\infty,+}^{\times}} \int_{F_{\infty,+}^{\times}} \left(\sum_{\varpi \in \mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |\mathscr{D}_{(e,f,g)} W_{\varpi,\mathfrak{t}}(y;v,p)|^2 \right)^{1/2} \right.$$
$$\left. \left(\sum_{\varpi \in \mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} |W_{\varpi,\mathfrak{t}}(y';v,p)|^2 \right)^{1/2} d_{\infty}^{\times} y \, d_{\infty}^{\times} y' \right)^{1/2}.$$

Now we apply [Mag, Theorem 1] with $a = 6, b = 2, c = 0$ in the first sum, and with $a = 0, b = 2, c = 0$ in the second sum. Together with (15), this implies that the integrand is

$$\ll \mathscr{N}(\mathfrak{q})^{\varepsilon} \mathscr{N}(v,p)^{4\alpha_2} \prod_{j=1}^{r} \min(|y_j|^{1/4}, |y_j|^{-3/2}) \min(|y_j'|^{1/4}, |y_j'|^{-3/2})$$
$$\cdot \prod_{j=r+1}^{r+s} \min(|y_j|^{3/4}, |y_j|^{-1}) \min(|y_j'|^{3/4}, |y_j'|^{-1})$$

with some positive integer α_2 depending only on F. Altogether, the Mellin part (20) is

$$\ll \mathscr{N}(\mathfrak{q})^{\varepsilon} \mathscr{N}(v,p)^{4\alpha_2} \mathscr{N}(v',p')^{-3/2}. \tag{22}$$

4.4.2. *Estimate of the arithmetic part.* Our next goal is to give a bound on (21), which is uniform in v', p'. Fix v', p' and consider

$$\sum_{\varpi \in \mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t} | \mathfrak{c}\mathfrak{c}_{\varpi}^{-1}} \left| \sum_{q \in \mathfrak{q}\mathfrak{n} \cap \mathscr{B}(\xi)} \frac{\lambda_{\varpi}^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s} |q_j|^{-v_j'} \prod_{j=r+1}^{r+s} \left(\frac{q_j}{|q_j|} \right)^{-p_j'} \right|^2. \tag{23}$$

Following [BH10, p.45], introduce, for any ideal $\mathfrak{a} \subseteq \mathfrak{o}$,

$$f(\mathfrak{a}; v', p') = \sum_{\substack{q \in \mathscr{B}(\xi) \\ (q) = \mathfrak{a}\mathfrak{n}}} \prod_{j=1}^{r+s} |q_j|^{-v_j'} \prod_{j=r+1}^{r+s} \left(\frac{q_j}{|q_j|} \right)^{-p_j'}.$$

The number of possible units ε for which $q\varepsilon \in \mathscr{B}$ is $\ll_{F,\varepsilon} \mathscr{N}(\mathfrak{q})^{\varepsilon}$ (recall the argument in Section 4.3.2), hence

$$|f(\mathfrak{a}; v', p')| \ll_{F,\varepsilon} \mathscr{N}(\mathfrak{q})^{\varepsilon}. \tag{24}$$

With this notation, we can rewrite the innermost sum in (23) as

$$\sum_{q \in \mathfrak{q}\mathfrak{n} \cap \mathscr{B}(\xi)} \frac{\lambda_{\varpi}^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s} |q_j|^{-v_j'} \prod_{j=r+1}^{r+s} \left(\frac{q_j}{|q_j|} \right)^{-p_j'} = \sum_{\mathscr{N}(\mathfrak{m}) \ll LY/\mathscr{N}(\mathfrak{q}\mathfrak{n})} \frac{\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathscr{N}(\mathfrak{m}\mathfrak{q})}} f(\mathfrak{m}\mathfrak{q}; v', p'),$$

where $\ll$ in the sum means that we may choose a constant depending only on F such that this holds. Now on the right-hand side, for each occurring $\mathfrak{m}$, transfer each prime factor dividing both $\mathfrak{m}$ and $\mathfrak{q}$ from $\mathfrak{m}$ to $\mathfrak{q}$. This does not affect the summand (since it depends only on the product $\mathfrak{m}\mathfrak{q}$) and lets us write

$$\sum_{\mathscr{N}(\mathfrak{m})\ll LY/\mathscr{N}(\mathfrak{q}\mathfrak{n})} \frac{\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q})}{\sqrt{\mathscr{N}(\mathfrak{m}\mathfrak{q})}} f(\mathfrak{m}\mathfrak{q};\nu',p') = \sum_{\mathfrak{q}|\mathfrak{q}'|\mathfrak{q}^{\infty}} \sum_{\substack{\mathscr{N}(\mathfrak{m})\ll LY/\mathscr{N}(\mathfrak{q}'\mathfrak{n}) \\ \gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}} \frac{\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')}{\sqrt{\mathscr{N}(\mathfrak{m}\mathfrak{q}')}} f(\mathfrak{m}\mathfrak{q}';\nu',p'). \tag{25}$$

The following lemma (which is based on [BHM07, pp.73-74]) expresses $\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}')$.

Lemma 8. *Assume* $\mathfrak{m}$ *and* $\mathfrak{q}'$ *are coprime ideals in* $\mathfrak{o}$. *Then*

$$\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}') = \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}).$$

Proof. We follow [BHM07, pp.73-74]. (At [BH10, p.45], [BHM07, pp.73-74] is adapted incorrectly. The corrected version can be found in the erratum of [BH10].)

By [Mag, (2.13-14)], we have

$$\begin{aligned}
\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}') &= \sum_{\mathfrak{s}|\gcd(\mathfrak{m}\mathfrak{q}',\mathfrak{t})} \alpha_{\mathfrak{t},\mathfrak{s}}\mathscr{N}(\mathfrak{s})^{1/2}\lambda_{\varpi}(\mathfrak{m}\mathfrak{q}'\mathfrak{s}^{-1}) \\
&= \sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t}) \\ \mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})}} \alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathscr{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_{\varpi}(\mathfrak{m}\mathfrak{q}'\mathfrak{s}_1^{-1}\mathfrak{s}_2^{-1}) \\
&= \sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t}) \\ \mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})}} \alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathscr{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_{\varpi}(\mathfrak{q}'\mathfrak{s}_2^{-1})\lambda_{\varpi}(\mathfrak{m}\mathfrak{s}_1^{-1}),
\end{aligned}$$

where the last equation holds by $\gcd(\mathfrak{m},\mathfrak{q}')=\mathfrak{o}$ and the multiplicativity of Hecke eigenvalues.

Inverting the multiplicativity relation, we see that

$$\begin{aligned}
\lambda_{\varpi}(\mathfrak{q}'\mathfrak{s}_2^{-1}) &= \lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\cdot\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1}) \\
&= \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1})} \mu(\mathfrak{b})\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_{\varpi}(\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{s}_2^{-1}\mathfrak{b}^{-1}).
\end{aligned}$$

Writing this into the above display, we obtain that

$$\begin{aligned}
\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\mathfrak{q}') &= \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \\
&\qquad \sum_{\substack{\mathfrak{s}_1|\gcd(\mathfrak{m},\mathfrak{t}) \\ \mathfrak{s}_2|\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}}} \alpha_{\mathfrak{t},\mathfrak{s}_1\mathfrak{s}_2}\mathscr{N}(\mathfrak{s}_1\mathfrak{s}_2)^{1/2}\lambda_{\varpi}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}\mathfrak{s}_1^{-1}\mathfrak{s}_2^{-1}) \\
&= \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \\
&\qquad \sum_{\mathfrak{s}|\gcd(\mathfrak{t},\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1})} \alpha_{\mathfrak{t},\mathfrak{s}}\mathscr{N}(\mathfrak{s})^{1/2}\lambda_{\varpi}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}\mathfrak{s}^{-1}) \\
&= \sum_{\mathfrak{b}|\gcd(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1},\gcd(\mathfrak{q}',\mathfrak{t}))} \mu(\mathfrak{b})\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1})\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m}\gcd(\mathfrak{q}',\mathfrak{t})\mathfrak{b}^{-1}),
\end{aligned}$$

which completes the proof. □

Using this in (25), and noting

$$\lambda_{\pi}(\mathfrak{m}) \ll_{\varepsilon} \mathscr{N}(\mathfrak{m})^{\theta+\varepsilon},$$

we obtain

$$\lambda_{\varpi}(\mathfrak{q}'\gcd(\mathfrak{q}',\mathfrak{t})^{-1}\mathfrak{b}^{-1}) \ll \mathscr{N}(\mathfrak{q}')^{\theta+\varepsilon}.$$

We claim $\gcd(\mathfrak{q}',\mathfrak{t})|\mathfrak{c}_\pi$. Indeed, $\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}$ with $\mathfrak{c}=\mathfrak{c}_\pi\mathrm{lcm}((l_1),(l_2))$, where l_1,l_2 are primes not dividing $\mathfrak{q}$. Altogether, the q-sum in (23) can be estimated as

$$\ll \sum_{\mathfrak{q}|\mathfrak{q}'|\mathfrak{q}^\infty} \mathcal{N}(\mathfrak{q}')^{-1/2+\theta+\varepsilon} \sum_{\mathfrak{b}|\mathfrak{c}_\pi} \left| \sum_{\substack{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q}'\mathfrak{n}) \\ \gcd(\mathfrak{m},\mathfrak{q})=\mathfrak{o}}} \frac{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}\mathfrak{b})}{\sqrt{\mathcal{N}(\mathfrak{m})}} f(\mathfrak{m}\mathfrak{q}';v',p') \right|. \tag{26}$$

Now take the function h defined in Section 3.1 with $a_j=\mathcal{N}(\mathfrak{q})^{2\varepsilon}$ at real, $a_j=\mathcal{N}(\mathfrak{q})^{\varepsilon}$ at complex places, $b_j=\sqrt{a_j}$ at all archimedean places, finally $a_j'=-1$ at complex places. This has the property that it gives weight $\gg 1$ to representations in $\mathscr{C}(\mathfrak{c},\varepsilon)$.

$$\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2$$
$$\ll \sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} h(\mathbf{r}_\varpi) \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2.$$

In the summation over ϖ, multiply by a factor C_ϖ^{-1}, which is $\gg \mathcal{N}(\mathfrak{q})^{-\varepsilon}$ by [BH10, Section 2.9] and [Mag13b, Chapter 3]. We also add the analogous nonnegative contribution of the Eisenstein spectrum (denoted by CSC).

Therefore, using (24), (26) estimates the ϖ-sum of (23) as

$$\begin{aligned}
&\sum_{\varpi\in\mathscr{C}(\mathfrak{c},\varepsilon)} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} \left| \sum_{q\in\mathfrak{q}\mathfrak{n}\cap\mathscr{B}(\xi)} \frac{\lambda_\varpi^{\mathfrak{t}}(q\mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q\mathfrak{n}^{-1})}} \prod_{j=1}^{r+s}|q_j|^{-v_j'} \prod_{j=r+1}^{r+s}\left(\frac{q_j}{|q_j|}\right)^{-p_j'} \right|^2 \\
&\ll \mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon} \max_{\mathfrak{b}_1,\mathfrak{b}_2|\mathfrak{c}_\pi} \sum_{\mathcal{N}(\mathfrak{m}_1),\mathcal{N}(\mathfrak{m}_2)\ll LY/\mathcal{N}(\mathfrak{q})} \frac{1}{\sqrt{\mathcal{N}(\mathfrak{m}_1\mathfrak{m}_2)}} \\
&\qquad \left| \sum_{\varpi\in\mathscr{C}(\mathfrak{c})} C_\varpi^{-1} \sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\varpi^{-1}} h(\mathbf{r}_\varpi)\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}_1\mathfrak{b}_1)\overline{\lambda_\varpi^{\mathfrak{t}}(\mathfrak{m}_2\mathfrak{b}_2)} + CSC \right|.
\end{aligned} \tag{27}$$

We apply the Kuznetsov formula to estimate the last line of (27), with $\alpha=\alpha'=1$, $\mathfrak{a}^{-1}=\mathfrak{m}_1\mathfrak{b}_1$, $\mathfrak{a}'^{-1}=\mathfrak{m}_2\mathfrak{b}_2$. The delta term is, up to a constant multiple,

$$[K(\mathfrak{o}):K(\mathfrak{c})]\Delta(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2)\int h(\mathbf{r}_\varpi)d\mu.$$

Here, by Section 3.1, the integral of h gives $\ll \mathcal{N}(\mathfrak{q})^{2(r+s)\varepsilon}$, and also $[K(\mathfrak{o}):K(\mathfrak{c})]\ll L^2\mathcal{N}(\mathfrak{q})^{\varepsilon}$ by (9). When $\Delta(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2)\neq 0$, $\mathcal{N}(\mathfrak{m}_1)\asymp_{F,\pi}\mathcal{N}(\mathfrak{m}_2)$, so the sum over $\mathfrak{m}_1,\mathfrak{m}_2$ can be replaced by a sum over $\mathfrak{m}$. Using (8), we see that $LY/\mathcal{N}(\mathfrak{q})\ll\mathcal{N}(\mathfrak{q})^{\varepsilon}L$, and taking into account also (9), we obtain that

$$\sum_{\mathcal{N}(\mathfrak{m})\ll LY/\mathcal{N}(\mathfrak{q})} \frac{1}{\mathcal{N}(\mathfrak{m})} \ll \mathcal{N}(\mathfrak{q})^{\varepsilon}.$$

Altogether, the delta term of the geometric side of the Kuznetsov formula contributes

$$\ll \mathcal{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}L^2 \tag{28}$$

to the right-hand side of (27).

As for the Kloosterman term, similarly to (3), we have to estimate

$$\max_{\mathfrak{a}\in C}\sum_{\varepsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}}\sum_{0\neq c\in\mathfrak{m}_1^{-1}\mathfrak{b}_1^{-1}\mathfrak{a}\mathfrak{c}}\frac{\mathscr{N}((\gcd(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2,c\mathfrak{m}_1\mathfrak{b}_1\mathfrak{a}^{-1})))^{1/2}}{\mathscr{N}(c\mathfrak{m}_1\mathfrak{b}_1\mathfrak{a}^{-1})^{1/2-\varepsilon}}$$
$$\cdot\prod_{j\leq r}\min(1,|\varepsilon_j\gamma_{\mathfrak{a},j}/c_j|^{1/2})\prod_{j>r}\min(1,|\varepsilon_j\gamma_{\mathfrak{a},j}/c_j|),$$

where $\gamma_\mathfrak{a}$ is a totally positive generator of the ideal $\mathfrak{a}^2(\mathfrak{m}_1\mathfrak{b}_1)^{-1}\mathfrak{m}_2\mathfrak{b}_2$, C is a fixed set of narrow class representatives (depending only on F and the narrow class of $(\mathfrak{m}_1\mathfrak{b}_1)^{-1}\mathfrak{m}_2\mathfrak{b}_2$) with the property that such a $\gamma_\mathfrak{a}$ exists for each $\mathfrak{a}\in C$. The sum over $\varepsilon\in\mathfrak{o}_+^\times/\mathfrak{o}^{2\times}$ is negligible. Now take a totally positive $\beta\in\mathfrak{o}$ such that $(\beta)\supseteq\mathfrak{m}_1\mathfrak{b}_1$, $\mathscr{N}((\beta))\gg\mathscr{N}(\mathfrak{m}_1\mathfrak{b}_1)$, and then the above is

$$\ll\max_{\mathfrak{a}\in C}\sum_{0\neq c\in\mathfrak{a}\mathfrak{c}}\frac{\mathscr{N}((\gcd(\mathfrak{m}_1\mathfrak{b}_1,\mathfrak{m}_2\mathfrak{b}_2,c\mathfrak{a}^{-1})))^{1/2}}{\mathscr{N}(c\mathfrak{a}^{-1})^{1/2-\varepsilon}}$$
$$\cdot\prod_{j\leq r}\min(1,|\gamma_{\mathfrak{a},j}\beta_j|^{1/4}/|c_j|^{1/2})\prod_{j>r}\min(1,|\gamma_{\mathfrak{a},j}\beta_j|^{1/2}/|c_j|),$$

Then the same method as in the proof of Lemma 3 shows that the previous display can be estimated as

$$\ll\mathscr{N}((\gamma_\mathfrak{a}\beta))^{1/4+\varepsilon}\mathscr{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2+\varepsilon}\mathscr{N}(\mathfrak{c})^{-1-\varepsilon}.$$

The last factor $\mathscr{N}(\mathfrak{c})^{-1-\varepsilon}$ cancels $[K(\mathfrak{o}):K(\mathfrak{c})]$. Noting that $\mathscr{N}((\gamma_\mathfrak{a}\beta))\ll\mathscr{N}(\mathfrak{m}_1\mathfrak{m}_2)$, we see that the Kloosterman term contributes

$$\ll\mathscr{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}\sum_{\mathscr{N}(\mathfrak{m}_1),\mathscr{N}(\mathfrak{m}_2)\ll LY/\mathscr{N}(\mathfrak{q})}\mathscr{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2+\varepsilon}\mathscr{N}(\mathfrak{m}_1\mathfrak{m}_2)^{-1/4+\varepsilon}$$

to the right-hand side of (27). Obviously

$$\mathscr{N}(\gcd(\mathfrak{m}_1,\mathfrak{m}_2,\mathfrak{c}))^{1/2}\leq\mathscr{N}(\gcd(\mathfrak{m}_1,\mathfrak{c}))^{1/4}\mathscr{N}(\gcd(\mathfrak{m}_2,\mathfrak{c}))^{1/4},$$

so the above display is (using also (8) and (9) again)

$$\ll\mathscr{N}(\mathfrak{q})^{-1+2\theta+2\varepsilon}\left(\sum_{\mathscr{N}(\mathfrak{m})\ll LY/\mathscr{N}(\mathfrak{q})}\left(\frac{\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))}{\mathscr{N}(\mathfrak{m})}\right)^{1/4}\right)^2.$$

Here, if $\mathfrak{m}$ is divisible by l_1 or l_2, then $\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))\ll L$ (by (9), an ideal of norm $\ll\mathscr{N}(\mathfrak{q})^\varepsilon L$ cannot have two different prime divisors l_1,l_2), this happens at most for $\mathscr{N}(\mathfrak{q})^\varepsilon$ many $\mathfrak{m}$'s. For other $\mathfrak{m}$'s, $\mathscr{N}(\gcd(\mathfrak{m},\mathfrak{c}))\ll 1$. Therefore, the Kloosterman contribution to (27) is

(29) $$\ll\mathscr{N}(\mathfrak{q})^{-1+2\theta+\varepsilon}L^{3/2}.$$

Taking square-roots, we obtain from (28) and (29) that the arithmetic part (21) is

(30) $$\ll\mathscr{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L.$$

4.4.3. *Summing up in the cuspidal spectrum.* Inside $\mathscr{C}(\mathfrak{c},\varepsilon)$, (19), (22) and (30) show that the contribution (17) is

$$\ll\sum_{p'\in\mathbf{Z}^s}\int_{(i\mathbf{R})^{r+s}}\mathscr{N}(v,p)^{4\alpha_2}\mathscr{N}(v',p')^{-3/2}\mathscr{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L|dv'_\infty|$$
$$\ll\mathscr{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L\mathscr{N}(v,p)^{4\alpha_2},$$

and this bound holds (with the implicit constant multiplied by 2^r) without restricting the summation in (19) to a specific sign ξ.

Now we concentrate on representations outside $\mathscr{C}(\mathfrak{c},\varepsilon)$. First of all, from Lemma 3, we see that

$$\lambda_\varpi^\mathfrak{t}(q\mathfrak{n}^{-1})\ll L^{1/2+\varepsilon}\mathscr{N}((q))^{1/4+\varepsilon}\mathscr{N}(\mathbf{r}_\varpi),$$

therefore, with a large c' (depending on ε), we may write (using (9)), outside $\mathscr{C}(\mathfrak{c},\varepsilon)$,

$$\frac{\lambda^{\mathfrak{t}}_{\varpi}(q\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(q\mathfrak{n}^{-1})}} \ll L^{1/2+\varepsilon}\mathscr{N}(\mathfrak{q})^{-1/4}\mathscr{N}(\mathbf{r}_{\varpi}) \ll \mathscr{N}(\mathbf{r}_{\varpi})^{c'}.$$

Now by Cauchy-Schwarz, outside $\mathscr{C}(\mathfrak{c},\varepsilon)$, the cuspidal contribution is, with some c much larger than c',

$$\ll \left(\sum_{0\neq q\in\mathfrak{qn}\cap\mathscr{B}}\sum_{\varpi\notin\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}}\mathscr{N}(\mathbf{r}_{\varpi})^{2(c'-c)}\right)^{1/2}$$
$$\cdot\left(\sum_{0\neq q\in\mathfrak{qn}\cap\mathscr{B}}\sum_{\varpi\notin\mathscr{C}(\mathfrak{c},\varepsilon)}\sum_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_{\varpi}^{-1}}\left|\mathscr{N}(\mathbf{r}_{\varpi})^{c}W_{\varpi,\mathfrak{t}}\left(\frac{q}{(LY)^{1/(r+2s)}};v,p\right)\right|^{2}\right)^{1/2}.$$

The first factor is $\ll_k \mathscr{N}(\mathfrak{q})^{-k}$ for any $k\in\mathbf{N}$, if $c-c'$ is large enough, as it follows from Corollary 4. As for the second factor, apply [Mag, Theorem 1] with $a=0$, $b=0$ and the above c. The number of q's in $\mathfrak{qn}\cap\mathscr{B}$ is $O_F(LY)$. Then together with (15), we see that the second factor is $\ll \mathscr{N}(\mathfrak{q})^{-1+\varepsilon}\mathscr{N}(v,p)^{4\alpha_3}$ with some positive integer α_3 depending only on F. To match Y and L with $\mathscr{N}(\mathfrak{q})$, we use (8) and (9) throughout.

Altogether, the cuspidal spectrum has contribution

$$\ll \mathscr{N}(v,p)^{4\max(\alpha_2,\alpha_3)}\mathscr{N}(\mathfrak{q})^{-1/2+\theta+\varepsilon}L. \tag{31}$$

4.5. **Choice of the amplification length.** Set $\alpha=\max(\alpha_1,\alpha_2,\alpha_3)$. Summing trivially over l_1,l_2, and using (12), (14), (16) and (31), we see

$$ODC \ll \mathscr{N}(v,p)^{4\alpha}\mathscr{N}(\mathfrak{q})^{1/2+\theta+\varepsilon}L.$$

This estimate, together with (11) and through (10), gives rise to

$$|\mathscr{L}_{\chi_{\mathrm{fin}}}(v,p)|^2 \ll \mathscr{N}(v,p)^{4\alpha}(\mathscr{N}(\mathfrak{q})^{1+\varepsilon}L^{-1}+\mathscr{N}(\mathfrak{q})^{1/2+\theta+\varepsilon}L),$$
$$|\mathscr{L}_{\chi_{\mathrm{fin}}}(v,p)| \ll \mathscr{N}(v,p)^{2\alpha}(\mathscr{N}(\mathfrak{q})^{1/2+\varepsilon}L^{-1/2}+\mathscr{N}(\mathfrak{q})^{1/4+\theta/2+\varepsilon}L^{1/2}).$$

We see that the optimal choice is $L=\mathscr{N}(\mathfrak{q})^{1/4-\theta/2}$, which meets the condition (9). With this, we obtain the bound

$$|\mathscr{L}_{\chi_{\mathrm{fin}}}(v,p)| \ll \mathscr{N}(v,p)^{2\alpha}\mathscr{N}(\mathfrak{q})^{3/8+\theta/4+\varepsilon}. \tag{32}$$

4.6. **Completion of the proof.** In the derivation of the subconvex bound on $L(1/2,\pi\otimes\chi)$, the starting point is [BH10, (75)], a consequence of the approximate functional equation [Har02, Theorem 2.1] (see also [Mag13b, Section 3.3]): there is a constant $c=c(F,\pi,\chi_\infty,\varepsilon)>0$ and a smooth function $V:(0,\infty)\to\mathbf{C}$ supported on $[1/2,2]$, satisfying $V^{(j)}(y)\ll_{F,\pi,\chi_\infty,j}1$ for each nonnegative integer j, such that

$$L(1/2,\pi\otimes\chi)\ll_{F,\pi,\chi_\infty,\varepsilon}\mathscr{N}(\mathfrak{q})^{\varepsilon}\max_{Y\leq c\mathscr{N}(\mathfrak{q})^{1+\varepsilon}}\left|\sum_{0\neq\mathfrak{m}\subseteq\mathfrak{o}}\frac{\lambda_\pi(\mathfrak{m})\chi(\mathfrak{m})}{\sqrt{\mathscr{N}(\mathfrak{m})}}V\left(\frac{\mathscr{N}(\mathfrak{m})}{Y}\right)\right|. \tag{33}$$

First of all, we split up the sum on the right-hand side of (33) over ideals according to their narrow class (with representatives $\mathfrak{n}$ satisfying (4)). Then

$$L(1/2,\pi\otimes\chi)\ll_{F,\pi,\chi_\infty,\varepsilon}\mathscr{N}(\mathfrak{q})^{\varepsilon}\max_{Y\leq c\mathscr{N}(\mathfrak{q})^{1+\varepsilon}}\left|\sum_{0<<t\in\mathfrak{n}\pmod{\mathfrak{o}_+^{\times}}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi(t\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(t\mathfrak{n}^{-1})}}V\left(\frac{|t|_\infty}{Y}\right)\right|$$

for some $c=c(F,\pi,\chi_\infty,\varepsilon)$, hence (8) is satisfied. Here, by the partition of unity introduced in the beginning of this section, the sum on the right-hand side can be rewritten as

$$\sum_{0<<t\in\mathfrak{n}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi(t\mathfrak{n}^{-1})}{\sqrt{\mathscr{N}(t\mathfrak{n}^{-1})}}G(t_\infty)V\left(\frac{|t|_\infty}{Y}\right)W\left(\frac{t_\infty}{Y^{1/(r+2s)}}\right),$$

where W is a smooth nonnegative function which is 1 on $[[c_1,c_2]]$ and supported on $[[c_3,c_4]]$. Now introducing the Mellin transform

$$\widehat{V}(v,p)=\int_{F_{\infty,+}^{\times}} G(y)V(y)\chi_\infty(y)\prod_{j=1}^{r+s}|y_j|^{v_j}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{p_j}d^{\times}y,$$

we have, by Mellin inversion, that the above display is

$$\ll_F \sum_{p\in\mathbf{Z}^s}\int_{v\in(i\mathbf{R})^{r+s}}\widehat{V}(v,p)\sum_{0<<t\in\mathfrak{n}}\frac{\lambda_\pi(t\mathfrak{n}^{-1})\chi_{\mathrm{fin}}(t)}{\sqrt{\mathcal{N}(t\mathfrak{n}^{-1})}}W_{v,p}\left(\frac{t}{Y^{1/(r+2s)}}\right)dv,$$

where

$$W_{v,p}(y)=W(y)\prod_{j=1}^{r+s}|y_j|^{-v_j}\prod_{j=r+1}^{r+s}\left(\frac{y_j}{|y_j|}\right)^{-p_j}d^{\times}y.$$

Since $F(y)$, $V(y)$, $W(y)$ are all smooth and compactly supported, we see that

$$\widehat{V}(v,p)\ll_{F,\pi,\chi_\infty,\varepsilon,\beta}\mathcal{N}(v,p)^{-\beta}$$

for all $\beta\in\mathbf{N}$ and also that the family of $W_{v,p}$'s satisfies (i) and (ii). Then

$$L(1/2,\pi\otimes\chi)\ll_{F,\pi,\chi_\infty,\varepsilon,\beta}\sum_{p\in\mathbf{Z}^s}\int_{v\in(i\mathbf{R})^{r+s}}\mathscr{L}_{\chi_{\mathrm{fin}}}(v,p)\mathcal{N}(v,p)^{-\beta}dv$$

with $\mathscr{L}$ of (7) satisfying all conditions we needed in its estimate. Now taking a β which is much larger than 2α, (32) completes the proof.

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