

# THE SPECTRAL DECOMPOSITION OF SHIFTED CONVOLUTION SUMS OVER NUMBER FIELDS

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ABSTRACT. Let  $\pi_1, \pi_2$  be cuspidal automorphic representations of  $\mathrm{GL}_2$  over a number field  $F$  with Hecke eigenvalues  $\lambda_{\pi_1}(\mathfrak{m}), \lambda_{\pi_2}(\mathfrak{m})$ . For nonzero integers  $l_1, l_2 \in F$  and compactly supported functions  $W_1, W_2$  on  $F_\infty^\times$ , a spectral decomposition of the shifted convolution sum

$$\sum_{\substack{l_1 t_1 - l_2 t_2 = q \\ 0 \neq t_1, t_2 \in \mathfrak{n}}} \frac{\lambda_{\pi_1}(t_1 \mathfrak{n}^{-1}) \overline{\lambda_{\pi_2}(t_2 \mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1 t_2 \mathfrak{n}^{-2})}} W_1(l_1 t_1) \overline{W_2(l_2 t_2)}$$

is obtained for any nonzero fractional ideal  $\mathfrak{n}$  and any nonzero element  $q \in \mathfrak{n}$ .

## 1. INTRODUCTION

Assume that  $\pi_1, \pi_2$  are cuspidal automorphic representations of  $\mathrm{GL}_2$  over a number field  $F$  with Hecke eigenvalues  $\lambda_{\pi_1}(\mathfrak{m}), \lambda_{\pi_2}(\mathfrak{m})$ . Fix a nonzero fractional ideal  $\mathfrak{n}$  and some nonzero element  $q \in \mathfrak{n}$ . The so-called shifted convolution sums

$$(1) \quad \sum_{\substack{l_1 t_1 - l_2 t_2 = q \\ 0 \neq t_1, t_2 \in \mathfrak{n}}} \frac{\lambda_{\pi_1}(t_1 \mathfrak{n}^{-1}) \overline{\lambda_{\pi_2}(t_2 \mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1 t_2 \mathfrak{n}^{-2})}} W_1(l_1 t_1) \overline{W_2(l_2 t_2)}$$

(for integers  $l_1, l_2 \in F$  and rapidly decaying functions  $W_1, W_2$  on  $F_\infty^\times$ ) proved to be important in analytic number theory: they are variants of the classical additive divisor sums and have several applications. It was shown by Blomer and Harcos in [BH08] and [BH10] that if  $F$  is totally real, then (1) decomposes spectrally over the full automorphic spectrum, and the spectral components can be estimated in terms of higher order Sobolev norms of  $W_1, W_2$ . In this paper, we work out the spectral decomposition and the relevant estimates over any number field  $F$ .

In 1965, Selberg [Sel65] proved that for classical holomorphic cusp forms  $\pi_1, \pi_2$  on the upper half-plane, and for any integer  $h > 0$ , the Dirichlet series

$$\sum_{\substack{m, n \geq 1 \\ m - n = h}} \frac{\lambda_{\pi_1}(m) \lambda_{\pi_2}(n) (mn)^\beta}{(m + n)^{s + \beta}}$$

is holomorphic for  $\Re s > 1/2$ , assuming  $\beta$  is a sufficiently large integer. This result was extended to arbitrary cusp forms over a totally real number field in [BH08] and [BH10] by decomposing the corresponding shifted convolution sums spectrally.

Shifted convolution sums have numerous applications to moment bounds for  $L$ -functions. For example, in the investigation of the fourth moment of the Riemann zeta function on the critical line, off-diagonal terms lead to the problem of the asymptotic behavior of additive divisor sums (see [Ing26], [Ing27], [Est31], [HB79], [Mot94], [Mot97]). Analogously, the second moment of an  $L$ -function corresponding to a  $\mathrm{GL}_2$  cusp form leads to shifted convolution sums of Hecke eigenvalues (see [Goo81a], [Goo81b]).

Shifted convolution sums also play a central role in the subconvexity problem of  $L$ -functions, as they arise naturally from averaging in short families or by employing amplifiers (which is an arithmetic way to

shorten the family). This program was initiated by Duke, Friedlander and Iwaniec [DFI93] who applied the circle method, then continued by Cogdell, Piatetski-Shapiro and Sarnak [CPSS] who used the spectral decomposition for holomorphic Hilbert modular forms, and by Blomer and Harcos [BH10] who established a Burgess type subconvex bound for twisted  $\mathrm{GL}_2$   $L$ -functions over totally real number fields (see also [Sar01] and [Ven10]). The author extended this result to any number field in his PhD thesis [Mag13b]. We remark that the same estimate was simultaneously established by Wu [Wu14] by a different method based on the work of Michel and Venkatesh [MV10].

**1.0.1. Organization of the paper.** In Section 2, we introduce the notions of automorphic theory which are necessary for the results in this paper, and we conclude by stating the spectral decomposition in Theorem 1. In Section 3, we derive some bounds on cusp forms and the corresponding Kirillov vectors. Finally, in Section 4, we prove Theorem 1.

## 2. BACKGROUND ON AUTOMORPHIC THEORY

**2.1. Notations.** First we introduce the notations we shall use later. We advise the reader to consult [Wei74] for the arising notions.

**2.1.1. The number field.** Throughout this paper,  $X \ll_A Y$  means that  $|X| \leq cY$  for some constant  $c > 0$  depending only on  $A$ .

Let  $F$  be a number field, a finite algebraic extension of  $\mathbf{Q}$ . Assuming  $F$  has  $r$  real and  $s$  complex places, we will throughout denote the corresponding archimedean completions by  $F_1, \dots, F_{r+s}$ , where  $F_1, \dots, F_r$  are all isomorphic to  $\mathbf{R}$  and  $F_{r+1}, \dots, F_{r+s}$  are all isomorphic to  $\mathbf{C}$  as topological fields. Let  $F_\infty$  stand for the direct sum of these fields (as rings),  $F_\infty^\times$  for its multiplicative group,  $F_{\infty,+}^\times$  for the totally positive elements (which are positive at each real place), and  $F_{\infty,+}^{\mathrm{diag}}$  for  $\{(a_1, \dots, a_{r+s}) \in F_{\infty,+}^\times : a_1 = \dots = a_{r+s}\}$ .

Denote by  $\mathfrak{o}$  the ring of integers of  $F$ . The ideals and fractional ideals will be denoted by gothic characters  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ , the prime ideals by  $\mathfrak{p}$  and we keep  $\mathfrak{d}$  for the different and  $D_F$  for the discriminant of  $F$ . Each prime ideal  $\mathfrak{p}$  determines a non-archimedean place and a corresponding completion  $F_{\mathfrak{p}}$ . At such a place, we denote by  $\mathfrak{o}_{\mathfrak{p}}$  the maximal compact subring.

Write  $\mathbf{A}$  for the adèle ring of  $F$ . Given an adèle  $a$ ,  $a_j$  denotes its projection to  $F_j$  for  $1 \leq j \leq r+s$ , and  $a_{\mathfrak{p}}$  the same to  $F_{\mathfrak{p}}$  for a prime ideal  $\mathfrak{p}$ . We will also use the subscripts  $j, \mathfrak{p}$  for the projections of other adelic objects to the place corresponding to  $j, \mathfrak{p}$ , respectively. The subscripts  $\infty$  and  $\mathrm{fin}$  stand for the projections to  $F_\infty$  and  $\prod_{\mathfrak{p}} F_{\mathfrak{p}}$ .

The absolute norm (module) of adeles will be denoted by  $|\cdot|$ , while  $|\cdot|_j$  and  $|\cdot|_{\mathfrak{p}}$  will stand for the norm (module) at single places. Sometimes we will need  $|\cdot|_\infty$ , which is the product of the archimedean norms. (At this point, we call the reader's attention to the notational ambiguity that for a real or complex number  $y$ , we keep the conventional  $|y|$  for its ordinary absolute value. We hope this will not lead to confusion. Note that at real places,  $|y|_j = |y|$ , while at complex places,  $|y|_j = |y|^2$ .) For a fractional ideal  $\mathfrak{a}$ ,  $\mathcal{N}(\mathfrak{a})$  will denote its absolute norm, defined as  $\mathcal{N}(\mathfrak{a}) = |a|^{-1}$ , where  $a$  is any finite representing idele for  $\mathfrak{a}$ . When  $a$  is a finite idele, we may also write  $\mathcal{N}(a)$  for  $|a|^{-1}$ .

We define an additive character  $\psi$  on  $\mathbf{A}$ : it is required to be trivial on  $F$  (embedded diagonally); on  $F_\infty$ :

$$\psi_\infty(x) = \exp(2\pi i \mathrm{Tr}(x)) = \exp(2\pi i(x_1 + \dots + x_r + x_{r+1} + \overline{x_{r+1}} + \dots + x_{r+s} + \overline{x_{r+s}}));$$

while on  $F_{\mathfrak{p}}$ : it is trivial on  $\mathfrak{d}_{\mathfrak{p}}^{-1}$  but not on  $\mathfrak{d}_{\mathfrak{p}}^{-1}\mathfrak{p}^{-1}$ .

2.1.2. *Matrix groups.* Given a ring  $R$ , we define the following subgroups of  $\mathrm{GL}_2(R)$ :

$$\begin{aligned} Z(R) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in R^\times \right\}, \\ B(R) &= \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in R^\times, b \in R \right\}, \\ N(R) &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in R \right\}. \end{aligned}$$

Assume  $0 \neq \mathfrak{n}_p, \mathfrak{c}_p \subseteq \mathfrak{o}_p$ . Then let

$$K_p(\mathfrak{n}_p, \mathfrak{c}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathfrak{o}_p, b \in (\mathfrak{n}_p \mathfrak{d}_p)^{-1}, c \in \mathfrak{n}_p \mathfrak{d}_p \mathfrak{c}_p, ad - bc \in \mathfrak{o}_p^\times \right\},$$

moreover in the special case  $\mathfrak{n}_p = \mathfrak{o}_p$ , we simply write  $K_p(\mathfrak{c}_p)$  instead of  $K_p(\mathfrak{o}_p, \mathfrak{c}_p)$ . For ideals  $0 \neq \mathfrak{n}, \mathfrak{c} \subseteq \mathfrak{o}$ , let

$$K(\mathfrak{n}, \mathfrak{c}) = \prod_{\mathfrak{p}} K_p(\mathfrak{n}_p, \mathfrak{c}_p), \quad K(\mathfrak{c}) = \prod_{\mathfrak{p}} K_p(\mathfrak{c}_p),$$

and taking the archimedean places into account, let

$$K_\infty = \prod_{j=1}^r \mathrm{SO}_2(\mathbf{R}) \times \prod_{j=r+1}^{r+s} \mathrm{SU}_2(\mathbf{C}), \quad K = K_\infty \times K(\mathfrak{o}) \subseteq \mathrm{GL}_2(\mathbf{A}).$$

Finally, for  $0 \neq \mathfrak{n}, \mathfrak{c} \subseteq \mathfrak{o}$ , let

$$\Gamma(\mathfrak{n}, \mathfrak{c}) = \left\{ g_\infty \in \mathrm{GL}_2(F_\infty) : \exists g_{\mathrm{fin}} \in \prod_{\mathfrak{p}} K_p(\mathfrak{n}_p, \mathfrak{c}_p) \text{ such that } g_\infty g_{\mathrm{fin}} \in \mathrm{GL}_2(F) \right\}.$$

We note that the choice of the subgroups  $K$  is not canonical (they can be conjugated arbitrarily), our normalization follows [BH10].

2.1.3. *Archimedean matrix coefficients.* On  $K_\infty$ , we define the matrix coefficients (see [Bum04, p.8]). Again, it is more convenient to give them on the factors. At a real place, on  $\mathrm{SO}_2(\mathbf{R})$ , for a given integer  $q$ , set

$$\Phi_q \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = \exp(iq\theta).$$

At a complex place, on  $\mathrm{SU}_2(\mathbf{C})$ , we introduce the parametrization

$$\mathrm{SU}_2(\mathbf{C}) = \left\{ k[\alpha, \beta] = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Assume now that the integers or half-integers  $p, q, l$  satisfy  $|p|, |q| \leq l$  and  $p \equiv q \equiv l \pmod{1}$ . Then the matrix coefficient  $\Phi_{p,q}^l$  is defined via

$$\sum_{|p| \leq l} \Phi_{p,q}^l(k[\alpha, \beta]) x^{l-p} = (\alpha x - \bar{\beta})^{l-q} (\beta x + \bar{\alpha})^{l+q},$$

where this equation is understood in the polynomial ring  $\mathbf{C}[x]$ , see [BM03, (3.18)] and [LG04, (2.28)]. Note that

$$\|\Phi_{p,q}^l\|_{\mathrm{SU}_2(\mathbf{C})} = \left( \int_{\mathrm{SU}_2(\mathbf{C})} |\Phi_{p,q}^l(k)|^2 dk \right)^{1/2} = \frac{1}{\sqrt{2l+1}} \binom{2l}{l-p}^{1/2} \binom{2l}{l-q}^{-1/2}$$

by [LG04, (2.35)], where the Haar measure on  $\mathrm{SU}_2(\mathbf{C})$  is the probability measure.

2.1.4. *Measures.* On  $F_p$ , we normalize the Haar measure such that  $\mathfrak{o}_p$  has measure 1. On  $F_\infty$ , we use the Haar measure  $|D_F|^{-1/2} dx_1 \cdots dx_r |dx_{r+1} \wedge d\bar{x}_{r+1}| \cdots |dx_{r+s} \wedge d\bar{x}_{r+s}|$ . On  $\mathbf{A}$ , we use the Haar measure  $dx$ , the product of these measures, this induces a Haar probability measure on  $F \backslash \mathbf{A}$  (see [Wei74, Chapter V, Proposition 7]).

On  $\mathbf{R}^\times$ , we use the Haar measure  $d_{\mathbf{R}}^\times y = dy/|y|$ , this gives rise to a Haar measure on  $\mathbf{C}^\times$  as  $d_{\mathbf{C}}^\times y = d_{\mathbf{R}}^\times |y| d\theta/2\pi$ , where  $\exp(i\theta) = y/|y|$ . On  $F_\infty^\times$ , we use the product  $d_\infty^\times y$  of these measures. On  $F_p^\times$ , we normalize the Haar measure such that  $\mathfrak{o}_p^\times$  has measure 1. The product  $d^\times y$  of these measures is a Haar measure on  $\mathbf{A}^\times$ , inducing some Haar measure on  $F^\times \backslash \mathbf{A}^\times$ .

On  $K$  and its factors, we use the Haar probability measures. On  $Z(F_\infty) \backslash \mathrm{GL}_2(F_\infty)$ , we use the Haar measure which satisfies

$$\int_{Z(F_\infty) \backslash \mathrm{GL}_2(F_\infty)} f(g) dg = \int_{(\mathbf{R}^\times)^r \times (\mathbf{R}_+^\times)^s} \int_{F_\infty} \int_{K_\infty} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k\right) dk dx \frac{d_\infty^\times y}{\prod_{j=1}^{r+s} |y_j|}.$$

Recalling  $|y|_\infty = \prod_{j=1}^r |y_j| \prod_{j=r+1}^{r+s} |y_j|^2$ , it follows that on  $F_\infty^\times$ ,  $d_\infty^\times y = \mathrm{const.} dy/|y|_\infty$ .

On  $\mathrm{GL}_2(F_p)$  we normalize the Haar measure such that  $K(\mathfrak{o}_p)$  has measure 1. On the factor space  $Z(F_\infty) \backslash \mathrm{GL}_2(\mathbf{A})$ , we use the product of these measures, which, on the factor  $Z(\mathbf{A}) \backslash \mathrm{GL}_2(\mathbf{A})$ , restricts as

$$\int_{Z(\mathbf{A}) \backslash \mathrm{GL}_2(\mathbf{A})} f(g) dg = \int_{\mathbf{A}^\times} \int_{\mathbf{A}} \int_K f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k\right) dk dx \frac{d^\times y}{|y|}.$$

Compare this with [BH10, p.6] and [GJ79, (3.10)].

**2.2. Spectral decomposition and Eisenstein series.** We review some basic facts about the automorphic theory of  $\mathrm{GL}_2$  that we shall use later. In the setup, we follow the work of Blomer and Harcos [BH10, Sections 2.2-2.7], even when it is not emphasized. Since our aim is to extend the spectral decomposition of [BH10] from totally real number fields to all number fields, we will always pay special attention to complex places.

For a Hecke character  $\omega$ , we denote by  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$  the Hilbert space of functions  $\phi : \mathrm{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$  satisfying

$$|\phi|^2 = \langle \phi, \phi \rangle < \infty, \text{ where } \langle \phi_1, \phi_2 \rangle = \int_{Z(\mathbf{A}) \backslash \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A})} \phi_1(g) \overline{\phi_2(g)} dg;$$

$$\forall z \in \mathbf{A}^\times, \gamma \in \mathrm{GL}_2(F), g \in \mathrm{GL}_2(\mathbf{A}) : \phi\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \gamma g\right) = \omega(z) \phi(g).$$

On  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$ , the group  $\mathrm{GL}_2(\mathbf{A})$  acts via right translations. From now on, without loss of generality, we assume that  $\omega$  is trivial on  $F_{\infty,+}^{\mathrm{diag}}$  (see [BH10, p.6]).

In this section, following [BH10, Section 2.2] closely, we give a short exposition of the spectral decomposition of the Hilbert space  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$ . For a detailed discussion, consult [GJ79, Sections 2-5].

First,  $\phi \in L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$  is called cuspidal if for almost every  $g \in \mathrm{GL}_2(\mathbf{A})$ ,

$$\int_{F \backslash \mathbf{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

The closed subspace generated by cuspidal functions is an invariant subspace  $L_{\mathrm{cusp}}$  decomposing into a countable sum of irreducible representations  $V_\pi$ , each  $\pi$  occuring with multiplicity one (see [GJ79, Section 2] and [JL70, Proposition 11.1.1]). Therefore, denoting the set of cuspidal representations by  $\mathcal{C}_\omega$ , we may write

$$L_{\mathrm{cusp}} = \bigoplus_{\pi \in \mathcal{C}_\omega} V_\pi,$$

where the irreducible representations on the right-hand side are distinct.

To any Hecke character  $\chi$  with  $\chi^2 = \omega$ , we can associate a one-dimensional representation  $V_\chi$  generated by  $g \mapsto \chi(\det g)$ , these sum up to

$$L_{\text{sp}} = \bigoplus_{\chi^2 = \omega} V_\chi.$$

For details, see [GJ79, Sections 3-4].

Now

$$L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}), \omega) = L_{\text{cusp}} \oplus L_{\text{sp}} \oplus L_{\text{cont}},$$

where  $L_{\text{cont}}$  can be described in terms of Eisenstein series.

Take Hecke quasicharacters  $\chi_1, \chi_2 : F^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{C}^\times$  satisfying  $\chi_1 \chi_2 = \omega$ . Denote by  $H(\chi_1, \chi_2)$  the space of functions  $\varphi : \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$  satisfying

$$\int_K |\varphi(k)|^2 dk < \infty$$

and

$$\varphi \left( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \right) = \chi_1(a) \chi_2(b) \left| \frac{a}{b} \right|^{1/2} \varphi(g), \quad x \in \mathbf{A}, a, b \in \mathbf{A}^\times.$$

In particular,  $H(\chi_1, \chi_2)$  can be identified with the set of functions  $\varphi \in L^2(K)$  satisfying

$$\varphi \left( \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \right) = \chi_1(a) \chi_2(b) \varphi(g), \quad \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in K.$$

There is a unique  $s \in \mathbf{C}$  such that  $\chi_1(a) = |a|_\infty^s$  and  $\chi_2(a) = |a|_\infty^{-s}$  for  $a \in F_{\infty,+}^{\text{diag}}$ , hence introduce

$$H(s) = \bigoplus_{\substack{\chi_1 \chi_2 = \omega \\ \chi_1 \chi_2^{-1} = |\cdot|_\infty^{2s} \text{ on } F_{\infty,+}^{\text{diag}}}} H(\chi_1, \chi_2).$$

Now regard the space  $H = \int_{s \in \mathbf{C}} H(s) ds$  as a holomorphic fibre bundle over base  $\mathbf{C}$ . Given a section  $\varphi \in H$ ,  $\varphi(s) \in H(s)$  and  $\varphi(s, g) \in \mathbf{C}$ . The bundle  $H$  is trivial, since any  $\varphi(0) \in H(0)$  extends to a section  $\varphi \in H$  satisfying  $\varphi(s, g) = \varphi(0, g) H(g)^s$ , where  $H(g)$  is the height function defined at [GJ79, p.219]. (One may think of this as a deformation of the function  $\varphi$ .)

Define

$$L'_{\text{cont}} = \int_0^\infty H(iy) dy,$$

and equip it with the inner product

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \frac{2}{\pi} \int_0^\infty \langle \phi_1(iy), \phi_2(iy) \rangle dy \\ &= \frac{2}{\pi} \int_0^\infty \int_{F^\times \backslash \mathbf{A}^1} \int_K \phi_1 \left( iy, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) \overline{\phi_2 \left( iy, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right)} dk dady, \end{aligned}$$

where  $\mathbf{A}^1$  stands for the group of ideles of norm 1 (see [GJ79, (3.15)]). Then there is an intertwining operator  $S : L_{\text{cont}} \rightarrow L'_{\text{cont}}$  given by [GJ79, (4.23)] on a dense subspace. Now combining this with the theory of Eisenstein series [GJ79, Section 5], we obtain the spectral decomposition of  $L_{\text{cont}}$ .

For  $\varphi \in H$ , and for  $\Re s > 1/2$ , define the Eisenstein series

$$E(\varphi(s), g) = \sum_{\gamma \in B(F) \backslash \text{GL}_2(F)} \varphi(s, \gamma g)$$

on  $\text{GL}_2(\mathbf{A})$ . This is a holomorphic function which continues meromorphically to  $s \in \mathbf{C}$ , with no poles on the line  $\Re s = 0$ . Now for  $y \in \mathbf{R}^\times$ , consider the complex vector space

$$V(iy) = \{E(\varphi(iy)) : \varphi(iy) \in H(iy)\}$$

with the inner product

$$\langle E(\varphi_1(iy)), E(\varphi_2(iy)) \rangle = \langle \varphi_1(iy), \varphi_2(iy) \rangle.$$

As above,

$$V(iy) = \bigoplus_{\substack{\chi_1 \chi_2 = \omega \\ \chi_1 \chi_2^{-1} = |\cdot|_{\infty}^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}}} V_{\chi_1, \chi_2},$$

with

$$V_{\chi_1, \chi_2} = \{E(\varphi(iy)) : \varphi(iy) \in H(\chi_1, \chi_2)\}.$$

Here,  $V(iy) = V(-iy)$  by [GJ79, (4.3), (4.24), (5.15)]. Therefore, we have a  $\text{GL}_2(\mathbf{A})$ -invariant decomposition

$$L_{\text{cont}} = \int_0^\infty V(iy) dy = \int_0^\infty \bigoplus_{\substack{\chi_1 \chi_2 = \omega \\ \chi_1 \chi_2^{-1} = |\cdot|_{\infty}^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}}} V_{\chi_1, \chi_2} dy.$$

In fact, [GJ79, (4.24), (5.15-18)] implies that for  $\phi \in L_{\text{cont}}$ , taking  $S\phi = \phi \in L'_{\text{cont}}$ ,

$$\phi(g) = \frac{1}{\pi} \int_0^\infty E(\varphi(iy), g) dy,$$

and also Plancherel holds, that is,

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle &= \frac{1}{\pi} \int_0^\infty \langle E(\varphi(iy), g), \phi_2 \rangle dy \\ &= \frac{2}{\pi} \int_0^\infty \langle \varphi_1(iy), \varphi_2(iy) \rangle dy = \frac{2}{\pi} \int_0^\infty \langle E(\varphi_1(iy)), E(\varphi_2(iy)) \rangle dy. \end{aligned}$$

To summarize,

$$(2) \quad L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}), \omega) = \bigoplus_{\pi \in \mathcal{E}_\omega} V_\pi \oplus \bigoplus_{\chi^2 = \omega} V_\chi \oplus \int_0^\infty \bigoplus_{\substack{\chi_1 \chi_2 = \omega \\ \chi_1 \chi_2^{-1} = |\cdot|_{\infty}^{2iy} \text{ on } F_{\infty,+}^{\text{diag}}}} V_{\chi_1, \chi_2} dy,$$

a function on the left-hand side decomposes into a convergent sum and integral of functions from the spaces appearing on the right-hand side, and also Plancherel holds.

For the Eisenstein spectrum, we introduce the notation  $\int_{\mathcal{E}_\omega} V_\varpi d\varpi$ , where  $\mathcal{E}_\omega$  is the set of unordered pairs of Hecke characters  $\{\chi_1, \chi_2\}$  which are nontrivial on  $F_{\infty,+}^{\text{diag}}$  and satisfy  $\chi_1 \chi_2 = \omega$ .

**2.3. Derivations and weights.** In this section, we review the action of the Lie algebra  $\mathfrak{sl}_2(F_\infty)$  on the space  $L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}), \omega)$ , following [BH10, Sections 2.3 and 2.10] at real places, [BM03, Section 3] and [LG04, Chapter 2] at complex places.

First we give a real basis such that each basis element is 0 for all but one place  $F_j$ . At this exceptional place, we use the following elements. For a real place ( $j \leq r$ ), let

$$(3) \quad \mathbf{H}_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{R}_j = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{L}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

while for a complex place ( $j > r$ ), let

$$(4) \quad \begin{aligned} \mathbf{H}_{1,j} &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \mathbf{V}_{1,j} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \mathbf{W}_{1,j} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathbf{H}_{2,j} &= \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \mathbf{V}_{2,j} &= \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & \mathbf{W}_{2,j} &= \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned}$$

An element  $X \in \mathfrak{sl}_2(F_\infty)$  acts as a right-differentiation on a function  $\phi : \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$  via

$$(X\phi)(g) = \left. \frac{d}{dt} \phi(g \exp(tX)) \right|_{t=0}.$$

Let  $\mathfrak{g} = \mathfrak{sl}_2(F_\infty) \otimes_{\mathbf{R}} \mathbf{C}$  be the complexified Lie algebra and set  $U(\mathfrak{g})$  for its universal enveloping algebra, consisting of higher-order right-differentiations with complex coefficients.

The above-defined first-order differentiations give rise to local Casimir elements

$$(5) \quad \begin{aligned} \Omega_j &= -\frac{1}{4} (\mathbf{H}_j^2 - 2\mathbf{H}_j + 4\mathbf{R}_j \mathbf{L}_j), \\ \Omega_{\pm,j} &= \frac{1}{8} ((\mathbf{H}_{1,j} \mp \mathbf{H}_{2,j})^2 + (\mathbf{V}_{1,j} \mp \mathbf{W}_{2,j})^2 - (\mathbf{W}_{1,j} \mp \mathbf{V}_{2,j})^2) \end{aligned}$$

at real and complex places, respectively.

On an irreducible unitary representation  $(\pi, V_\pi)$ , these local Casimir elements act as scalars, that is, for  $\phi \in V_\pi^\infty$ ,  $\Omega_j \phi = \lambda_j \phi$ ,  $\Omega_{+,j} \phi = \lambda_{+,j} \phi$ ,  $\Omega_{-,j} \phi = \lambda_{-,j} \phi$  with

$$(6) \quad \lambda_j = \frac{1}{4} - v_j^2, \quad \lambda_{\pm,j} = \frac{1}{8} ((v_j \mp p_j)^2 - 1),$$

where the parameters can be described as follows. At each place, the representation can be either even or odd (according to the action of the element which is  $-\text{id}$  at the corresponding place and  $\text{id}$  at all other places). At real places, there are three families of representations: principal series  $v_j \in i\mathbf{R}$ , complementary series  $v_j \in [-\theta, \theta]$  (only in the even case), and discrete series  $v_j \in 1/2 + \mathbf{Z}$  in the even case and  $v_j \in \mathbf{Z}$  in the odd case. At complex places, there are two families of representations: principal series  $v_j \in i\mathbf{R}, p_j \in \mathbf{Z}$  in the even case and  $v_j \in i\mathbf{R}, p_j \in 1/2 + \mathbf{Z}$  in the odd case, and complementary series  $v_j \in [-2\theta, 2\theta], p_j = 0$  (only in the even case). Here,  $\theta$  is a constant towards the Ramanujan-Petersson conjecture, according to the current state of art (see [BB11]),  $\theta = 7/64$  is admissible.

For some  $\mathcal{D} \in U(\mathfrak{g})$  and a smooth vector  $\phi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}), \omega)$ , recalling the spectral decomposition (2),

$$\phi = \sum_{\pi \in \mathcal{C}_\omega} \phi_\pi + \sum_{\chi^2 = \omega} \phi_\chi + \int_{\mathcal{E}_\omega} \phi_\varpi d\varpi,$$

we have

$$(7) \quad \|\mathcal{D}\phi\|^2 = \sum_{\pi \in \mathcal{C}_\omega} \|\mathcal{D}\phi_\pi\|^2 + \sum_{\chi^2 = \omega} \|\mathcal{D}\phi_\chi\|^2 + \int_{\mathcal{E}_\omega} \|\mathcal{D}\phi_\varpi\|^2 d\varpi,$$

see [CPS90, Sections 1.2-4] with references to [DM78]. Compare (7) also with [BH08, (33)] and [BH10, (84)].

We focus on the local subgroups  $\text{SO}_2(\mathbf{R})$  (for  $j \leq r$ ) and  $\text{SU}_2(\mathbf{C})$  (for  $j > r$ ), they are compact, connected and (modulo the center) maximal subgroups of  $\text{GL}_2(\mathbf{R})$ ,  $\text{GL}_2(\mathbf{C})$ , respectively, with these properties. The corresponding Lie algebras are  $\mathfrak{so}_2(\mathbf{R})$  and  $\mathfrak{su}_2(\mathbf{C})$ , and define

$$(8) \quad \Omega_{\mathfrak{k},j} = \mathbf{R}_j - \mathbf{L}_j, \quad \Omega_{\mathfrak{k},j} = -\frac{1}{2} (\mathbf{H}_{2,j}^2 + \mathbf{W}_{1,j}^2 + \mathbf{W}_{2,j}^2),$$

at real and complex places, respectively. At a complex place,  $\Omega_{\mathfrak{k},j}$  is the Casimir element (see [Sug90, Definition 9 on p.72]).

We now define the weight set  $W(\pi)$ . For  $j \leq r$ , let  $q_j$  be any integer of the same parity as the representation at the corresponding place, with the only restriction  $|q_j| \geq 2|v_j| + 1$  in the discrete series. For  $j > r$ , let  $(l_j, q_j)$  be any pair of numbers satisfying  $|q_j| \leq l_j \geq |p_j|$  and  $p_j \equiv q_j \equiv l_j \pmod{1}$ . Now set

$$(9) \quad \mathbf{w} = (q_1, \dots, q_r, (l_{r+1}, q_{r+1}), \dots, (l_{r+s}, q_{r+s}))$$

and denote by  $W(\pi)$  the set of  $\mathbf{w}$ 's satisfying the above condition.

For a given  $\mathbf{w} \in W(\pi)$ , we say that  $\phi : \text{GL}_2(\mathbf{A}) \rightarrow \mathbf{C}$  is of weight  $\mathbf{w}$ , if for  $j \leq r$ ,

$$(10) \quad \Omega_{\mathfrak{k},j} \phi = i q_j \phi$$

and for  $j > r$ ,

$$(11) \quad \mathbf{H}_{2,j}\phi = -iq_j\phi, \quad \Omega_{\mathfrak{t},j}\phi = \frac{1}{2}(l_j^2 + l_j)\phi,$$

for the action of  $\Omega_{\mathfrak{t},j}$  at complex places, see [Sug90, Chapter II, Proposition 5.15].

Note that  $W(\pi)$ , through  $(q_1, \dots, q_r, l_{r+1}, \dots, l_{r+s})$ , lists all irreducible representations of  $K_\infty$  occuring in  $\pi$ , while  $(q_{r+1}, \dots, q_{r+s})$  is to single out a one-dimensional space from each such representation.

Similarly, introduce the notation

$$(12) \quad \mathbf{r} = (v_1 \dots, v_r, (v_{r+1}, p_{r+1}), \dots, (v_{r+s}, p_{r+s})),$$

and also its norm

$$\mathcal{N}(\mathbf{r}) = \prod_{j=1}^r (1 + |v_j|) \prod_{j=r+1}^{r+s} (1 + |v_j| + |p_j|)^2,$$

compare this with [MV10, Section 3.1.8].

## 2.4. Cuspidal spectrum.

2.4.1. *Analytic conductor, newforms and oldforms.* Let  $V_\pi$  be a cuspidal representation in  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$ . By the tensor product theorem (see [Bum97, Section 3.4] or [Fla79]),

$$(13) \quad V_\pi = \bigotimes_v V_{\pi_v}$$

as a restricted tensor product with respect to the family  $\{K_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}})\}$  (by [Bum97, Theorem 3.3.4], irreducible cuspidal representations are admissible).

For an ideal  $\mathfrak{c} \subseteq \mathfrak{c}_\omega$  (with  $\mathfrak{c}_\omega$  standing for the conductor of  $\omega$ ), let

$$V_\pi(\mathfrak{c}) = \left\{ \phi \in V_\pi : \phi \left( g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \omega_{\mathfrak{c}}(d)\phi(g), \text{ if } g \in \mathrm{GL}_2(\mathbf{A}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathfrak{c}) \right\},$$

where  $\omega_{\mathfrak{c}}(x) = \prod_{\mathfrak{p}|\mathfrak{c}} \omega_{\mathfrak{p}}(x)$ . Then  $\mathfrak{c}' \subseteq \mathfrak{c}$  implies  $V_\pi(\mathfrak{c}') \supseteq V_\pi(\mathfrak{c})$  (see [BH10, p.9]).

By [Miy71, Corollary 2(a) of Theorem 2], there is a nonzero ideal  $\mathfrak{c}_\pi$  such that  $V_\pi(\mathfrak{c})$  is nontrivial if and only if  $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ . Now the analytic conductor of the representation is defined as

$$C(\pi) = \mathcal{N}(\mathfrak{c}_\pi) \mathcal{N}(\mathbf{r}).$$

Introducing also

$$V_{\pi, \mathbf{w}}(\mathfrak{c}) = \{ \phi \in V_\pi(\mathfrak{c}) : \phi \text{ is of weight } \mathbf{w} \}$$

for  $\mathbf{w} \in W(\pi)$ , [Miy71, Corollary 2(b) of Theorem 2] states that for any  $\mathbf{w} \in W(\pi)$ ,  $V_{\pi, \mathbf{w}}(\mathfrak{c}_\pi)$  is one-dimensional, that is, restricting  $V_\pi(\mathfrak{c}_\pi)$  to  $K_\infty$ , each irreducible representation of  $K_\infty$  listed in  $W(\pi)$  appears with multiplicity one. A nontrivial element of  $V_{\pi, \mathbf{w}}(\mathfrak{c}_\pi)$  is called a newform of weight  $\mathbf{w}$ .

Now consider an ideal  $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ , and take any ideal  $\mathfrak{t}$  such that  $\mathfrak{t}\mathfrak{c}_\pi \supseteq \mathfrak{c}$ . Fixing some finite idele  $t \in \mathbf{A}_{\mathrm{fin}}^\times$  representing  $\mathfrak{t}$ , we obtain an isometric embedding

$$(14) \quad R_{\mathfrak{t}} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c}), \quad (R_{\mathfrak{t}}\phi)(g) = \phi \left( g \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Then combining [Miy71, Corollary 2(c) of Theorem 2] with [Cas73, Corollary on p.306] and (13), we see the decompositions

$$V_\pi(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_{\mathfrak{t}}V_\pi(\mathfrak{c}_\pi), \quad V_{\pi, \mathbf{w}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t}|\mathfrak{c}\mathfrak{c}_\pi^{-1}} R_{\mathfrak{t}}V_{\pi, \mathbf{w}}(\mathfrak{c}_\pi),$$

which are not orthogonal in general. However, for nonzero ideals  $\mathfrak{t}_1, \mathfrak{t}_2$ ,

$$\langle R_{\mathfrak{t}_1}\phi_1, R_{\mathfrak{t}_2}\phi_2 \rangle = \langle \phi_1, \phi_2 \rangle C(\mathfrak{t}_1, \mathfrak{t}_2, \pi),$$



with the constant factor  $C(t_1, t_2, \pi)$  depending only on  $t_1, t_2, \pi$ , but not on  $\mathbf{w}$  (see [Mag13b, Chapter 3]). This allows us to use the Gram-Schmidt method, obtaining complex numbers  $\alpha_{t, \mathfrak{s}}$  (with  $\alpha_{o, o} = 1$ ) for any pair of ideals  $\mathfrak{s} | t | \mathfrak{c} \mathfrak{c}_\pi^{-1}$  such that the isometries

$$R^t = \sum_{\mathfrak{s} | t} \alpha_{t, \mathfrak{s}} R_{\mathfrak{s}} : V_\pi(\mathfrak{c}_\pi) \hookrightarrow V_\pi(\mathfrak{c}), \quad t | \mathfrak{c} \mathfrak{c}_\pi^{-1},$$

give rise to the orthogonal decompositions

$$(15) \quad V_\pi(\mathfrak{c}) = \bigoplus_{t | \mathfrak{c} \mathfrak{c}_\pi^{-1}} R^t V_\pi(\mathfrak{c}_\pi), \quad V_{\pi, \mathbf{w}}(\mathfrak{c}) = \bigoplus_{t | \mathfrak{c} \mathfrak{c}_\pi^{-1}} R^t V_{\pi, \mathbf{w}}(\mathfrak{c}_\pi).$$

**2.4.2. Whittaker functions and the Fourier-Whittaker expansion.** For a given  $\mathbf{r}, \mathbf{w}$  (recall (9) and (12)), we define the Whittaker function as the product of Whittaker functions at archimedean places. The important property of these functions is that they are the exponentially decaying eigenfunctions of the Casimir operators  $\Omega, \Omega_\pm$ , therefore, they emerge in the Fourier expansion of automorphic forms (see [Bum97, Section 3.5]).

At real places,

$$(16) \quad \mathcal{W}_{q, \mathbf{v}}(y) = \frac{i^{\text{sign}(y) \frac{q}{2}} W_{\text{sign}(y) \frac{q}{2}, \mathbf{v}}(4\pi|y|)}{(\Gamma(\frac{1}{2} - \mathbf{v} + \text{sign}(y) \frac{q}{2}) \Gamma(\frac{1}{2} + \mathbf{v} + \text{sign}(y) \frac{q}{2}))^{1/2}},$$

$W$  denoting the classical Whittaker function (see [WW96, Chapter XVI]). This is taken from [BH10, (23)].

At complex places, we first define the Whittaker function on the positive real axis via

$$(17) \quad \mathcal{W}_{(l, q), (\mathbf{v}, p)}(|y|) = \frac{\sqrt{8(2l+1)}}{(2\pi)^{\Re \mathbf{v}}} \binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} \sqrt{\left| \frac{\Gamma(l+1+\mathbf{v})}{\Gamma(l+1-\mathbf{v})} \right|} \\ \cdot (-1)^{l-p} (2\pi)^{\mathbf{v}} i^{-p-q} w_q^l(\mathbf{v}, p; |y|),$$

where

$$(18) \quad w_q^l(\mathbf{v}, p; |y|) = \sum_{k=0}^{l - \frac{1}{2}(|q+p| + |q-p|)} (-1)^k \xi_p^l(q, k) \frac{(2\pi|y|)^{l+1-k}}{\Gamma(l+1+\mathbf{v}-k)} K_{\mathbf{v}+l-|q+p|-k}(4\pi|y|),$$

$K$  denoting the  $K$ -Bessel function, and

$$(19) \quad \xi_p^l(q, k) = \frac{k!(2l-k)!}{(l-p)!(l+p)!} \binom{l - \frac{1}{2}(|q+p| + |q-p|)}{k} \binom{l - \frac{1}{2}(|q+p| - |q-p|)}{k}.$$

Then extend this to  $y \in \mathbf{C}^\times$  to satisfy

$$\mathcal{W}_{(l, q), (\mathbf{v}, p)}(ye^{i\theta}) = e^{-iq\theta} \mathcal{W}_{(l, q), (\mathbf{v}, p)}(y), \quad y \in \mathbf{C}^\times, \theta \in \mathbf{R}.$$

This definition is borrowed from [BM03, Section 5] and [LG04, Section 4.1], apart from the first line, which is a normalization to gain the right  $L^2$ -norm.

In both cases, the occurring numbers  $\mathbf{v}, p, q, l$  are those given by the representation and weight data, encoded in the action of the elements  $\Omega, \Omega_\pm, \Omega_{\mathfrak{t}}, \mathbf{H}_2$  (recall (4), (5), (6), (8), (9), (12)).

Finally, define the archimedean Whittaker function as

$$\mathcal{W}_{\mathbf{w}, \mathbf{r}}(y) = \prod_{j \leq r} \mathcal{W}_{q_j, \mathbf{v}_j}(y_j) \prod_{j > r} \mathcal{W}_{(l_j, q_j), (\mathbf{v}_j, p_j)}(y_j).$$

With the given normalization, for a fixed  $\mathbf{r}$ ,

$$(20) \quad \int_{F_\infty^\times} \mathcal{W}_{\mathbf{w}, \mathbf{r}}(y) \overline{\mathcal{W}_{\mathbf{w}', \mathbf{r}}(y)} d_\infty^\times y = \delta_{\mathbf{w}, \mathbf{w}'}.$$

This can be seen as the product of the analogous results at single places. For real places, see [BH10, (25)] and [BM05, Section 4]. As for complex places, see [Mag13a, Lemma 2] and [Mag13b, Lemma 8.1 and Lemma 8.2].

Now we extend [BH10, Section 2.5] to our more general situation. For any  $\pi \in \mathcal{C}_\omega$ ,  $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ , any function  $\phi \in V_{\pi, \mathbf{w}}(\mathfrak{c})$  can be expanded into Fourier series as follows. There exists a character  $\varepsilon_\pi : \{\pm 1\}^r \rightarrow \{\pm 1\}$  depending only on  $\pi$  such that

$$(21) \quad \phi \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \sum_{t \in F^\times} \rho_\phi(t y_{\text{fin}}) \varepsilon_\pi(\text{sign}(t y_\infty)) \mathcal{W}_{\mathbf{w}, \mathbf{r}}(t y_\infty) \psi(tx).$$

Note that  $\varepsilon_\pi$  is not well-defined, if we are in the discrete series and that the coefficient  $\rho(t y_{\text{fin}})$  depends only on the fractional ideal generated by  $t y_{\text{fin}}$ . Moreover, it is zero, if this fractional ideal is nonintegral. For the proof of this, see [DFI02, Section 4], [KM96, Sections 1-3] or [Mag13b, Proposition 2.1].

Now assume that  $\mathfrak{c} = \mathfrak{c}_\pi$ , i.e.  $\phi$  is a newform of weight  $\mathbf{w}$ . In this case, the coefficients  $\rho_\pi(\mathfrak{m})$  are proportional to the Hecke eigenvalues  $\lambda_\pi(\mathfrak{m})$ :

$$\rho_\phi(\mathfrak{m}) = \frac{\lambda_\pi(\mathfrak{m})}{\sqrt{\mathcal{N}(\mathfrak{m})}} \rho_\phi(\mathfrak{o}).$$

Setting

$$W_\phi(y) = \rho_\phi(\mathfrak{o}) \varepsilon_\pi(\text{sign}(y)) \mathcal{W}_{\mathbf{w}, \mathbf{r}}(y), \quad y \in F_\infty^\times,$$

we obtain

$$(22) \quad \phi \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \sum_{t \in F^\times} \frac{\lambda_\pi(t y_{\text{fin}})}{\sqrt{\mathcal{N}(t y_{\text{fin}})}} W_\phi(t y_\infty) \psi(tx).$$

**2.4.3. The archimedean Kirillov model.** Now fixing  $y_{\text{fin}} = (1, 1, \dots)$ , we can single out the term corresponding to  $t = 1$ :

$$(23) \quad W_\phi(y) = \int_{F \setminus \mathbf{A}} \phi \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) \psi(-x) dx.$$

In the case of arbitrary (i.e. non-necessarily pure weight) smooth functions in  $V_\pi(\mathfrak{c}_\pi)$ , this latter formula can be considered as the definition of the mapping  $\phi \mapsto W_\phi$ . The image is a dense subspace in  $L^2(F_\infty^\times, d_\infty^\times y)$ , moreover, there is a positive constant  $C_\pi$  depending only on  $\pi$  and satisfying

$$(24) \quad C(\pi)^{-\varepsilon} \ll_\varepsilon C_\pi \ll_\varepsilon C(\pi)^\varepsilon$$

such that

$$(25) \quad \langle \phi_1, \phi_2 \rangle = C_\pi \langle W_{\phi_1}, W_{\phi_2} \rangle,$$

where the scalar product on the left-hand side is the scalar product in  $L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A}), \omega)$ , while on the right-hand side, it is the scalar product in  $L^2(F_\infty^\times, d_\infty^\times y)$ . For the proof of these facts, see [Mag13b, Proposition 2.2 and Section 3]. The map  $\phi \mapsto W_\phi$  is therefore surjective from  $V_\pi(\mathfrak{c}_\pi)$  to  $L^2(F_\infty^\times, d_\infty^\times y)$ .

Now turn to the general case  $\mathfrak{c} \subseteq \mathfrak{c}_\pi$ . Using the isometries  $R^t$ , (22) gives rise to, for every  $\phi \in R^t V_\pi(\mathfrak{c}_\pi)$ ,

$$(26) \quad \phi \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \sum_{t \in F^\times} \frac{\lambda_\pi^t(t y_{\text{fin}})}{\sqrt{\mathcal{N}(t y_{\text{fin}})}} W_\phi(t y_\infty) \psi(tx),$$

with

$$(27) \quad W_\phi = W_{(R^t)^{-1}\phi}, \quad \lambda_\pi^t(\mathfrak{m}) = \sum_{\mathfrak{s} | \text{gcd}(\mathfrak{t}, \mathfrak{m})} \alpha_{\mathfrak{t}, \mathfrak{s}} \mathcal{N}(\mathfrak{s})^{1/2} \lambda_\pi(\mathfrak{m} \mathfrak{s}^{-1}).$$

**2.5. Eisenstein spectrum.** In this section, we develop the theory of Eisenstein series. Since Eisenstein series will show up only in the spectral decomposition of an automorphic function of trivial central character, we assume temporarily that  $\omega = 1$ . From now on, let  $\chi \in \mathcal{E}_1$  be a Hecke character which is nontrivial on  $F_{\infty, +}^{\text{diag}}$ .

2.5.1. *Analytic conductor, newforms and oldforms.* Similarly to the cuspidal case, for any ideal  $\mathfrak{c} \subseteq \mathfrak{o}$ , define

$$V_{\chi, \chi^{-1}}(\mathfrak{c}) = \left\{ \phi \in V_{\chi, \chi^{-1}} : \phi \left( g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \phi(g), \text{ if } g \in \mathrm{GL}_2(\mathbf{A}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathfrak{c}) \right\}.$$

Using that  $V_{\chi, \chi^{-1}}$  and  $H(\chi, \chi^{-1})$  are isomorphic as  $\mathrm{GL}_2(\mathbf{A})$ -representations, we have

$$V_{\chi, \chi^{-1}}(\mathfrak{c}) = \{E(\phi(iy), \cdot) \in V_{\chi, \chi^{-1}} : \phi \in H(\chi, \chi^{-1}, \mathfrak{c})\}$$

with

$$H(\chi, \chi^{-1}, \mathfrak{c}) = \left\{ \phi \in H(\chi, \chi^{-1}) : \phi \left( g \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \phi(g), \text{ if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(\mathfrak{c}) \right\}.$$

Analogously to (13), we have

$$H(\chi, \chi^{-1}) = \bigotimes_v H_v(\chi, \chi^{-1}),$$

a restricted tensor product with respect to the family  $\{K_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}})\}$  again, the admissibility of  $H(\chi, \chi^{-1})$  is straight-forward.

Assume  $\chi$  has conductor  $\mathfrak{c}_{\chi}$ . The following is taken from [BH10, Section 2.6].

**Proposition 2.1.** *For any non-archimedean place  $\mathfrak{p}$ , set  $d = v_{\mathfrak{p}}(\mathfrak{d})$  and  $m = v_{\mathfrak{p}}(\mathfrak{c}_{\chi})$ , and fix some  $\varpi$  such that  $v_{\mathfrak{p}}(\varpi) = 1$ . Then for any integer  $n \geq 0$ , the complex vector space  $H_{\mathfrak{p}}(\chi, \chi^{-1}, \mathfrak{p}^n)$  has dimension  $\max(0, n - 2m + 1)$ . For  $n \geq 2m$ , an orthogonal basis is  $\{\varphi_{\mathfrak{p}, j} : 0 \leq j \leq n - 2m\}$  with functions  $\varphi_{\mathfrak{p}, j}$  defined as follows.*

If  $m = 0$  and  $k = \begin{pmatrix} * & * \\ b\varpi^d & * \end{pmatrix} \in K_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}})$ , let

$$\varphi_{\mathfrak{p}, 0}(k) = 1; \quad \varphi_{\mathfrak{p}, 1}(k) = \begin{cases} \mathcal{N}(\mathfrak{p})^{-1/2}, & \text{if } v_{\mathfrak{p}}(b) = 0, \\ -\mathcal{N}(\mathfrak{p})^{1/2}, & \text{if } v_{\mathfrak{p}}(b) \geq 1; \end{cases}$$

while for  $j \geq 2$ ,

$$\varphi_{\mathfrak{p}, j}(k) = \begin{cases} 0, & v_{\mathfrak{p}}(b) \leq j - 2, \\ -\mathcal{N}(\mathfrak{p})^{j/2-1}, & \text{if } v_{\mathfrak{p}}(b) = j - 1, \\ \mathcal{N}(\mathfrak{p})^{j/2} \left(1 - \frac{1}{\mathcal{N}(\mathfrak{p})}\right), & \text{if } v_{\mathfrak{p}}(b) \geq j. \end{cases}$$

If  $m > 0$  and  $k = \begin{pmatrix} a & * \\ b\varpi^d & * \end{pmatrix} \in K_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}})$ , let

$$\varphi_{\mathfrak{p}, j}(k) = \begin{cases} \mathcal{N}(\mathfrak{p})^{(m+j)/2} \chi_{\mathfrak{p}}(ab^{-1}), & \text{if } v_{\mathfrak{p}}(b) = m + j, \\ 0, & \text{if } v_{\mathfrak{p}}(b) \neq m + j. \end{cases}$$

Moreover,

$$1 - \frac{1}{\mathcal{N}(\mathfrak{p})} \leq \|\varphi_{\mathfrak{p}, j}\| \leq 1.$$

*Proof.* See [BH10, Lemma 1 and Remark 7]. □

Therefore,  $\mathfrak{c}_{\chi, \chi^{-1}} = (\mathfrak{c}_{\chi})^2$  is the maximal ideal  $\mathfrak{c}$  such that  $V_{\chi, \chi^{-1}}(\mathfrak{c})$  and  $H(\chi, \chi^{-1}, \mathfrak{c})$  are nontrivial.

Now turn our attention to the archimedean quasifactors  $H_j(\chi, \chi^{-1})$ . They are always principal series representations and their parameter  $\mathbf{r}$  is the following. At real places,  $v_j \in i\mathbf{R}$  of (6) is the one satisfying  $\chi_j(a) = a^{v_j}$  for  $a \in \mathbf{R}_+$  (see [BM05, p.83]). At complex places,  $v_j \in i\mathbf{R}$  and  $p_j \in \mathbf{Z}/2$  of (6) are those satisfying  $\chi_j(ae^{i\theta}) = a^{v_j} e^{-ip_j\theta}$  for  $a \in \mathbf{R}_+$ ,  $\theta \in \mathbf{R}$  (see [BM03, Section 3] or [LG04, Section 2.3]). Now these give rise to the set  $W(\chi, \chi^{-1})$  of weights (those occurring in  $H_j(\chi, \chi^{-1})$ ): the only condition is  $|q_j| \leq l_j \geq |p_j|$  at complex places.

The analytic conductor is again defined as

$$(28) \quad C(\chi, \chi^{-1}) = \mathcal{N}(\mathfrak{c}_{\chi, \chi^{-1}}) \mathcal{N}(\mathbf{r}).$$

We can now give an orthogonal basis of  $H(\chi, \chi^{-1}, \mathfrak{c})$  for any  $\mathfrak{c} \subseteq \mathfrak{c}_\chi^2$ . Given  $\mathfrak{t} | \mathfrak{c} \mathfrak{c}_\chi^{-2}$  and any weight  $\mathbf{w} \in W(\chi, \chi^{-1})$ , let  $\phi^{\mathfrak{t}, \mathbf{w}}$  be the tensor product of the following local functions. At the archimedean places, with the notation of Section 2.1.3, let  $\phi_j^{\mathfrak{t}, \mathbf{w}}(k) = \Phi_{q_j}(k)$ ,  $\phi_j^{\mathfrak{t}, \mathbf{w}}(k) = \Phi_{p_j, q_j}^{l_j}(k) / \|\Phi_{p_j, q_j}^{l_j}\|_{\mathrm{SU}_2(\mathbb{C})}$  for  $k \in K_j$  with  $j \leq r$ ,  $j > r$ , respectively. At non-archimedean places, let  $\phi_p^{\mathfrak{t}, \mathbf{w}} = \phi_{p, v_p(\mathfrak{t})}$ . The global functions form an orthogonal basis of  $H(\chi, \chi^{-1}, \mathfrak{c})$  and this gives rise to an orthogonal basis in  $V_{\chi, \chi^{-1}}$  via the corresponding Eisenstein series  $\phi^{\mathfrak{t}, \mathbf{w}} = E(\phi^{\mathfrak{t}, \mathbf{w}})$ . Finally, defining  $R^{\mathfrak{t}} : V_{\chi, \chi^{-1}}(\mathfrak{c}_\chi^2) \hookrightarrow V_{\chi, \chi^{-1}}(\mathfrak{c})$  as  $\phi^{\mathfrak{o}, \mathbf{w}} / \|\phi^{\mathfrak{o}, \mathbf{w}}\| \mapsto \phi^{\mathfrak{t}, \mathbf{w}} / \|\phi^{\mathfrak{t}, \mathbf{w}}\|$  for all  $\mathbf{w}$ , we obtain the orthogonal decomposition

$$(29) \quad V_{\chi, \chi^{-1}}(\mathfrak{c}) = \bigoplus_{\mathfrak{t} | \mathfrak{c} \mathfrak{c}_\chi^{-2}} R^{\mathfrak{t}} V_{\chi, \chi^{-1}}(\mathfrak{c}_\chi^2).$$

**2.5.2. The Fourier-Whittaker expansion and the archimedean Kirillov model.** Similarly to cusp forms, Eisenstein series can also be expanded into Fourier-Whittaker series. Assume  $\varphi$  is one of the pure tensors defined above and  $\phi = E(\varphi)$ , where we dropped  $\mathfrak{t}$  and  $\mathbf{w}$  from the notation. Denoting by  $\rho_{E(\varphi), 0}(y)$  the constant term [GJ79, p.220], we obtain the Fourier-Whittaker expansion (see [BH10, Sections 2.6 and 2.7] and [Mag13b, Section 2.4])

$$(30) \quad \phi \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) = \rho_{E(\varphi), 0}(y) + \sum_{t \in F^\times} \frac{\lambda_{\chi, \chi^{-1}}^{\mathfrak{t}}(ty_{\mathrm{fin}})}{\sqrt{\mathcal{N}(ty_{\mathrm{fin}})}} W_{E(\varphi)}(ty_{\infty}) \psi(tx)$$

where the coefficients satisfy

$$\lambda_{\chi, \chi^{-1}}^{\mathfrak{t}}(\mathfrak{m}) \ll_{F, \varepsilon} \mathcal{N}(\mathrm{gcd}(\mathfrak{t}, \mathfrak{m})) \mathcal{N}(\mathfrak{m})^\varepsilon,$$

for all  $\mathfrak{m} \subseteq \mathfrak{o}$ . Also,

$$(31) \quad \|W_{E(\varphi)}\| \ll_{F, \varepsilon} \mathcal{N}(\mathfrak{t})^\varepsilon C(\chi, \chi^{-1})^\varepsilon \|\varphi\|,$$

where the norms are understood in the spaces  $L^2(F_\infty^\times, d_\infty^\times y)$  and  $L^2(K)$  (recall also (28)). Compare these with [BH10, (48-50)].

The mapping  $E(\varphi) \mapsto W_{E(\varphi)}$  has similar properties as in the cuspidal spectrum. In the special case  $\mathfrak{c} = \mathfrak{c}_\chi^2$ ,  $\mathfrak{t} = \mathfrak{t}_\chi = \mathfrak{o}$ ,  $E(\varphi)$  spans the space  $V_{\chi, \chi^{-1}, \mathbf{w}}(\mathfrak{c}_\chi^2)$  of newforms of weight  $\mathbf{w}$ . In this case, we have the alternative definition

$$(32) \quad W_{E(\varphi)}(y) = \int_{F \setminus \mathbf{A}} E(\varphi) \left( \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \right) \psi(-x) dx,$$

where  $y_{\mathrm{fin}} = (1, 1, \dots)$ . For  $\phi_1, \phi_2 \in V_{\chi, \chi^{-1}}(\mathfrak{c}_\chi^2)$ , we have

$$(33) \quad \langle \phi_1, \phi_2 \rangle = C_{\chi, \chi^{-1}} \langle W_{\phi_1}, W_{\phi_2} \rangle$$

with some positive constant  $C_{\chi, \chi^{-1}} \gg_{F, \varepsilon} C(\chi, \chi^{-1})^{-\varepsilon}$  depending only on  $\chi$ . Also,  $\lambda_{\chi, \chi^{-1}}$  specialize to Hecke eigenvalues.

**2.6. The spectral decomposition of shifted convolution sums.** Fix an ideal  $\mathfrak{c} \neq 0$ , and focus on  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}) / K(\mathfrak{c}))$ , the subspace consisting of functions that are right  $K(\mathfrak{c})$ -invariant. Its spectral decomposition is similar to (2), the only modification is the restriction of  $\mathcal{C}_\omega, \mathcal{E}_\omega$  to

$$\mathcal{C}_\omega(\mathfrak{c}) = \{\pi \in \mathcal{C}_\omega \mid \mathfrak{c} \subseteq \mathfrak{c}_\pi\}, \quad \mathcal{E}_\omega(\mathfrak{c}) = \{\chi \in \mathcal{E}_\omega \mid \mathfrak{c} \subseteq \mathfrak{c}_{\chi, \chi^{-1}}\}.$$

We write

$$(34) \quad \int_{(\mathfrak{c})} f_\omega d\omega = \sum_{\pi \in \mathcal{C}_\omega(\mathfrak{c})} f_\pi + \int_{\mathcal{E}_\omega(\mathfrak{c})} f_\omega d\omega,$$

if  $f$  is a function of the infinite-dimensional representations not orthogonal to the subspace  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}) / K(\mathfrak{c}), \omega)$ .

We further introduce the notion of Sobolev norms in the automorphic setup. Let  $d \geq 0$  be an integer. Assume that  $\phi \in L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$  is a function such that  $X_1 \dots X_d \phi$  exists for every sequence  $X_1, \dots, X_d$ ,

where each  $X_k$  is one of those differential operators given in (3) and (4). Then the Sobolev norm  $\|\phi\|_{S_d}$  of  $\phi$  is defined via

$$\|\phi\|_{S_d}^2 = \sum_{k=0}^d \sum_{\{X_1, \dots, X_k\} \in \{\mathbf{H}_j, \mathbf{R}_j, \mathbf{L}_j, \mathbf{H}_{1,j}, \mathbf{H}_{2,j}, \mathbf{V}_{1,j}, \mathbf{V}_{2,j}, \mathbf{W}_{1,j}, \mathbf{W}_{2,j}\}^k} \|X_1 \dots X_k \phi\|^2.$$

One can prove (consult [Ven10, Section 8.1]) that the action of  $U(\mathfrak{g})$  can be lifted up to the Kirillov model, i.e. for sufficiently smooth vectors  $W \in L^2(F_\infty^\times)$ , it makes sense to speak about

$$\|W\|_{S_d}^2 = \sum_{k=0}^d \sum_{\{X_1, \dots, X_k\} \in \{\mathbf{H}_j, \mathbf{R}_j, \mathbf{L}_j, \mathbf{H}_{1,j}, \mathbf{H}_{2,j}, \mathbf{V}_{1,j}, \mathbf{V}_{2,j}, \mathbf{W}_{1,j}, \mathbf{W}_{2,j}\}^k} \|X_1 \dots X_k W\|^2.$$

Now we are in the position to state the main result of the paper, a generalization of [BH10, Theorem 2] for arbitrary number fields.

**Theorem 1.** *Assume  $\pi_1, \pi_2$  are irreducible cuspidal representations of the same central character. Let  $l_1, l_2 \in \mathfrak{o} \setminus \{0\}$ , and set  $\mathfrak{c} = \text{lcm}(l_1 \mathfrak{c}_{\pi_1}, l_2 \mathfrak{c}_{\pi_2})$ . Let moreover  $W_1, W_2 : F_\infty^\times \rightarrow \mathbf{C}$  be arbitrary Schwarz functions, that is, they are smooth and tend to 0 faster than any power of  $y^{-1}$  or  $y$ , as  $y$  tends to  $\infty$  or 0, respectively. Then for any  $\varpi \in \mathcal{C}_1(\mathfrak{c}) \cup \mathcal{C}_1(\mathfrak{c})$  and  $\mathfrak{t} | \mathfrak{c} \mathfrak{c}_\pi^{-1}$ , there exists a function  $W_{\varpi, \mathfrak{t}} : F_\infty^\times \rightarrow \mathbf{C}^\times$  depending only on  $F, \pi_1, \pi_2, W_1, W_2, \varpi, \mathfrak{t}$  such that the following holds. For any  $Y \in (0, \infty)^{r+s}$ , any ideal  $\mathfrak{n} \subseteq \mathfrak{o}$  and any  $0 \neq q \in \mathfrak{n}$ , there is a spectral decomposition of the shifted convolution sum*

$$\begin{aligned} \sum_{l_1 t_1 - l_2 t_2 = q, 0 \neq t_1, t_2 \in \mathfrak{n}} \frac{\lambda_{\pi_1}(t_1 \mathfrak{n}^{-1}) \overline{\lambda_{\pi_2}(t_2 \mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1 t_2 \mathfrak{n}^{-2})}} W_1 \left( \left( \frac{(l_1 t_1)_j}{Y_j} \right)_j \right) \overline{W_2 \left( \left( \frac{(l_2 t_2)_j}{Y_j} \right)_j \right)} \\ = \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c} \mathfrak{c}_\pi^{-1}} \frac{\lambda_{\varpi}^{\mathfrak{t}}(q \mathfrak{n}^{-1})}{\sqrt{\mathcal{N}(q \mathfrak{n}^{-1})}} W_{\varpi, \mathfrak{t}} \left( \left( \frac{q_j}{Y_j} \right)_j \right) d\varpi, \end{aligned}$$

where  $\lambda_{\varpi}^{\mathfrak{t}}(\mathfrak{m})$  is given in (27).

Let  $P \in \mathbf{C}[x_1, \dots, x_{r+2s}]$  be a polynomial of degree at most  $a$  in each variable. Set then

$$\mathcal{D} = P \left( \left( y_j \frac{\partial}{\partial y_j} \right)_{j \leq r}, \left( y_j \frac{\partial}{\partial y_j} \right)_{j > r}, \left( \overline{y_j} \frac{\partial}{\partial \overline{y_j}} \right)_{j > r} \right).$$

Then for any  $0 < \varepsilon < 1/4$  and nonnegative integers  $b, c$ , we have, for all  $y \in F_\infty^\times$ ,

$$\begin{aligned} \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c} \mathfrak{c}_\pi^{-1}} (\mathcal{N}(\mathbf{r}_\varpi))^{2c} |\mathcal{D} W_{\varpi, \mathfrak{t}}(y)|^2 d\varpi \ll_{F, \varepsilon, \pi_1, \pi_2, a, b, c, P} \mathcal{N}((l_1 l_2))^\varepsilon \|W_1\|_{S_\alpha}^2 \|W_2\|_{S_\alpha}^2 \\ \cdot \prod_{j=1}^r (|y_j|^{1-\varepsilon} + |y_j|^{1-2\theta-\varepsilon}) (\min(1, |y_j|^{-2b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/2} + |y_j|^2) (\min(1, |y_j|^{-2b})) \end{aligned}$$

with  $\alpha = (a + b + 2c + 20)r + (a + b + 2c + 44)s + 4$ .

From the  $L^2$ -bound presented in Theorem 1, one can easily deduce  $L^1$ -bounds (see [Mag13b, Chapter 6]).

### 3. ANALYSIS OF SMOOTH AUTOMORPHIC VECTORS

Assume that we are given a smooth automorphic vector  $\phi$  appearing in an automorphic representation. The aim of this section is to give a pointwise estimate for the associated Kirillov vector  $W_\phi$ , and, when  $\phi$  is a cuspidal newform, the supremum norm of  $\phi$ , both in terms of some Sobolev norm of  $\phi$ .

**3.1. Bounds on Bessel functions.** About the classical  $J$ -Bessel function of parameter  $p \in \mathbf{Z}/2$ , record the bounds

$$(35) \quad |J_{2p}(x)| \leq 1 \text{ for all } x \in (0, \infty), \quad |J_{2p}(x)| \ll x^{-1/2} \text{ for all } x \in (\max(1/2, (2p)^2), \infty),$$

see [Wat95, 2.2(1)] and [GR07, 8.451(1-8)].

Now we define and estimate a function  $j$  that later will turn out to be the Bessel function of a certain representation (after a simple transformation of the argument).

**Lemma 3.1.** *Assume  $v \in \mathbf{C}$  and  $p \in \mathbf{Z}/2$  are given such that either  $\Re v = 0$  (principal series) or  $\Re v \neq 0$ ,  $\Im v = 0$ ,  $|v| \leq 2\theta = 7/32$ ,  $p = 0$  (complementary series). Define*

$$(36) \quad j(t) = 4\pi|t|^2 \int_0^\infty y^{2v} \left( \frac{yt + y^{-1}\bar{t}}{|yt + y^{-1}\bar{t}|} \right)^{2p} J_{2p}(2\pi|yt + y^{-1}\bar{t}|) d_{\mathbf{R}}^\times y.$$

Then  $j(t)$  is an even function of  $t \in \mathbf{C}^\times$  satisfying the bound

$$(37) \quad j(t) \ll |t|^2(1 + |t|^{-1/2})(1 + |p|).$$

*Proof.* It is clear that  $j(t) = j(-t)$ , so we are left to prove (37). Assume first that  $p \neq 0$ , which implies that we are in the principal series. Then trivially

$$j(t) \ll |t|^2 \int_0^\infty |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y.$$

The integral is invariant under  $y \leftrightarrow 1/y$ , so we have

$$j(t) \ll |t|^2 \int_1^\infty |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y.$$

Here

$$\int_1^2 |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y \ll 1$$

and

$$\int_2^{\max(\frac{4p^2}{\pi|t|}, 2)} |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y \ll \max\left(\log\left(\frac{4p^2}{\pi|t|}\right), 0\right)$$

by  $|J_{2p}(x)| \leq 1$  of (35). On the remaining domain,  $y \geq 2$ , hence  $|yt + y^{-1}\bar{t}| \geq y|t|/2$ . Moreover, since  $y \geq 4p^2/(\pi|t|)$ , we have  $2\pi|yt + y^{-1}\bar{t}| \geq (2p)^2 > 1/2$ , so we may apply  $|J_{2p}(x)| \ll x^{-1/2}$  of (35), obtaining

$$\int_{\max(\frac{4p^2}{\pi|t|}, 2)}^\infty |J_{2p}(2\pi|yt + y^{-1}\bar{t}|)| d_{\mathbf{R}}^\times y \ll 1 + |t|^{-1/2}.$$

Altogether,

$$j(t) \ll |t|^2 \left( 1 + |t|^{-1/2} + \max\left(\log\left(\frac{4p^2}{\pi|t|}\right), 0\right) \right),$$

which obviously implies

$$(38) \quad j(t) \ll |t|^2(1 + |t|^{-1/2})(1 + |p|).$$

If  $p = 0$ , in particular, in the complementary series, a similar calculation yields (using also that  $2|\Re v| \leq 7/16$ )

$$(39) \quad j(t) \ll |t|^2(1 + |t|^{-1/2}).$$

(This time the integral might not be invariant under  $y \leftrightarrow 1/y$ , however, replacing  $y^{2\Re v}$  by  $y^{2|\Re v|}$ , we may write  $\int_1^\infty$  in place of  $\int_0^\infty$ ; and the domain of integration  $[1, \infty]$  is splitted up as  $[1, 2] \cup [2, \max(1/|t|, 2)] \cup [\max(1/|t|, 2), \infty]$ .)

Collecting the bounds (38), (39), we arrive at (37).  $\square$

**3.2. Bounds on Whittaker functions.** We would like to give estimates on the Whittaker functions defined in (16) and (17). At real places, we refer to [BH08].

**Lemma 3.2.** *For all  $v$ ,*

$$(40) \quad \mathcal{W}_{q,v}(y) \ll |y|^{1/2} \left( \frac{|y|}{|q| + |v| + 1} \right)^{-1 - |\Re v|} \exp \left( -\frac{|y|}{|q| + |v| + 1} \right).$$

*For  $v \in (\mathbf{Z}/2) \cup i\mathbf{R}$  and for any  $0 < \varepsilon < 1/4$ ,*

$$(41) \quad \mathcal{W}_{q,v}(y) \ll_{\varepsilon} |y|^{1/2 - \varepsilon} (|q| + |v| + 1).$$

*For  $v \in (-1/2, 1/2)$  and for any  $0 < \varepsilon < 1$ ,*

$$(42) \quad \mathcal{W}_{q,v}(y) \ll_{\varepsilon} |y|^{1/2 - |v| - \varepsilon} (|q| + |v| + 1)^{1 + |v|}.$$

*Proof.* See [BH08, (24-26)] (and also [BH10, (26-28)]). □

At complex places, introduce

$$\mathbf{J}_{(l,q),(v,p)}(y) = \mathcal{W}_{(l,q),(v,p)}(y) \left( \frac{\sqrt{8(2l+1)}}{(2\pi)^{\Re v}} \binom{2l}{l-q}^{\frac{1}{2}} \binom{2l}{l-p}^{-\frac{1}{2}} \sqrt{\left| \frac{\Gamma(l+1+v)}{\Gamma(l+1-v)} \right|} \right)^{-1},$$

the unnormalized Whittaker function appearing in [BM03, Section 5] and [LG04, Section 4.1]; our function  $\mathbf{J}_{(l,q),(v,p)}(y)$  is the same as  $\mathbf{J}_1 \phi_{l,q}(v, p)(a(y))$  in [LG04]. The advantage of this unnormalized function is its regularity in  $v$ . Note that  $\mathbf{J}_{(l,q),(v,p)}$  is nothing else but (17) without its first line.

**Lemma 3.3.** *For  $0 < |y| \leq 1$  and  $\varepsilon > 0$ ,*

$$(43) \quad \mathcal{W}_{(l,q),(v,p)}(y) \ll_{\varepsilon} |y|^{1 - |\Re v| - \varepsilon} (1 + |p| + l)^{1 + |p|/2}.$$

*For  $|y| \geq (l^4 + 1)(|v|^2 + 1)$ ,*

$$(44) \quad \mathcal{W}_{(l,q),(v,p)}(y) \ll \exp \left( -\frac{|y|}{|v| + l + 1} \right).$$

*Proof.* It is clear from the definition and the fact  $|\Re v| \leq 7/32$  that

$$\mathcal{W}_{(l,q),(v,p)}(y) \ll \mathbf{J}_{(l,q),(v,p)}(y) (1 + |p| + l)^{1 + |p|/2}.$$

Together with [LG04, (4.28)], this shows the bound (43). As for (44), take  $|y| \geq (l^4 + 1)(|v|^2 + 1)$ . We first estimate  $\mathbf{J}_{(l,q),(v,p)}$  from its expression in terms of  $K$ -Bessel functions (recall (18) and (19)). We trivially have

$$\xi_p^l(q, k), (2\pi|y|)^{l+1-k}, (1+l)(1+|p|+l)^{1+|p|/2} \ll e^{|y|/(3(|v|+l+1))}$$

for the binomial factor, for the power of  $|y|$ , and for the summation over  $k$  together with the transition factor from  $\mathbf{J}_{(l,q),(v,p)}$  to  $\mathcal{W}_{(l,q),(v,p)}$ . Now we would like to estimate

$$\frac{K_{v+l-|q+p|-k}(4\pi|y|)}{\Gamma(l+1+v-k)},$$

where  $0 \leq k \leq l - \max(|p|, |q|)$ . Instead of this, we may write

$$\frac{K_{v+a}(4\pi|y|)}{\Gamma(b+1+v)},$$

where  $0 \leq a \leq b \leq l$ : in the principal series  $\Re v = 0$ , this is justified by  $K_s(x) = K_{-s}(x)$  (see [Wat95, 3.7(6)]) and  $|\Gamma(x)| = |\Gamma(\bar{x})|$ , hence take  $b = l - k$ , then  $a = |l - k - |q + p||$  (and we conjugate  $v$ , if  $l - k < |q + p|$ ),  $0 \leq a \leq b \leq l$  follows from the constraint on  $k$ ; while in the complementary series,  $p = 0$  implies  $l - |q + p| - k \geq 0$ ,

from which  $0 \leq a \leq b \leq l$  is satisfied by setting  $b = l - k$ ,  $a = l - k - |q + p|$ . By Basset's integral [Wat95, §6.16],

$$\frac{K_{v+a}(4\pi|y|)}{\Gamma(b+1+v)} = \frac{\Gamma(v+a+1/2)}{\Gamma(v+b+1)} \frac{1}{2\sqrt{\pi}(2\pi|y|)^{v+a}} \int_{-\infty}^{\infty} \frac{e^{-i4\pi|y|t}}{(1+t^2)^{v+a+1/2}} dt.$$

From Stirling's formula, we see that the quotient of the  $\Gamma$ -factors is  $O(1)$ . As for the rest, integrating by parts, then shifting the contour to  $\Im t = -(|v| + a + 2)^{-1}$  (similarly as in [BM05, (4.2-5)]),

$$\frac{1}{2\sqrt{\pi}(2\pi|y|)^{v+a}} \int_{-\infty}^{\infty} \frac{e^{-i4\pi|y|t}}{(1+t^2)^{v+a+1/2}} dt \ll \frac{|v| + a + 1}{|y|^{v+a-1}} \exp\left(\frac{-(3+1/3)\pi|y|}{|v| + a + 1}\right).$$

Here,  $|v| + a + 1 \ll |y|^{1/2}$ , so as above,

$$|v| + a + 1 \ll e^{|y|/(|v|+a+1)}, \quad |y|^{-v-a+1} \ll e^{|y|/(3(|v|+a+1))},$$

giving

$$\frac{K_{v+l-|q+p|-j}(4\pi|y|)}{\Gamma(l+1+v-j)} \ll \exp\left(-\frac{2|y|}{|v|+l+1}\right).$$

Altogether

$$\mathcal{W}_{(l,q),(v,p)}(y) \ll \exp\left(-\frac{|y|}{|v|+l+1}\right)$$

as claimed. □

Now borrowing an idea from [BH08, p.330], we give a further bound on  $\mathcal{W}_{(l,q),(v,p)}$ .

**Lemma 3.4.** *For all  $y \in \mathbf{C}^\times$ ,*

$$(45) \quad \mathcal{W}_{(l,q),(v,p)}(y) \ll (|y|^{3/4} + |y|)(l^4 + 1)(|v|^2 + 1)(|p| + 1).$$

*Proof.* Our starting point is a special Jacquet-Langlands functional equation

$$(46) \quad \mathcal{W}_{(l,q),(v,p)}(y) = \kappa(p, l, q) \pi \int_{\mathbf{C}^\times} j(\sqrt{t}) \mathcal{W}_{(l,-q),(v,p)}(t/y) d_{\mathbf{C}}^\times t,$$

where  $j$  is defined in (36) and  $|\kappa(p, l, q)| = 1$ . This is proved in [BM02, Theorem 2 and (3)] in a different formulation, one is straight-forward from the other using [LG04, (2.30), (2.43) and (4.2)]. Note that in [BM02], it is stated only for the principal series (i.e.  $\Re v = 0$ ) and even representations (i.e.  $p \in \mathbf{Z}$ ), but the result extends to the complementary series by analytic continuation, the odd case can be handled similarly (see [BM02, p.90]). Also note that  $j(\sqrt{t})$  does not lead to confusion, since  $j(t)$  is an even function of  $t$  (by Lemma 3.1).

In (46), split up the integral as

$$\begin{aligned} \mathcal{W}_{(l,q),(v,p)}(y) &\ll \overbrace{\int_{0 < |t| < |y|(l^4+1)(|v|^2+1)} j(\sqrt{t}) \mathcal{W}_{(l,-q),(v,p)}(t/y) d_{\mathbf{C}}^\times t}^{\text{I}} \\ &\quad + \overbrace{\int_{|t| \geq |y|(l^4+1)(|v|^2+1)} j(\sqrt{t}) \mathcal{W}_{(l,-q),(v,p)}(t/y) d_{\mathbf{C}}^\times t}^{\text{II}}. \end{aligned}$$

First estimate I. Using Cauchy-Schwarz,

$$\begin{aligned} \text{I} &\ll \left( \int_{0 < |t| < |y|(l^4+1)(|v|^2+1)} |j(\sqrt{t})|^2 d_{\mathbf{C}}^\times t \right)^{1/2} \\ &\quad \cdot \left( \int_{0 < |t| < |y|(l^4+1)(|v|^2+1)} |\mathcal{W}_{(l,-q),(v,p)}(t/y)|^2 d_{\mathbf{C}}^\times t \right)^{1/2}. \end{aligned}$$



The second factor is at most 1, since the Whittaker functions have  $L^2$ -norm 1 (recall (20) and the remark after that). In the first factor, we may apply (37), giving

$$I \ll \max(|y|, |y|^{3/4})(l^4 + 1)(|v|^2 + 1)(|p| + 1).$$

In the second term II, we apply Lemma 3.3 together with (37). This gives

$$II \ll \max(|y|, |y|^{3/4})(|v| + l + 1)(|p| + 1).$$

Summing up, we arrive at (45).  $\square$

**3.3. A bound on the supremum norm of a cusp form.** The aim of this section is to give a bound of the form  $\|\phi\|_{\text{sup}} \ll_{F,\pi} \|\phi\|_{S_d}$ , where  $\phi$  is a sufficiently smooth newform in the cuspidal representation  $\pi$ , and the order  $d$  depends only on  $F$ .

**Proposition 3.5.** *Let  $(\pi, V_\pi)$  be an irreducible cuspidal representation. Assume that  $\phi \in V_\pi(\mathfrak{c}_\pi)$  such that  $\|\phi\|_{S_{14r+36s}}$  exists. Then*

$$\|\phi\|_\infty = \sup_{g \in \text{GL}_2(\mathbf{A})} |\phi(g)| \ll_{F,\pi} \|\phi\|_{S_{14r+36s}}.$$

*Proof.* We follow the proof of [BH10, Lemma 5]. Note that there is a correction made later in its erratum, which we also build in. First assume  $\phi \in V_\pi(\mathfrak{c}_\pi)$  is of pure weight  $\mathbf{w}$ . Let  $\eta_1, \dots, \eta_h \in \mathbf{A}_{\text{fin}}^\times$  be finite ideles representing the ideal classes. By strong approximation [Bum97, Theorem 3.3.1], there exist  $\gamma \in \text{GL}_2(F)$ ,  $g' \in \text{GL}_2(F_\infty)$ ,  $k \in K(\mathfrak{o})$  such that for some  $1 \leq j \leq h$ ,

$$g = \gamma \left( g' \times \begin{pmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} k \right).$$

Now decompose  $g'$  as

$$g' = z\gamma' a_{j'} \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k',$$

where  $a_{j'} \in \text{GL}_2(F)$  (regarded as an element of  $\text{GL}_2(F_\infty)$ ) is from a fixed set  $\{a_1, \dots, a_{2r_h}\}$ ,  $y' > \delta$  at all archimedean places, where  $\delta > 0$  is fixed (depending only on  $F$ ),  $z \in Z(F_\infty)$ ,  $\gamma' \in \text{SL}_2(\mathfrak{o})$ ,  $k' \in K_\infty$ , this can be done by [Mag13b, Lemma 4.9]. From now on, we regard  $z$  as an element in  $Z(\mathbf{A})$ , therefore we have

$$g = z\gamma\gamma' a_{j'} \left( \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k' \times a_{j'}^{-1} \gamma'^{-1} \begin{pmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} k \right).$$

Here,  $a_{j'}^{-1} \gamma'^{-1} \begin{pmatrix} \eta_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} k$  lies in a fixed compact subset of  $\text{GL}_2(\mathbf{A}_{\text{fin}})$ , which can be covered with finitely many left cosets of the open subgroup  $K(\mathfrak{c}_\pi)$ . Therefore

$$g = z\gamma^* \left( \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \times m \right) (k_\infty^* \times k_{\text{fin}}^*),$$

where  $\gamma^* \in \text{GL}_2(F)$ ,  $k^* = k_\infty^* \times k_{\text{fin}}^* \in K_\infty \times K(\mathfrak{c}_\pi)$ , and  $m \in \text{GL}_2(\mathbf{A}_{\text{fin}})$  runs through a finite set depending only on  $F$  and  $\mathfrak{c}_\pi$ ,  $y' > \delta$  at all archimedean places.

Now let  $\phi_m(g) = \phi(gm)$ . Obviously,  $\phi$  and  $\phi_m$  have the same supremum and Sobolev norms, and when  $g$  decomposes as above,

$$(47) \quad \begin{aligned} |\phi(g)| &= \left| \phi_m \left( \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} k_\infty^* \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right| \\ &\ll_F \left| \phi_m \left( \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right| \prod_{j=r+1}^{r+s} (l_j + 1)^7, \end{aligned}$$

where we applied [Mag13b, Lemma 4.8] in the last estimate.

The function  $\phi_m$  can be regarded as a classical automorphic function on  $\mathrm{GL}_2(F_\infty)$  (see [Mag13b, Section 4.3]). Therefore, analogously to (21), we see that  $\phi_m$  (as a function on  $\mathrm{GL}_2(F_\infty)$ ) can be expanded into Fourier series

$$(48) \quad \phi_m \left( \begin{pmatrix} y' & x' \\ 0 & 1 \end{pmatrix} \right) = \sum_{0 \neq t \in \mathfrak{f}} a(t) \mathscr{W}_{\mathbf{w}, \mathbf{r}}(ty') \psi_\infty(tx'),$$

where  $\mathfrak{f}$  is a fractional ideal (regarded as a lattice in  $F_\infty$ ) depending only on  $F$  and  $\pi$ . Here,

$$a(t) \ll_{F, \pi} \|\phi\| \prod_{j \leq r} (1 + |q_j|^5) \prod_{j > r} (1 + l_j^{10}),$$

by [Mag13b, Lemma 4.11].

Now (47) and (48) give

$$(49) \quad |\phi(g)| \ll_{F, \pi} \|\phi\| \prod_{j \leq r} (1 + |q_j|^5) \prod_{j > r} (1 + l_j^{17}) \sum_{0 \neq t \in \mathfrak{f}} |\mathscr{W}_{\mathbf{w}, \mathbf{r}}(ty')|.$$

We turn our attention to  $\sum_{0 \neq t \in \mathfrak{f}} |\mathscr{W}_{\mathbf{w}, \mathbf{r}}(ty')|$ .

From (40), (41), (42), (44) and (45), we see that

$$(50) \quad \begin{aligned} \mathscr{W}_{q, v}(y) &\ll_{F, \pi} (|q|^3 + 1) \exp \left( -\frac{|y|}{2(q^2 + 1)(|v|^2 + 1)} \right), \\ \mathscr{W}_{(l, q), (v, p)}(y) &\ll_{F, \pi} (l^8 + 1) \exp \left( -\frac{|y|}{2(l^4 + 1)(|v|^2 + 1)} \right) \end{aligned}$$

hold for all  $y \neq 0$ , at real and complex places, respectively.

Setting  $A_j = |q_j|^3 + 1$ ,  $B_j = 2(q_j^2 + 1)(|v_j|^2 + 1)$  at real places, and  $A_j = l_j^8 + 1$ ,  $B_j = 2(l_j^4 + 1)(|v_j|^2 + 1)$  at complex places, (50) and a simple calculation yields

$$\sum_{0 \neq t \in \mathfrak{f}} |\mathscr{W}_{\mathbf{w}, \mathbf{r}}(ty')| \ll_{F, \mathfrak{f}} \prod_{j=1}^{r+s} A_j B_j^{\deg[F_j: \mathbf{R}]},$$

where we used that  $|y'_j| > \delta$  at all places, and also the fact that a lattice  $L$  in  $F_\infty$  contains  $O_L(N^{r+2s})$  points of supremum norm  $\leq N$ .

Therefore,

$$\sum_{0 \neq t \in \mathfrak{f}} |\mathscr{W}_{\mathbf{w}, \mathbf{r}}(ty')| \ll_{F, \pi} \prod_{j=1}^r (|q_j|^5 + 1) \prod_{j=r+1}^{r+s} (l_j^{16} + 1),$$

which, together with (49), give rise to

$$(51) \quad |\phi(g)| \ll_{F, \pi} \|\phi\| \prod_{j \leq r} (1 + q_j^{10}) \prod_{j > r} (1 + l_j^{33}).$$

Assume now a sufficiently smooth  $\phi \in V_\pi$  is not necessarily of pure weight. We may decompose it as

$$(52) \quad \phi = \sum_{\mathbf{w} \in W(\pi)} b_{\mathbf{w}} \phi_{\mathbf{w}},$$

where  $\phi_{\mathbf{w}}$  is a weight  $\mathbf{w}$  function of norm 1 in  $V_\pi$ . Let us follow the common practice and using the smoothness of  $\phi$ , estimate  $b_{\mathbf{w}}$  in terms of  $\sup \mathbf{w} = \max(|q_1|, \dots, |q_r|, l_{r+1}, \dots, l_{r+s})$ . Using Parseval, then (10) and (11), we find, for any nonnegative integer  $k$ ,

$$(53) \quad b_{\mathbf{w}} = \langle \phi, \phi_{\mathbf{w}} \rangle \ll_k \frac{1}{(1 + (\sup \mathbf{w}))^{2k}} \langle \Omega_{t, j}^k \phi, \phi_{\mathbf{w}} \rangle \ll_k \frac{1}{(1 + (\sup \mathbf{w}))^{2k}} \|\phi\|_{S_{2k}},$$

where  $j$  is the index of an archimedean place, where the maximum (in the definition of  $\sup \mathbf{w}$ ) is attained. Together with (51) and (52), this implies

$$|\phi(g)| \ll_{F, \pi, k} \sum_{\mathbf{w} \in W(\pi)} (1 + \sup \mathbf{w})^{10r+33s-2k} \|\phi\|_{S_{2k}}.$$

Here, choosing  $k = 7r + 18s$ , we obtain the statement by noting that  $\sup \mathbf{w}$  attains the positive integer  $N$  on a set of cardinality  $O_F(N^{r+2s-1})$ .  $\square$

### 3.4. A bound on Kirillov vectors.

**Proposition 3.6.** *Let  $(\pi, V_\pi)$  be an irreducible automorphic representation occurring in  $L^2(\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbf{A}), \omega)$ . Let  $\mathfrak{t} \subseteq \mathfrak{o}$  be an ideal,  $a, b, c$  be nonnegative integers,  $0 < \varepsilon < 1/4$ . Let  $P \in \mathbb{C}[x_1, \dots, x_{r+2s}]$  be a polynomial of degree at most  $a$  in each variable. Set then*

$$\mathcal{D} = P \left( \left( y_j \frac{\partial}{\partial y_j} \right)_{j \leq r}, \left( y_j \frac{\partial}{\partial y_j} \right)_{j > r}, \left( \bar{y}_j \frac{\partial}{\partial \bar{y}_j} \right)_{j > r} \right).$$

Assume  $\phi \in R^t V_\pi(\mathfrak{c}_\pi)$  such that  $\|\phi\|_{S_{(a+b+2c+6)r+(a+b+2c+8)s+a+b+2c+4}}$  exists. Then  $\mathcal{D}W_\phi$  exists and

$$\begin{aligned} \mathcal{D}W_\phi(y) &\ll_{a,b,c,P,F,\varepsilon} \|\phi\|_{S_{(a+b+2c+6)r+(a+b+2c+8)s+a+b+2c+4}} \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^{-c} \\ &\cdot \prod_{j=1}^r (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) (\min(1, |y_j|^{-b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/4} + |y_j|) (\min(1, |y_j|^{-b})). \end{aligned}$$

*Proof.* We follow the proof of [BH10, Lemma 4]. First assume  $\phi \in R^t V_\pi(\mathfrak{c}_\pi)$  is of pure weight  $\mathbf{w}$ . Then we may write

$$|W_\phi(y)| = \|W_\phi\| \cdot |\mathcal{W}_{\mathbf{w}, \mathbf{r}}(y)|.$$

Using (24), (25), (31), (33), the remark after that, and the estimates (41), (42), (45), we have, for  $0 < \varepsilon < 1/4$ ,

$$\begin{aligned} W_\phi(y) &\ll_{F,\varepsilon} \|\phi\| \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^\varepsilon \prod_{j=1}^r (1 + |\mathbf{v}_j| + |q_j|)^{1+\theta} (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \\ &\cdot \prod_{j=r+1}^{r+s} (1 + |p_j|)(1 + |\mathbf{v}_j|^2)(1 + l_j^4)(|y_j|^{3/4} + |y_j|). \end{aligned}$$

This gives

$$\begin{aligned} W_\phi(y) &\ll_{F,\varepsilon} \|\phi\| \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^2 \prod_{j=1}^r (1 + |q_j|)^{1+\theta} (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \\ &\cdot \prod_{j=r+1}^{r+s} (1 + l_j^4)(|y_j|^{3/4} + |y_j|). \end{aligned}$$

Now take an arbitrary  $\phi \in R^t V_\pi(\mathfrak{c}_\pi)$ , which is sufficiently smooth. Then recalling (52) and (53), in

$$\phi = \sum_{\mathbf{w} \in W(\pi)} b_{\mathbf{w}} \phi_{\mathbf{w}}, \quad \|\phi_{\mathbf{w}}\| = 1,$$

we have

$$b_{\mathbf{w}} \ll_k \frac{1}{(1 + (\sup \mathbf{w}))^{2k}} \|\phi\|_{S_{2k}}.$$

Now choosing  $k = 3r + 4s$ , we obtain

$$\begin{aligned} (54) \quad W_\phi(y) &\ll_{F,\varepsilon} \|\phi\|_{S_{2(3r+4s)}} \mathcal{N}(\mathfrak{t})^\varepsilon \mathcal{N}(\mathfrak{c}_\pi)^\varepsilon \mathcal{N}(\mathbf{r})^2 \prod_{j=1}^r (|y_j|^{1/2-\varepsilon} + |y_j|^{1/2-\theta-\varepsilon}) \\ &\cdot \prod_{j=r+1}^{r+s} (|y_j|^{3/4} + |y_j|). \end{aligned}$$

The differential operators given in (3) and (4) act on the sufficiently smooth Kirillov vectors. We record the action of some of them (neglecting some absolute scalars for simplicity). Of course,  $\Omega_{j(\pm)}$  act by  $\lambda_{(\pm)}$ . From (23) and (32), it is easy to derive that  $\mathbf{R}_j, \mathbf{V}_{1,j} + \mathbf{W}_{1,j}, \mathbf{V}_{2,j} + \mathbf{W}_{2,j}$  act via a multiplication by  $y_j, \Re y_j, \Im y_j$ ,

respectively; finally  $\mathbf{H}_j$  by  $y_j(\partial/\partial y_j)$ , and  $\mathbf{H}_{1,j}, \mathbf{H}_{2,j}$  by  $y_j(\partial/\partial y_j) + \overline{y_j}(\partial/\partial \overline{y_j})$ ,  $iy_j(\partial/\partial y_j) - i\overline{y_j}(\partial/\partial \overline{y_j})$ , respectively.

Now assume given  $a, b, c$  and the polynomial  $P$  as in the statement. Then

$$\mathcal{D} = \text{const}_{F, \mathcal{D}} P \left( (\mathbf{H}_j)_{j \leq r}, ((\mathbf{H}_{1,j} - i\mathbf{H}_{2,j})/2)_{j > r}, ((\mathbf{H}_{1,j} + i\mathbf{H}_{2,j})/2)_{j > r} \right),$$

and define the differential operator

$$\mathcal{D}' = \left( \prod_{j \leq r} \Omega_j^{c+2} \prod_{j > r} \Omega_{j,+}^{c+2} \right) \left( \prod_{\substack{1 \leq j \leq r \\ |y_j| \geq 1}} \mathbf{R}_j^b \right) \left( \prod_{\substack{r+1 \leq j \leq r+s \\ |y_j| \geq 1 \\ |\Re y_j| \geq |\Im y_j|}} (\mathbf{V}_{1,j} + \mathbf{W}_{1,j})^b \right) \left( \prod_{\substack{r+1 \leq j \leq r+s \\ |y_j| \geq 1 \\ |\Re y_j| < |\Im y_j|}} (\mathbf{V}_{2,j} + \mathbf{W}_{2,j})^b \right).$$

Applying (54) to  $\mathcal{D}' \mathcal{D} \phi$ , we obtain the statement.  $\square$

#### 4. PROOF OF THEOREM 1

Using the archimedean Kirillov model, we see that there exist functions  $\phi_1 \in V_{\pi_1}(\mathfrak{c}_{\pi_1})$ ,  $\phi_2 \in V_{\pi_2}(\mathfrak{c}_{\pi_2})$  such that  $W_{\phi_1} = W_1$ ,  $W_{\phi_2} = W_2$ . Set then

$$\Phi = R_{(l_1)} \phi_1 R_{(l_2)} \overline{\phi_2}.$$

Then since  $\mathfrak{c} = \text{lcm}(l_1 \mathfrak{c}_{\pi_1}, l_2 \mathfrak{c}_{\pi_2})$ , we see that  $\Phi$  is right  $K(\mathfrak{c})$ -invariant. Also, since  $W_1, W_2$  are from the Schwarz space,  $\phi_1, \phi_2$  are smooth and have finite Sobolev norms of arbitrarily large order, the same hold for  $\Phi \in L^2(\text{GL}_2(F) \backslash \text{GL}_2(\mathbf{A})/K(\mathfrak{c}), 1)$  (use (24), (25) and Proposition 3.5 together with [Ven10, Lemma 8.4]). Then by (2), (15), (29) and (34), we can decompose  $\Phi$  as

$$(55) \quad \Phi = \Phi_{\text{sp}} + \int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}_{\pi}^{-1}} \Phi_{\mathfrak{w}, \mathfrak{t}} d\mathfrak{w},$$

where  $\Phi_{\mathfrak{w}, \mathfrak{t}} \in R^t(V_{\mathfrak{w}}(\mathfrak{c}_{\mathfrak{w}}))$  and  $\Phi_{\text{sp}}$  is the orthogonal projection of  $\Phi$  to  $L_{\text{sp}}$ . Now set  $W_{\mathfrak{w}, \mathfrak{t}} = W_{\Phi_{\mathfrak{w}, \mathfrak{t}}}$ . We claim this fulfills the property stated in Theorem 1. Given  $Y \in (0, \infty)^{r+s}$ ,  $\mathfrak{n} \subseteq \mathfrak{o}$ ,  $0 \neq q \in \mathfrak{n}$ , let  $(y_{\text{fin}}) = \mathfrak{n}$ , and  $y_{\infty} = Y$ . We compute

$$(56) \quad \int_{F \backslash \mathbf{A}} \Phi \left( \begin{pmatrix} y^{-1} & x \\ 0 & 1 \end{pmatrix} \right) \psi(-qx) dx$$

in two ways. On the one hand, we use (55). Here,  $q \neq 0$  implies that  $\Phi_{\text{sp}}$  has zero contribution to (56), and we obtain that (56) equals

$$\int_{(\mathfrak{c})} \sum_{\mathfrak{t} | \mathfrak{c}_{\mathfrak{w}}^{-1}} \frac{\lambda_{\mathfrak{w}}^{\mathfrak{t}}(qn^{-1})}{\sqrt{\mathcal{N}(qn^{-1})}} W_{\mathfrak{w}, \mathfrak{t}} \left( \left( \frac{q_j}{Y_j} \right)_j \right) d\mathfrak{w}$$

from (26) and (30). On the other hand, using (14) and (22) together with the choice of  $\phi_1, \phi_2$ , we obtain that (56) equals

$$\sum_{l_1 t_1 - l_2 t_2 = q, 0 \neq t_1, t_2 \in \mathfrak{n}} \frac{\lambda_{\pi_1}(t_1 \mathfrak{n}^{-1}) \overline{\lambda_{\pi_2}(t_2 \mathfrak{n}^{-1})}}{\sqrt{\mathcal{N}(t_1 t_2 \mathfrak{n}^{-2})}} W_1 \left( \left( \frac{(l_1 t_1)_j}{Y_j} \right)_j \right) \overline{W_2 \left( \left( \frac{(l_2 t_2)_j}{Y_j} \right)_j \right)}.$$

The equality of the last two displays is exactly the statement about the spectral decomposition.

We still have to prove the inequalities concerning the  $L^2$ -norm of the derivatives. By Proposition 3.6 and a consequence of (7) (see [BH10, (85)]), we have

$$\int_{(c)} \sum_{t|cc_{\bar{\omega}}^{-1}} (\mathcal{N}(\mathbf{r}_{\bar{\omega}}))^{2c} |\mathcal{D}W_{\bar{\omega},t}(y)|^2 d\bar{\omega} \ll_{F,\varepsilon,\pi_1,\pi_2,a,b,c,P} \mathcal{N}(l_1 l_2)^\varepsilon \|\Phi\|_{S_\beta}^2$$

$$\cdot \prod_{j=1}^r (|y_j|^{1-\varepsilon} + |y_j|^{1-2\theta-\varepsilon})(\min(1, |y_j|^{-2b})) \prod_{j=r+1}^{r+s} (|y_j|^{3/2} + |y_j|^2)(\min(1, |y_j|^{-2b}))$$

with  $\beta = 2(3r + 4s + 2) + (r + s)(a + b + 2c)$ . For any differential operator  $\mathcal{D}' \in U(\mathfrak{g})$  of order  $k$ , we have

$$\|\mathcal{D}'\phi_1\|_\infty \ll_{F,\pi_1} \|\phi_1\|_{S_{k+2(7r+18s)}}, \quad \|\mathcal{D}'\phi_2\|_\infty \ll_{F,\pi_2} \|\phi_2\|_{S_{k+2(7r+18s)}}$$

by Proposition 3.5. Since  $Z(\mathbf{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A})$  has finite volume, and the operators  $R_{(l_1,2)}$  do not affect Sobolev norms,  $\|\Phi\|_{S_\beta} \ll_{F,\pi_1,\pi_2} \|\phi_1\|_{S_{\beta+2(7r+18s)}} \|\phi_2\|_{S_{\beta+2(7r+18s)}}$ . Now (24) completes the proof of Theorem 1.

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