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An Axiomatization of the Proportional Rule

in Financial Networks

PÉTER CSÓKA – P. JEAN-JACQUES HERINGS

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Authors:

Péter Csóka
research fellow
"Momentum" Game Theory Research Group
Centre for Economic and Regional Studies, Hungarian Academy of Sciences
and Corvinus University of Budapest
E-mail: peter.csoka@uni-corvinus.hu

P. Jean-Jacques Herings
Department of Economics, Maastricht University, The Netherlands
E-mail: P.Herings@maastrichtuniversity.nl

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Abstract

The most important rule to determine payments in real-life bankruptcy problems is the proportional rule. Many bankruptcy problems are characterized by network aspects and default may occur as a result of contagion. Indeed, in financial networks with defaulting agents, the values of the agents' assets are endogenous as they depend on the extent to which claims on other agents can be collected. These network aspects make an axiomatic analysis challenging. This paper is the first to provide an axiomatization of the proportional rule in financial networks. Our two central axioms are impartiality and non-manipulability by identical agents. The other axioms are claims boundedness, limited liability, priority of creditors, and continuity.

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Az arányos csődszabály axiomatizálása pénzügyi hálózatokban

Csóka Péter - P. Jean-Jacques Herings

Összefoglaló

Valós csődproblémákban a fizetések meghatározására használt legfontosabb módszer az arányos csődszabály. Sok csődproblémában megjelenik a hálózati aspektus és a csőd fertőzés miatt is bekövetkezhet. Ha pénzügyi hálózatban mennek csődbe szereplők, akkor ezek eszközeinek értéke endogén módon határozódik meg, mivel az is számít, hogy a többi szereplőtől mennyire lehet a követeléseket behajtani. Ezek a hálózati hatások megnehezítik az axiomatikus elemzést. Tanulmányunkban először adunk axiomatizációt pénzügyi hálózatokban az arányos csődszabályra. Két központi axiómánk a pártatlanság és az azonos szereplők általi manipulálhatatlanság. A további szükséges axiómák: a követelések mint felső korlát, a korlátolt felelősség, a hitelezők elsőbbsége és a folytonosság.

Tárgyszavak: pénzügyi hálózatok, rendszerkockázat, csődszabályok, arányos csődszabály

JEL kódok: C71, G10

An Axiomatization of the Proportional Rule in Financial Networks[*]

Péter Csóka[†]   P. Jean-Jacques Herings[‡]

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Abstract

The most important rule to determine payments in real-life bankruptcy problems is the proportional rule. Many bankruptcy problems are characterized by network aspects and default may occur as a result of contagion. Indeed, in financial networks with defaulting agents, the values of the agents' assets are endogenous as they depend on the extent to which claims on other agents can be collected. These network aspects make an axiomatic analysis challenging. This paper is the first to provide an axiomatization of the proportional rule in financial networks. Our two central axioms are impartiality and non-manipulability by identical agents. The other axioms are claims boundedness, limited liability, priority of creditors, and continuity.

Keywords: Financial networks, systemic risk, bankruptcy rules, proportional rule.

JEL Classification: C71, G10.

1 Introduction

The principle of proportionality plays an important role in bankruptcy law across the globe. The EC Council Regulation on insolvency proceedings states that[1]

> Every creditor should be able to keep what he has received in the course of insolvency proceedings but should be entitled only to participate in the distribution of total assets in other proceedings if creditors with the same standing have obtained the same proportion of their claims.

The principle of proportionality is also important for American bankruptcy law, according to which claimants of equal status should receive payments proportional to the value of their liabilities, see Kaminski (2000).

Given the prominence of the proportional rule in practice, it is important to understand its crucial features by finding an axiomatization. Starting with the seminal paper of O'Neill (1982), the literature that takes an axiomatic approach to the bankruptcy problem assumes there is a single bankrupt agent while the other agents have claims on his estate. We refer to this class of problems as claims problems. The central question is how this estate should be divided over the claims and the axiomatic approach has provided firm underpinnings for a number of well-known division rules. See Thomson (2003), Thomson (2013), and Thomson (2015) for an overview of this stream of the literature.

Notable axiomatizations of the proportional rule for claims problems are given by Young (1988), Moreno-Ternero (2006), Ju, Miyagawa, and Sakai (2007), and Thomson (2016) by notions like self duality, composition up, composition down, non-manipulability and merging-and splitting proofness. For related axiomatizations, see Tasnádi (2002) on probabilistic rationing methods and Moulin (2016) on proportional assignment and rationing of goods with different characteristics.

Recent crisis on financial markets related to the Lehman bankruptcy as well as sovereign debt problems of European countries have spurred an extensive literature on systemic risk that takes a network perspective to the bankruptcy problem, starting with the contribution by Eisenberg and Noe (2001). The literature that is based on this model, either extending it (Cifuentes, Ferrucci, and Shin, 2005; Shin, 2008; Rogers and Veraart, 2013; Schuldenzucker, Seuken, and Battiston, 2016), or using it to relate the number and magnitude of defaults to the network topology (Gai and Kapadia, 2010; Elliott, Golub, and Jackson, 2014; Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015; Capponi, Chen, and Yao, 2013; Glasserman and Young, 2015), or measuring systemic risk (Chen, Iyengar, and Moallemi, 2013; Demange, 2017) uses the proportional rule to determine the mutual payments by the

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[1]Council Regulation (EC) No 1346/2000 of 29 May 2000 on insolvency proceedings: `http://eur-lex.europa.eu/legal-content/EN/TXT/HTML/?uri=CELEX:02000R1346-20140709&qid=1471509284560`

agents. For an overview of this stream of the literature, we refer to the excellent survey by Glasserman and Young (2016).

The aim of this paper is to provide axiomatic foundations for the use of the proportional rule for bankruptcy problems in financial networks. A financial network consists of a set of agents, with each agent being characterized by his endowments and his liabilities towards the other agents. A bankruptcy rule determines the actual payments of agents to each other, collected in a payment matrix. More technically, a bankruptcy rule is simply a function that assigns to each financial network a payment matrix. To clearly distinguish concepts, we use the terminology bankruptcy rule for financial network problems and division rule for the simpler context of claims problems.

An agent is in fundamental default if he cannot fully pay his liabilities, even if he receives full payments on all his claims from the other agents. In a network setting, a default can also result from contagion, where an agent defaults only because other agents are not fully paying their liabilities to him. Because of these mutual dependencies, it is not trivial to define the proportional rule in a financial network and following the seminal contribution by Eisenberg and Noe (2001) one proceeds as follows. First, one determines the asset value of an agent, the value of his endowments together with the payments as collected from the other agents. Under the proportional rule, an agent spends his asset value in a proportional way over his liabilities, up to the value of those liabilities. Subtracting the payments as made by an agent from his asset value yields an agent's equity. Because of the mutual dependencies caused by the network aspect, one has to solve a system of equations to determine the actual payments. The agents' asset values and equities are therefore determined endogenously in a financial network using the proportional rule.

In real life, entities frequently merge or create spin-offs. When mergers or spin-offs do not generate added value, they should not influence the payments made to and received from agents not involved in them. The axiom which requires that the merger of a set of agents or the split of an agent into multiple agents should not affect the payment matrix is called non-manipulability.

The proportional rule does not satisfy non-manipulability. As an example, one expects the merger of a bankrupt and a solvent agent to affect the payment matrix generated by the proportional rule, since part of the assets of the solvent agent that were not seized before can now be used for making payments related to claims on the bankrupt agent. The other way around, if an agent is allowed to create a new entity that receives all its liabilities but none of its claims or endowments, then the agent is clearly going to benefit since it will end up in paying none of its liabilities. This kind of manipulation is illegal in reality, since in winding up or in insolvency proceedings, the borrower is not allowed to do anything that would threaten directly or indirectly the payments to its lenders.

We show that non-manipulability is incompatible with any reasonable bankruptcy rule in financial networks by proving that there is no bankruptcy rule satisfying non-manipulability, claims boundedness, limited liability, and priority of creditors. Claims boundedness expresses that no agent pays an amount in excess of his liabilities. A bankruptcy rule satisfies limited liability if it leads to a payment matrix such that none of the agents ends up with negative equity. Priority of creditors is satisfied if the only circumstance under which an agent is allowed to default is when his equity is equal to zero. This impossibility result also shows financial networks to be quite different from claims problems, where this form of non-manipulability is compatible with many division rules and has been used by Moreno-Ternero (2006) and Ju, Miyagawa, and Sakai (2007) to axiomatize the proportional rule.

We significantly weaken non-manipulability to non-manipulability by identical agents. Agents are identical if they have the same endowments, claims, and liabilities. This is a very strong requirement and implies, for instance, that the mutual liabilities of the identical agents are equal to zero. Non-manipulability by identical agents requires that the merger of a group of agents that are identical or the split of an agent into multiple identical agents should not affect the payment matrix. Our other main substantive axiom is impartiality, requiring that two agents with the same claim on an agent should receive the same payment from him. We show that the two main axioms together with claims boundedness, limited liability, priority of creditors, and impartiality it is shown to axiomatize the proportional rule on the rational domain, i.e. the domain of financial networks with the values of all primitives expressed by rational numbers. When the primitives of the financial network are denoted in real numbers, we have to add continuity as an axiom to obtain an axiomatization of the proportional rule.

The way the proportional rule for claims problems is extended to financial networks can be used to extend any division rule. The resulting bankruptcy rules consist of computing each agent's asset value and then making payments in accordance with the given division rule. Although clearing payment matrices in financial networks are unique when proportional division rules are used, this is not the case in general. Adapting the proof of Csóka and Herings (2016) for the discrete case, we show that there exists a least clearing payment matrix and a greatest clearing payment matrix. We select the greatest clearing payment matrix to define the bankruptcy rule and formulate a programming problem to calculate it.

In the demonstration that our axioms are independent, we show the surprising result that the constrained equal awards rule for financial networks does not satisfy continuity, whereas the constrained equal losses rule does. Under the constrained equal awards division rule, for a recent axiomatization of its weighted version see Flores-Szwagrzak (2015), all

claimants get the same amount, up to the value of their claim. The constrained equal losses division rule is its dual and imposes that all claimants face the same loss, up to the value of their claim. We show that constrained equal losses division rules lead to unique clearing payment matrices, but constrained equal awards division rules do not.

Groote Schaarsberg, Reijnierse, and Borm (2013) also extend division rules for claims problems to financial networks, but focus on the resulting equity rather than payment matrices. They show that equity is uniquely determined and they characterize equity corresponding to the Aumann-Maschler division rule within the class of division rule based bankruptcy rules.

Not all bankruptcy rules are based on division rules. For instance, a bankruptcy rule that consists of pairwise netting all mutual claims first and next applying the proportional rule to the resulting liabilities is not in this class, since in this case payments do not only depend on the asset value of an agent and his liabilities, but also on his claims towards other agents. Our axiomatization does not impose any a priori structure on bankruptcy rules. Absence of pairwise netting, for instance, is therefore a consequence of our axioms rather than an assumption.

Finally, we would like to mention an emerging literature on the extension of the bankruptcy literature to network problems as appearing in operations research. Bjørndal and Jörnsten (2010) analyze generalized bankruptcy problems with multiple estates as flow sharing problems and define the nucleolus and the constrained egalitarian solution for such problems. Moulin and Sethuraman (2013) consider bipartite rationing problems, where agents can have claims on a subset of unrelated estates. They consider whether rules for single resource problems can be consistently extended to their framework.

The rest of the paper is organized as follows. Section 2 defines financial networks and the proportional and the pairwise netting proportional rules. Section 3 defines our axioms and discusses non-manipulability in particular. Section 4 provides our axiomatization result on the rational domain. In Section 5, we define bankruptcy rules that are based on division rules and pay particular attention to the constrained equal awards and constrained equal losses rules. In Section 7 we introduce the axiom of continuity and provide the axiomatization on the real domain. Section 8 concludes.

2 Financial Networks

Let $\mathbb{N}$ be the set of all potential agents and $\mathcal{N}$ the collection of non-empty, finite subsets of $\mathbb{N}$.

A *financial network* is a triple (N, z, L) with the following interpretation.

The set of *agents* in the financial network is given by $N \in \mathcal{N}$.

The vector $z \in \mathbb{Q}_{++}^N$ represents the *endowments* of the agents, which are strictly positive rational numbers. The endowment of an agent includes all his tangible and intangible assets, but excludes the claims and liabilities the agent has towards the other agents. In the main part of the paper, we assume that all endowments, claims, and liabilities are expressed as rational numbers, unless explicitly noted otherwise. At the end of the paper, we treat the real-valued case.

The non-negative *liability matrix* $L \in \mathbb{Q}_+^{N \times N}$ describes the mutual claims of the agents. Its entry L_{ij} is the liability of agent i towards agent j or, equivalently, the claim of agent j on agent i. We make the normalizing assumption that $L_{ii} = 0$. In general, it can occur that agent i has a liability towards agent j and vice versa, so it may happen that simultaneously $L_{ij} > 0$ and $L_{ji} > 0$.

The set of all matrices in $\mathbb{Q}_+^{N \times N}$ with a zero diagonal is denoted by $\mathcal{M}(N)$. The union over all finite sets of agents of these matrices is denoted by $\mathcal{M} = \cup_{N \in \mathcal{N}} \mathcal{M}(N)$. The partial order $\leq$ on $\mathcal{M}(N)$ is defined in the usual way: For $P, P' \in \mathcal{M}(N)$ it holds that $P \leq P'$ if and only if $P_{ij} \leq P'_{ij}$ for all $(i,j) \in N \times N$. For $P \in \mathcal{M}(N)$ and $i \in N$, let $P_i \in \mathbb{Q}^N$ denote row i of P. For $P_i, P'_i \in \mathbb{Q}^N$, we write $P_i < P'_i$ if $P_{ij} \leq P'_{ij}$ for all $j \in N$ and there is $k \in N$ such that $P_{ik} < P'_{ik}$. Column $i \in N$ of the matrix P is denoted by P^i.

The set of all financial networks is denoted by $\mathcal{F}$.

Consider a financial network $(N, z, L) \in \mathcal{F}$. A *payment matrix* $P \in \mathcal{M}(N)$ describes the mutual payments to be made by the agents, that is, P_{ij} is the monetary amount to be paid by agent $i \in N$ to agent $j \in N$. Given a payment matrix $P \in \mathcal{M}(N)$, the *asset value* $a_i(N, z, P)$ of agent $i \in N$ is given by

$$a_i(N, z, P) = z_i + \sum_{j \in N} P_{ji}.$$

Subtracting the payments as made by an agent from his asset value yields an agent's equity. The *equity* $e_i(N, z, P)$ of an agent $i \in N$ is given by

$$e_i(N, z, P) = a_i(N, z, P) - \sum_{j \in N} P_{ij} = z_i + \sum_{j \in N} (P_{ji} - P_{ij}).$$

It follows immediately from the above expression that the sum over agents of their equities is the same as the sum over agents of their initial endowments.

A bankruptcy rule b associates to each financial network $(N, z, L) \in \mathcal{F}$ a payment matrix $P \in \mathcal{M}(N)$. More formally, we have the following definition.

Definition 2.1. A *bankruptcy rule* is a function $b : \mathcal{F} \to \mathcal{M}$ such that for every $(N, z, L) \in \mathcal{F}$ it holds that $b(N, z, L) \in \mathcal{M}(N)$.

The analysis of financial networks is complicated because of the mutual liability structure and the contagion effects of default. A much simpler framework is provided by the

frequently studied class of claims problems. In such a problem, an estate $E \in \mathbb{Q}_+$ has to be divided over a set of claimants $N \in \mathcal{N}$ having a vector of claims $c \in \mathbb{Q}_+^N$. To clearly distinguish concepts, we use the terminology division rule rather than bankruptcy rule in the context of claims problems. The *proportional division rule* $d^{\mathrm{p}} : \mathbb{Q}_+ \times \mathbb{Q}_+^N \to \mathbb{Q}_+^N$ assigns to claimant $j \in N$ the amount

$$
d_j^{\mathrm{p}}(E, c) = \begin{cases} 0, & \text{if } c_j = 0, \\ \min\{\frac{c_j}{\sum_{k \in N} c_k} E, c_j\}, & \text{otherwise.} \end{cases}
$$

Under the proportional division rule, the estate is divided in a proportional way over the claimants, up to the value of those claims.

For financial networks, the *proportional rule* $p : \mathcal{F} \to \mathcal{M}$ is the bankruptcy rule that takes for every agent the value of the estate equal to his asset value and next uses the proportional division rule to spend his asset value in a proportional way over his liabilities.

Definition 2.2. The *proportional rule* is the function $p : \mathcal{F} \to \mathcal{M}$ such that for every $(N, z, L) \in \mathcal{F}$ it holds that $p(N, z, L) = P$, where P solves the following system of equations:

$$
P_{ij} = d_j^{\mathrm{p}}(a_i(N, z, P), L_i), \quad i, j \in N. \tag{2.1}
$$

In system of equations (2.1) agent i is treated as a claimant on his own estate $a_i(N, z, P)$ with a claim equal to $L_{ii} = 0$ and therefore receives a payment from himself equal to zero. Using the definition of $d_j^{\mathrm{p}}(a_i(N, z, P), L_i)$, we can write the system of equations in (2.1) more explicitly as the following system of equations for $i, j \in N$,

$$
P_{ij} = \begin{cases} 0, & \text{if } L_{ij} = 0, \\ \min\{\frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, P), L_{ij}\}, & \text{otherwise.} \end{cases} \tag{2.2}
$$

The next theorem states that the system of equations (2.2) has a unique solution and it belongs to the rational payment matrices $\mathcal{M}(N)$, so the proportional rule p is well-defined by (2.1).

Theorem 2.3. *Let $(N, z, L) \in \mathcal{F}$ be a financial network. The system of equations (2.2) has a unique solution and the solution belongs to $\mathcal{M}(N)$.*

Proof. It follows from Theorem 2 in Eisenberg and Noe (2001) that the system of equations (2.2) has a unique real-valued solution, say P. It therefore remains to be shown that each entry of this matrix belongs to $\mathbb{Q}$.

Let $i \in N$ be such that $P_i = L_i$. Since $L_i \in \mathbb{Q}^N$, it follows trivially that $P_i \in \mathbb{Q}^N$.

Let $D \subset N$ be the set of defaulting agents, so $D = \{i \in N \mid P_i < L_i\}$. For every $i \in D$, we define the fraction of liabilities $\lambda_i \in (0, 1)$ that is paid by

$$
\lambda_i = \frac{\sum_{j \in N} P_{ij}}{\sum_{j \in N} L_{ij}}. \tag{2.3}
$$

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Notice that $i \in D$ implies $\sum_{j \in N} L_{ij} > 0$, so the denominator in (2.3) is well-defined and there is a one-one correspondence between λ_i and the payments P_i of agent i. Since endowments are strictly positive, it holds that $\lambda_i > 0$ for every $i \in D$.

For every $i \in D$ we have that $e_i(N, z, P) = 0$, since in this case $P_{ij} = \frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, P)$ for all $j \in N$, implying that $\sum_{j \in N} P_{ij} = a_i(N, z, P)$. Thus λ_i is the unique solution to the following system of equations:

$$(\sum_{j \in N} L_{ij})\lambda_i - \sum_{j \in D} L_{ji}\lambda_j = z_i + \sum_{j \in N \setminus D} L_{ji}, \quad i \in D.$$

Since this is a linear system of equations in λ with a unique solution and all the coefficients are rational, it follows that the solution must be rational too. $\qquad\square$

Section 5 discusses how division rules for claims problems can be turned into bankruptcy rules for financial networks more generally.

The proof above implies that the payments made under the proportional rule p in the financial network $F = (N, z, L) \in \mathcal{F}$ satisfy three properties. First, for every $i \in N$, there is $\lambda_i \in \mathbb{Q}_+ \cap (0, 1]$ such that row i of the payment matrix $p_i(F) = \lambda_i L_i$. Here λ_i is the fraction of agent i's liabilities that is going to be paid. Second, bankrupt and of course also solvent agents should not end up with negative equity, so for every $i \in N$, $e_i(N, z, p(F)) \geq 0$. Finally, an agent is not allowed to default when having positive equity: $p_i(F) < L_i$ implies $e_i(N, z, p(F)) = 0$. Taking these three properties into account, the payment matrix generated by the proportional rule can also be found as the solution to a linear programming problem. The linear programming formulation will turn out to be very useful in several of the proofs. Let $\mathbb{1}$ denote a vector of ones of appropriate dimension.

Theorem 2.4 (Eisenberg and Noe (2001)). *Let $(N, z, L) \in \mathcal{F}$ be a financial network and let P' solve the following linear programming problem:*

$$
\begin{aligned}
&\max_{P \in \mathbb{R}_+^{N \times N}, \lambda \in \mathbb{R}_+^N} \sum_{i \in N} \sum_{j \in N} P_{ij}, \\
&subject\ to \\
&\quad P_{ij} = \lambda_i L_{ij}, && i, j \in N, \\
&\quad \lambda_i \leq 1, && i \in N, \\
&\quad z + P^\top \mathbb{1} - P\mathbb{1} \geq 0.
\end{aligned}
\tag{2.4}
$$

Then it holds that $p(N, z, L) = P'$.

This result is presented as Lemma 4 in Eisenberg and Noe (2001). The first and second constraint in the linear program (2.4) guarantee that payments are proportional to the liabilities and that no agent receives more than his claim. The third constraint ensures

that no agent ends up with negative equity. The property that an agent is not allowed to default when having positive equity follows from the fact that the solution maximizes the objective function. Otherwise, it would be possible to increase the value of the objective function by having the defaulting agent make additional payments.

An alternative to the proportional rule is to first revise the claims by doing a round of pairwise netting and by next applying the proportional rule to the financial networks with the revised claims. The revised claims have the property that for every pair of agents $i, j \in N$ it holds that $L_{ij} = 0$ or $L_{ji} = 0$.

Definition 2.5. The *pairwise netting proportional rule* is the function $pnp : \mathcal{F} \to \mathcal{M}$, such that, for every $(N, z, L) \in \mathcal{F}$,

$$pnp(N, z, L) = \min\{L, L^\top\} + p(N, z, L - \min\{L, L^\top\}). \tag{2.5}$$

Under the pairwise netting proportional rule, first pairwise mutual payments are made resulting in pairwise netting of the liabilities, and next the remaining liabilities are settled using the proportional rule. Since both the matrix of pairwise mutual payments $\min\{L, L^\top\}$ and the revised matrix of liabilities $L - \min\{L, L^\top\}$ belong to the rational payment matrices $\mathcal{M}(N)$, and the proportional rule p leads to a payment matrix $p(N, z, L)$ in $\mathcal{M}(N)$, also the payments made under the pairwise netting proportional rule pnp belong to $\mathcal{M}(N)$. Notice that the expression in (2.5) can also be written as

$$pnp(N, z, L) = \min\{L, L^\top\} + p(N, z, \max\{0, L - L^\top\}).$$

In the following example, we illustrate that the proportional rule and the pairwise netting proportional rule may lead to different asset values and equities in financial networks.

Example 2.6. Consider the financial network $(N, z, L) \in \mathcal{F}$ with three agents $N = \{1, 2, 3\}$ and endowments and liabilities as in Table 1. Table 2 presents the payment matrix P resulting from the proportional rule p and the induced asset values and equities. Table 3 presents the pairwise netting amounts $\min\{L, L^\top\}$, the payment matrix resulting from the proportional rule applied to the revised problem $P' = p(N, z, L - \min\{L, L^\top\})$, the payment matrix $\overline{P}$, asset values $a(\overline{P})$, and equities $e(\overline{P})$ resulting from the pairwise netting proportional rule pnp.

The payment matrices P and $\overline{P}$ lead to different equities for agents 2 and 3. The reason is that the pairwise netting between agents 1 and 2 is equivalent to a full reimbursement of half the liability of agent 1 to agent 2. To the remaining liability of agent 1 to agent 2 and the liability of agent 1 to agent 3, the proportional rule is applied. Altogether this is better for agent 2 than applying the proportional rule to his entire claim on agent 1. The pairwise netting proportional rule has the undesirable feature that agent 1 makes a payment of 10 units to agent 2 and of 8 units to agent 3, even though both agents hold the same claim against agent 1.

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z	L		
12	0	12	12
6	6	0	0
6	0	0	0

Table 1: The endowments and liabilities in Example 2.6.

z	L			P			$a(N,z,P)$	$e(N,z,P)$
12	0	12	12	0	9	9	18	0
6	6	0	0	6	0	0	15	9
6	0	0	0	0	0	0	15	15

Table 2: The payment matrix, asset values and equities resulting from the proportional rule p in Example 2.6.

z	L			L'			$\min\{L,L^\top\}$			P'			$\overline{P}$			$a(N,z,\overline{P})$	$e(N,z,\overline{P})$
12	0	12	12	0	6	12	0	6	0	0	4	8	0	10	8	18	0
6	6	0	0	0	0	0	6	0	0	0	0	0	6	0	0	16	10
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	14	14

$$L' = L - \min\{L, L^\top\}$$
$$P' = p(N, z, L - \min\{L, L^\top\})$$

Table 3: The pairwise netting amounts $\min\{L, L^\top\}$, the payment matrix of the revised problem P', the payment matrix $\overline{P}$, asset values $a(\overline{P})$ and equities $e(\overline{P})$ resulting from the pairwise netting proportional rule pnp in Example 2.6.

3 Axioms

In this section we define and discuss the set of axioms characterizing the proportional rule in financial networks: claims boundedness, limited liability, priority of creditors, impartiality, and non-manipulability by identical agents.

Definition 3.1. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *claims boundedness* (B) if for every $F = (N, z, L) \in \mathcal{F}$ it holds that $b(F) \leq L$.

Claims boundedness expresses that no agent needs to pay an amount in excess of his liabilities.

Definition 3.2. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *limited liability* (L) if for every $F = (N, z, L) \in \mathcal{F}$, for every $i \in N$, we have that $e_i(N, z, b(F)) \geq 0$.

A bankruptcy rule satisfies limited liability if it leads to a payment matrix such that none of the agents ends up with negative equity.

Definition 3.3. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *priority of creditors* (P) if for every $F = (N, z, L) \in \mathcal{F}$, for every $i \in N$, if $b_i(F) < L_i$, then $e_i(N, z, b(F)) = 0$.

A bankruptcy rule satisfies priority of creditors if the only circumstance under which an agent is allowed to default is when his equity is equal to zero. The axioms of limited liability and priority of creditors are closely related to the notions of limited liability and absolute priority as introduced in Eisenberg and Noe (2001) in the context of the proportional rule. In Eisenberg and Noe (2001), limited liability and absolute priority are not formulated as properties of bankruptcy rules, but as requirements on the payment matrix following from proportional division rules.

Definition 3.4. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *impartiality* (I) if for every $F = (N, z, L) \in \mathcal{F}$, for every $i, j, k \in N$ such that $L_{ij} = L_{ik}$, it holds that $b_{ij}(F) = b_{ik}(F)$.

Impartiality requires that two agents j and k with the same claim on agent i should receive the same payment from i. It follows from Example 2.6 that Axiom I is not satisfied by the pairwise netting proportional rule, since there agent 1 makes a payment of 10 units to agent 2 and of 8 units to agent 3, even though both agents hold the same claim against agent 1.

For the class of claims problems, non-manipulability says that no group of agents can increase their total awards by merging their claims and that no single agent can increase his award by splitting his claim among dummy agents and himself. This axiom was introduced as strategy-proofness by O'Neill (1982). Strong non-manipulability, introduced as the additivity of claims property by Curiel, Maschler, and Tijs (1987), says that if an agent splits his claim and appears as several different claimants, or a group of agents merge their claims and appear as a single claimant, nothing changes for the other agents involved in the problem.

Our next axiom generalizes strong non-manipulability for claims problems to the setting of financial networks. Let a financial network $F = (N, z, L) \in \mathcal{F}$, an agent $j \in N$, and a set of agents $K \subset N \setminus \{j\}$ be given. The financial network $F' = (N', z', L')$ that results after a *take-over* by agent $j \in N$ of the endowments, claims, and liabilities of the agents

in the set $K \subset N \setminus \{j\}$ is denoted by $T(F, j, K)$, so

$$
\begin{aligned}
N' &= N \setminus K, \\
z'_j &= z_j + \textstyle\sum_{k \in K} z_k, \\
z'_i &= z_i, & i \in N' \setminus \{j\}, \\
L'_{ji} &= L_{ji} + \textstyle\sum_{k \in K} L_{ki}, & i \in N' \setminus \{j\}, \\
L'_{ij} &= L_{ij} + \textstyle\sum_{k \in K} L_{ik}, & i \in N' \setminus \{j\}, \\
L'_{hi} &= L_{hi}, & h, i \in N' \setminus \{j\}.
\end{aligned}
$$

We define the axiom of non-manipulability for financial networks in the following way.

Definition 3.5. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *non-manipulability* if for every $F = (N, z, L) \in \mathcal{F}$, for every $j \in N$, for every $K \subset N \setminus \{j\}$, the payments in the financial network $F' = (N', z', L') = T(F, j, K)$ satisfy

$$
\begin{aligned}
b_{ji}(F') &= b_{ji}(F) + \textstyle\sum_{k \in K} b_{ki}(F), & i \in N' \setminus \{j\}, \\
b_{ij}(F') &= b_{ij}(F) + \textstyle\sum_{k \in K} b_{ik}(F), & i \in N' \setminus \{j\}, \\
b_{hi}(F') &= b_{hi}(F), & h, i \in N' \setminus \{j\}.
\end{aligned}
$$

We have formulated non-manipulability as the requirement that the merger of a set of agents should not affect the payments made to and received from the agents that are not involved in the merger. Equivalently, we could have used the formulation that splitting an agent into multiple agents should not affect the payments made to and received from the agents that are not involved in the split. Another way to look at the definition is that under a non-manipulable bankruptcy rule, involved agents do neither benefit from a take-over nor from a split, giving two inequalities which result in the first two equalities in Definition 3.5. The feature that also mutual payments between agents not involved in the take-over or split do not change, the third line of equalities in Definition 3.5, makes this notion of non-manipulability particularly robust, since it rules out the possibility that agents outside a take-over or split would benefit from it and would be willing to make side-payments to induce it. To sum up, non-manipulability requires the merger of a set of agents or the split of an agent into multiple agents not to affect the payment matrix.

Although seemingly attractive, we argue next that the requirement of non-manipulability is too strong in financial networks. First, we show that it is not satisfied by the proportional rule.

Example 3.6. We start from the financial network $F = (N, z, L) \in \mathcal{F}$ of Example 2.6, presented in Table 4 for convenience.

Consider a split of agent 1 into agents 1 and 4, resulting in the financial network $F' = (N', z', L') = (N \cup \{4\}, z', L')$. In the split, agent 1 allocates half of his endowment and all of his liabilities to agent 4, but none of his claims. The financial network F' is

z	L			P			$a(N, z, P)$	$e(N, z, P)$
12	0	12	12	0	9	9	18	0
6	6	0	0	6	0	0	15	9
6	0	0	0	0	0	0	15	15

Table 4: The payment matrix, asset values, and equities resulting from the proportional rule p in Example 3.6 for the financial network $F = (N, z, L)$.

presented in Table 5, as well as the payment matrix P' resulting from the proportional rule p. Notice that $F = T(F', 1, \{4\})$.

z'	L'				P'				$a(N', z', P')$	$e(N', z', P')$
6	0	0	0	0	0	0	0	0	12	12
6	6	0	0	0	6	0	0	0	9	3
6	0	0	0	0	0	0	0	0	9	9
6	0	12	12	0	0	3	3	0	6	0

Table 5: The payment matrix, asset values, and equities resulting from the proportional rule p in Example 3.6 for the financial network $F' = (N', z', L')$.

Clearly, the proportional rule violates non-manipulability, since $P_{12} = 9 \neq 3 = P'_{12} + P'_{42}$ and $P_{13} = 9 \neq 3 = P'_{13} + P'_{43}$. Agent 4 has no claims and his liabilities exceed his endowment, so is sure to default on his liabilities. On the other hand, agent 1 has no liabilities, a positive endowment, and positive claims, so will be solvent for sure. Agent 1, who defaults in financial network F and has $e_1(N, z, p(F)) = 0$, has splitted in a solvent agent 1 and a defaulting agent 4 in financial network F' with resulting equity $e_1(N', z', p(F')) = 12$ and $e_4(N', z', p(F')) = 0$. Obviously, if a bankrupt agent is allowed to allocate all his liabilities to a spin-off and keeps his endowment and claims to himself, he will end up with positive equity himself and a bankrupt spin-off. This kind of manipulation is illegal, since in winding up or in insolvency proceedings, the borrower is not allowed to do anything that would threaten directly or indirectly the payments to its lenders.

We can generalize the findings of Example 3.6 to the following impossibility result.

Theorem 3.7. *There is no bankruptcy rule satisfying claims boundedness* (B), *limited liability* (L), *priority of creditors* (P), *and non-manipulability.*

Proof. Suppose b is a bankruptcy rule satisfying B, L, P, and non-manipulability. Let F and F' be the financial networks as defined in Example 3.6. We define $P = b(F)$ and $P' = b(F')$.

By B it holds that

$$P'_1 = (0, 0, 0, 0) \tag{3.1}$$

and

$$e_4(N', z', P') = z'_4 + \sum_{i \in N'} P'_{i4} - \sum_{i \in N'} P'_{4i} = 6 - \sum_{i \in N'} P'_{4i}.$$

If $P'_4 = L_4$, then evidently $e_4(N', z', P') < 0$, which would violate L. It therefore holds that $P'_4 \neq L_4$ and by B that $P'_4 < L_4$. Axiom P now implies that $\sum_{i \in N'} P'_{4i} = 6$. Axiom B implies $P'_{41} = 0$.

We now apply non-manipulability to derive that

$$
\begin{aligned}
P_{12} &= P'_{12} + P'_{42}, \\
P_{13} &= P'_{13} + P'_{43},
\end{aligned}
$$

so

$$\sum_{i \in N} P_{1i} = P_{11} + P_{12} + P_{13} = 0 + P'_{12} + P'_{42} + P'_{13} + P'_{43} = \sum_{i \in N} P'_{1i} + \sum_{i \in N} P'_{4i} = 6, \tag{3.2}$$

where the last equality comes from (3.1). It follows that

$$e_1(N, z, P) = z_1 + \sum_{i \in N} P_{i1} - \sum_{i \in N} P_{1i} \geq 12 + 0 - 6 = 6 > 0,$$

so P and B yields $P_1 = L_1$ and

$$\sum_{i \in N} P_{1i} = \sum_{i \in N} L_{1i} = 24,$$

a contradiction to (3.2). $\qquad \square$

We therefore impose the much weaker axiom of non-manipulability by identical agents, which requires that the merger of a group of agents that are identical, that is, have the same endowments, claims, and liabilities, should not affect the payments made to and received by the agents not involved in the merger, formally defined as follows.

Definition 3.8. A bankruptcy rule $b : \mathcal{F} \to \mathcal{M}$ satisfies *non-manipulability by identical agents* (N) if for every $F = (N, z, L) \in \mathcal{F}$, for every $j \in N$, for every $K \subset N \setminus \{j\}$ such that for all $k \in K$ we have that $z_k = z_j, L_k = L_j$, and $L^k = L^j$, the payments in the financial network $F' = (N', z', L') = T(F, j, K)$ satisfy

$$
\begin{aligned}
b_{ji}(F') &= b_{ji}(F) + \sum_{k \in K} b_{ki}(F), & i \in N' \setminus \{j\}, \\
b_{ij}(F') &= b_{ij}(F) + \sum_{k \in K} b_{ik}(F), & i \in N' \setminus \{j\}, \\
b_{hi}(F') &= b_{hi}(F), & h, i \in N' \setminus \{j\}.
\end{aligned}
$$

When using non-manipulability by identical agents, all agents $k \in K$ should be identical to agent j, that is, they should have the same endowment, claim, and liability vector. Since $L_{jj} = 0$ and $L_{kk} = 0$, the requirement $L_j = L_k$ implies $L_{jk} = L_{kj} = 0$. More generally, the same argument can be used to derive that

$$L_{k\ell} = 0, \quad k, \ell \in \{\{j\} \cup K\}. \tag{3.3}$$

The inequalities resulting from the requirement that a set of identical agents does not benefit from a merger and the requirement that an agent should not benefit from splitting into multiple identical agents can be reduced to the first two sets of equalities in Definition 3.8. The third set of equalities requires that also mutual payments between agents not involved in the merger or split should not change. To sum up, Axiom N requires that the merger of a group of agents that are identical or the split of an agent into multiple identical agents should not affect the payment matrix.

4 The Axiomatization on the Rational Domain

In this section, we show that the axioms of claims boundedness (B), limited liability (L), priority of creditors (P), impartiality (I), and non-manipulability by identical agents (N) characterize the proportional rule in financial networks. First, we show that for any financial network the proportional rule satisfies those axioms.

Theorem 4.1. *The proportional rule p satisfies* B, L, P, I, *and* N.

Proof. Consider the constraints in the linear programming formulation of the proportional rule in Theorem 2.4. Since, for every $i \in N$, $\lambda_i \leq 1$, it holds that B is satisfied. The constraint $z + P^\top \mathbb{1} - P\mathbb{1} \geq 0$ implies that L holds. Since, for every $i, j \in N$, the constraint $P_{ij} = \lambda_i L_{ij}$ is imposed, Axiom I follows.

Next, we show Axiom P to hold. Take any financial network $F = (N, z, L) \in \mathcal{F}$ and any agent $i \in N$ such that $p_i(F) < L_i$. Then, by the system of equations (2.2), we have that

$$p_{ij}(F) = \frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, p(F)),$$

so

$$\sum_{j \in N} p_{ij}(F) = \sum_{j \in N} \frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, p(F)) = a_i(N, z, p(F)),$$

implying that $e_i(N, z, p(F)) = 0$.

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To show Axiom N holds, take any financial network $F = (N, z, L) \in \mathcal{F}$, $j \in N$, and $K \subset N \setminus \{j\}$ such that

$$z_j = z_k, L_j = L_k, L^j = L^k, \quad k \in K, \tag{4.1}$$

implying, see (3.3), that there are no liabilities among agents in $\{j\} \cup K$,

$$L_{k\ell} = 0, \quad k, \ell \in \{j\} \cup K. \tag{4.2}$$

Consider the financial network $F' = (N', z', L') = T(F, j, K)$. We show that the payment matrix $P' \in \mathcal{M}(N')$ defined by

$$
\begin{array}{rcll}
P'_{ji} & = & p_{ji}(F) + \sum_{k \in K} p_{ki}(F), & i \in N' \setminus \{j\}, \\
P'_{ij} & = & p_{ij}(F) + \sum_{k \in K} p_{ik}(F), & i \in N' \setminus \{j\}, \\
P'_{hi} & = & p_{hi}(F), & h, i \in N' \setminus \{j\},
\end{array} \tag{4.3}
$$

is a solution to (2.2) in Definition 2.2 for F'. We have two main cases, 1 and 2, depending on whether the liability between a pair of agents is zero or not, and we have three subcases, a, b, and c, depending on the row in (4.3) under consideration.

Case 1a: $i \in N' \setminus \{j\}$, $L'_{ji} = 0$.
We have to show that $P'_{ji} = 0$. It holds that

$$0 = L'_{ji} = L_{ji} + \sum_{k \in K} L_{ki},$$

so $L_{ji} = 0$ and, for every $k \in K$, $L_{ki} = 0$. This implies that

$$P'_{ji} = p_{ji}(F) + \sum_{k \in K} p_{ki}(F) = 0,$$

where the first equality uses (4.3) and the second (2.2).

Case 1b: $i \in N' \setminus \{j\}$, $L'_{ij} = 0$.
We have to show that $P'_{ij} = 0$. It holds that

$$0 = L'_{ij} = L_{ij} + \sum_{k \in K} L_{ik},$$

so $L_{ij} = 0$ and, for every $k \in K$, $L_{ik} = 0$. This implies that

$$P'_{ij} = p_{ij}(F) + \sum_{k \in K} p_{ik}(F) = 0,$$

where the first equality uses (4.3) and the second (2.2).

Case 1c: $h, i \in N' \setminus \{j\}$, $L'_{hi} = 0$.
We have to show that $P'_{hi} = 0$. It holds that $L_{hi} = L'_{hi} = 0$, so $P'_{hi} = p_{hi}(F) = 0$, where the first equality uses (4.3) and the second (2.2).

Case 2a: $i \in N' \setminus \{j\}$, $L'_{ji} > 0$.

It holds that

$$L'_{ji} = L_{ji} + \sum_{k \in K} L_{ki} > 0.$$

Since for all $k \in K$ we have that $L_j = L_k$ by (4.1), it follows that $L_j = L_k > 0$ for all $k \in K$. Hence we are in the second case of (2.2) and by (4.3) we have that

$$\begin{aligned} P'_{ji} &= p_{ji}(F) + \sum_{k \in K} p_{ki}(F) \\ &= \min\left\{ \frac{L_{ji}}{\sum_{h \in N} L_{jh}} a_j(N, z, p(F)), L_{ji} \right\} + \sum_{k \in K} \min\left\{ \frac{L_{ki}}{\sum_{h \in N} L_{kh}} a_k(N, z, p(F)), L_{ki} \right\}. \end{aligned} \tag{4.4}$$

We have to show that

$$P'_{ji} = \min\left\{ \frac{L'_{ji}}{\sum_{h \in N'} L'_{jh}} a_j(N', z', P'), L'_{ji} \right\}. \tag{4.5}$$

Observe that for every $k \in K$

$$a_k(N, z, p(F)) = z_k + \sum_{h \in N} p_{hk}(F) = z_j + \sum_{h \in N} p_{hj}(F) = a_j(N, z, p(F)), \tag{4.6}$$

where the second equality follows from the fact that the agents in $\{j\} \cup K$ have the same endowments and claims by (4.1) and receive the same payments since the proportional rule satisfies I. Moreover, the agents in $\{j\} \cup K$ have the same liabilities in F by (4.1), hence $\sum_{h \in N} L_{jh} = \sum_{h \in N} L_{kh}$, and (4.4) can be written as

$$P'_{ji} = (|K| + 1) \min\left\{ \frac{L_{ji}}{\sum_{h \in N} L_{jh}} a_j(N, z, p(F)), L_{ji} \right\}. \tag{4.7}$$

Note that, for every $h \in N'$, it holds by (4.3) that

$$P'_{hj} = p_{hj}(F) + \sum_{k \in K} p_{hk}(F) = (|K| + 1) p_{hj}(F), \tag{4.8}$$

since the agents in $\{j\} \cup K$ have the same claims by (4.1) and the proportional rule satisfies I. By (4.1), it follows that

$$(|K| + 1) L_{ji} = L'_{ji}, \tag{4.9}$$

$$(|K| + 1) z_j = z'_j. \tag{4.10}$$

By (4.2) and since p satisfies L, we have that

$$a_j(N, z, p(F)) = z_j + \sum_{h \in N} p_{hj}(F) = z_j + \sum_{h \in N'} p_{hj}(F).$$

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Using this fact, (4.2), (4.8), (4.9), and (4.10), we can rewrite (4.7) as

$$P'_{ji} = \min \left\{ \frac{L'_{ji}}{\frac{1}{|K|+1} \sum_{h \in N'} L'_{jh}} \left(\frac{1}{|K|+1} z'_j + \frac{1}{|K|+1} \sum_{h \in N'} P'_{hj} \right), L'_{ji} \right\}$$

$$= \min \left\{ \frac{L'_{ji}}{\sum_{h \in N'} L'_{jh}} \left(z'_j + \sum_{h \in N'} P'_{hj} \right), L'_{ji} \right\},$$

thus (4.5) is satisfied.

$\quad$ *Case 2b:* $i \in N' \setminus \{j\}$, $L'_{ij} > 0$.

It holds that

$$L'_{ij} = L_{ij} + \sum_{k \in K} L_{ik} > 0.$$

Since for all $k \in K$ we have that $L^j = L^k$ by (4.1), it follows that $L^j = L^k > 0$ for all $k \in K$. Hence we are in the second case of (2.2) and by (4.3) we have that

$$P'_{ij} = p_{ij}(F) + \sum_{k \in K} p_{ik}(F)$$

$$= \min \left\{ \frac{L_{ij}}{\sum_{h \in N} L_{ih}} a_i(N, z, p(F)), L_{ij} \right\} + \sum_{k \in K} \min \left\{ \frac{L_{ik}}{\sum_{h \in N} L_{ih}} a_i(N, z, p(F)), L_{ik} \right\}$$

$$= (|K| + 1) \min \left\{ \frac{L_{ij}}{\sum_{h \in N} L_{ih}} a_i(N, z, p(F)), L_{ij} \right\}, \tag{4.11}$$

where the last equality follows from the fact that by (4.1) we have that $L_{ij} = L_{ik}$ for all $k \in K$.

$\quad$ We have to show that

$$P'_{ij} = \min \left\{ \frac{L'_{ij}}{\sum_{h \in N'} L'_{ih}} a_i(N', z', P'), L'_{ij} \right\}. \tag{4.12}$$

By (4.1) it follows that

$$(|K| + 1) L_{ij} = L'_{ij}. \tag{4.13}$$

It holds that

$$a_i(N, z, p(F)) = z_i + \sum_{h \in N} p_{hi}(F) = z'_i + \sum_{h \in N'} P'_{hi} = a_i(N', z', P'), \tag{4.14}$$

where we use $z_i = z'_i$ by (4.1) and the first and third case of (4.3) to derive the second equality. By definition of F' it follows that

$$\sum_{h \in N} L_{ih} = \sum_{h \in N'} L'_{ih}. \tag{4.15}$$

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Combining (4.11) with (4.13), (4.14), and (4.15), we obtain (4.12).

Case 2c: $h, i \in N' \setminus \{j\}$, $L'_{hi} > 0$.

It holds that $L'_{hi} = L_{hi} > 0$. Hence, we are in the second case of (2.2) and by (4.3) it holds that

$$
\begin{aligned}
P'_{hi} &= p_{hi}(F) \\
&= \min \left\{ \frac{L_{hi}}{\sum_{g \in N} L_{hg}} a_h(N, z, p(F)), L_{hi} \right\} \\
&= \min \left\{ \frac{L'_{hi}}{\sum_{g \in N'} L'_{hg}} a_h(N', z', P'), L'_{hi} \right\},
\end{aligned}
$$

where the third equality follows from $L'_{hi} = L_{hi}$, (4.14), and (4.15). $\qquad \square$

To show that the axioms of B, L, P, I, and N imply the proportional rule, we will use the following two lemmas. Lemma 4.2 considers the case where one liability of an agent is an integer multiple of another liability.

Lemma 4.2. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let $i, j, k \in N$ be such that $qL_{ij} = L_{ik}$ for some integer $q \geq 2$. Let b be a bankruptcy rule satisfying axioms I and N. Then we have $qb_{ij}(F) = b_{ik}(F)$.*

Proof. Let $F' = (N', z', L') \in \mathcal{F}$ be the financial network where agent k is splitted into q identical agents k and $\ell_1, \ldots, \ell_{q-1} \in \mathbb{N} \setminus N$, more precisely

$$
\begin{aligned}
N' &= N \cup \{\ell_1, \ldots, \ell_{q-1}\}, \\
z'_k &= z'_{\ell_1} = \cdots = z'_{\ell_{q-1}} = z_k/q, \\
L'_{ik} &= L'_{i\ell_1} = \cdots = L'_{i\ell_{q-1}} = L_{ik}/q, & i \in N \setminus \{k\}, \\
L'_{ki} &= L'_{\ell_1 i} = \cdots = L'_{\ell_{q-1} i} = L_{ki}/q, & i \in N \setminus \{k\}, \\
L'_{ij} &= 0, & i, j \in \{k, \ell_1, \ldots, \ell_{q-1}\}, \\
z'_i &= z_i, & i \in N \setminus \{k\}, \\
L'_{ij} &= L_{ij}, & i, j \in N \setminus \{k\}.
\end{aligned}
$$

Notice that $F = T(F', k, \{\ell_1, \ldots, \ell_{q-1}\})$. Then we have that

$$
b_{ik}(F) = q b_{ik}(F') = q b_{ij}(F') = q b_{ij}(F),
$$

where the first equality follows by Axiom N, the second equality by I, and the third equality again by N. $\qquad \square$

The next lemma treats the case where one liability of an agent is an arbitrary multiple of another liability.

Lemma 4.3. *Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let $i, j, k \in N$ and $q, r \in \mathbb{N}$ be such that $L_{ij} = (q/r)L_{ik}$. Let b be a bankruptcy rule satisfying* I *and* N. *Then we have $b_{ij}(F) = (q/r)b_{ik}(F)$.*

Proof. Without loss of generality, we assume $q < r$. Let $F' = (N', z', L') \in \mathcal{F}$ be the financial network where agent k is splitted into r identical agents k and $\ell_1, \ldots, \ell_{r-1}$, more precisely

$$
\begin{aligned}
N' &= N \cup \{\ell_1, \ldots, \ell_{r-1}\}, \\
z'_k &= z'_{\ell_1} = \cdots = z'_{\ell_{r-1}} = z_k/r, \\
L'_{ik} &= L'_{i\ell_1} = \cdots = L'_{i\ell_{r-1}} = L_{ik}/r, \quad i \in N \setminus \{k\}, \\
L'_{ki} &= L'_{\ell_1 i} = \cdots = L'_{\ell_{r-1} i} = L_{ki}/r, \quad i \in N \setminus \{k\}, \\
L'_{ij} &= 0, \qquad\qquad\qquad\qquad\qquad i, j \in \{k, \ell_1, \ldots, \ell_{r-1}\}, \\
z'_i &= z_i, \qquad\qquad\qquad\qquad\qquad i \in N \setminus \{k\}, \\
L'_{ij} &= L_{ij}, \qquad\qquad\qquad\qquad\quad i, j \in N \setminus \{k\}.
\end{aligned}
$$

Notice that $F = T(F', k, \{\ell_1, \ldots, \ell_{r-1}\})$. Then we have that

$$
b_{ik}(F) = r b_{ik}(F') = \tfrac{r}{q} b_{ij}(F') = \tfrac{r}{q} b_{ij}(F),
$$

where the first equality follows by Axiom N, the second equality by Lemma 4.2, and the third equality again by Axiom N. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem characterizes the proportional rule as the only bankruptcy rule satisfying B, L, P, I, and N.

Theorem 4.4. *If the bankruptcy rule b satisfies* B, L, P, I, *and* N, *then $b = p$.*

Proof. Let $F = (N, z, L) \in \mathcal{F}$ be a financial network and let b be a bankruptcy rule satisfying B, L, P, I, and N. We show that $b(P)$ is a solution to the system of equations (2.2). We consider two main cases.

Case 1: $i, j \in N$, $L_{ij} = 0$.
By B we have that $b_{ij}(F) \leq 0$ and from $b(F) \in \mathcal{M}(N)$ we get that $b_{ij}(F) = 0$.

Case 2: $i, j \in N$, $L_{ij} > 0$.
We have to show that

$$
b_{ij}(F) = \min\left\{ \frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, b(F)), L_{ij} \right\}.
$$

Case 2a: $a_i(N, z, b(F)) \geq \sum_{k \in N} L_{ik}$.
We have to show that $b_{ij}(F) = L_{ij}$. Suppose, on the contrary, that $b_{ij}(F) \neq L_{ij}$. Then by B we have that

$$
b_{ij}(F) < L_{ij}. \tag{4.16}
$$

By Axiom P, the assumption of Case 2a, Axiom B, and (4.16) we get that

$$0 = e_i(N, z, b(F)) = a_i(N, z, b(F)) - \sum_{k \in N} b_{ik}(F) > \sum_{k \in N} L_{ik} - \sum_{k \in N} L_{ik} = 0, \quad (4.17)$$

a contradiction. Consequently, it holds that $b_{ij}(F) = L_{ij}$.

Case 2b: $a_i(N, z, b(F)) < \sum_{k \in N} L_{ik}$.

We have to show that

$$b_{ij}(F) = \frac{L_{ij}}{\sum_{k \in N} L_{ik}} a_i(N, z, b(F)). \quad (4.18)$$

By Lemma 4.3 there exists $\pi_i \geq 0$ such that for all $k \in N$ we have that

$$b_{ik}(F) = \pi_i L_{ik}. \quad (4.19)$$

Thus we have to show that

$$\pi_i = \frac{a_i(N, z, b(F))}{\sum_{k \in N} L_{ik}}. \quad (4.20)$$

By L we have that

$$\sum_{k \in N} b_{ik}(F) \leq a_i(N, z, b(F)). \quad (4.21)$$

Suppose that

$$\sum_{k \in N} b_{ik}(F) < a_i(N, z, b(F)). \quad (4.22)$$

Then $e_i(N, z, b(F)) > 0$ and P implies that $\sum_{k \in N} b_{ik} = \sum_{k \in N} L_{ik}$, a contradiction to the assumption of Case 2b and (4.22). Consequently, (4.21) holds with equality and by (4.19) we have that

$$a_i(N, z, b(F)) = \sum_{k \in N} b_{ik}(F) = \sum_{k \in N} \pi_i L_{ik} = \pi_i \sum_{k \in N} L_{ik}, \quad (4.23)$$

implying (4.20) as desired. □

By combining Theorems 4.1 and 4.4 we obtain the following corollary.

Corollary 4.5. *The proportional rule p is the unique bankruptcy rule satisfying the axioms of* B, L, P, I, *and* N.

5 Division Rule Based Bankruptcy Rules

In Section 6, we show the independence of the axioms. To do so, we define a number of bankruptcy rules in this section. The definition of the proportional rule for financial networks is based on the proportional division rule for claims problems. We follow the approach in Csóka and Herings (2016) to extend division rules for claims problems into bankruptcy rules for financial networks. See also Groote Schaarsberg, Reijnierse, and Borm (2013) for a related approach focusing on equity rather than payment matrices. We are particularly interested in the constrained equal awards division rule and the constrained equal losses division rule. Under the constrained equal awards division rule, all claimants get the same amount, up to the value of their claim. The constrained equal losses division rule is its dual and imposes that all claimants face the same loss, up to the value of their claim.

In this section, it is convenient to allow for real-valued amounts. As a notational convention, when confusion could arise, we use an asterisk as a superscript when a function is defined on a real-valued domain and for sets of real-valued objects. Let some set of agents $N \in \mathcal{N}$ be given. A *division rule* on the real domain is a function $d^* : \mathbb{R}_+ \times \mathbb{R}_+^N \to \mathbb{R}_+^N$ such that, for every $j \in N$, $d_j^*(E, c) \leq c_j$ and $\sum_{j \in N} d_j^*(E, c) = \min\{E, \sum_{k \in N} c_k\}$. Moreover, for every $j \in N$, d_j^* is required to be weakly increasing in E. It is well-known that these properties of d^* imply it is continuous, see for instance Thomson (2003).

It is straightforward to extend the proportional division rule d^{p} for rational-valued claims problems of Section 2 to the proportional division rule $d^{*\mathrm{p}}$ for real-valued claims problems. Another example of a division rule is the constraint equal awards division rule. If $E > \sum_{j \in N} c_j$, then define $\lambda = \max_{j \in N} c_j$. Otherwise, define $\lambda \in [0, \max_{j \in N} c_j]$ as the unique solution to

$$\sum_{j \in N} \min\{c_j, \lambda\} = E.$$

The *constrained equal awards division rule* assigns to claimant $j \in N$ the amount

$$d_j^{*\mathrm{cea}}(E, c) = \min\{c_j, \lambda\}.$$

In a similar vein, if $E > \sum_{j \in N} c_j$, then define $\mu = 0$. Otherwise, define $\mu \in [0, \max_{j \in N} c_j]$ as the unique solution to

$$\sum_{j \in N} \max\{c_j - \mu, 0\} = E.$$

The *constrained equal losses division rule* assigns to claimant $j \in N$ the amount

$$d_j^{*\mathrm{cel}}(E, c) = \max\{c_j - \mu, 0\}.$$

The set of all matrices in $\mathbb{R}_+^{N \times N}$ with a zero diagonal is denoted by $\mathcal{M}^*(N)$. The union over all finite sets of agents of these matrices is denoted by $\mathcal{M}^* = \cup_{N \in \mathcal{N}} \mathcal{M}^*(N)$. The set of all financial networks (N, z, L) with set of agents $N \in \mathcal{N}$, endowments $z \in \mathbb{R}_{++}^N$, and liability matrix $L \in \mathcal{M}^*(N)$ is denoted by $\mathcal{F}^*$. The definition of a bankruptcy rule on the real domain is as follows.

Definition 5.1. A *bankruptcy rule* on the real domain is a function $b^* : \mathcal{F}^* \to \mathcal{M}^*$ such that for every $(N, z, L) \in \mathcal{F}^*$ it holds that $b^*(N, z, L) \in \mathcal{M}^*(N)$.

Using Definition 2.2 as based on $d^{*\mathrm{p}}$, we can extend the proportional rule p for rational-valued financial networks to a proportional rule $p^* : \mathcal{F}^* \to \mathcal{M}^*$ for real-valued financial networks. We can proceed in a similar way for general division rules. We extend the definitions of asset value $a_i(N, z, P)$ and equity $e_i(N, z, P)$ in a straightforward way to the real-valued setup.

Definition 5.2. Given a financial network $(N, z, L) \in \mathcal{F}^*$ and division rules $(d^{*i})_{i \in N}$, the payment matrix $P \in \mathcal{M}^*(N)$ is a *clearing payment matrix* if it solves the following system of equations:

$$P_{ij} = d_j^{*i}(a_i(N, z, P), L_i), \quad i, j \in N.$$

Unlike the case with proportional division rules, it is in general not guaranteed that the clearing payment matrix is uniquely determined by Definition 5.2. However, we will argue next that there is a uniquely defined least clearing payment matrix and a uniquely defined greatest clearing payment matrix.

A *lattice* is a partially ordered set in which every pair of elements has a supremum and an infimum. A *complete lattice* is a lattice in which every non-empty subset has a supremum and an infimum. The proof of the following result relies on Tarski's fixed point theorem (Tarski, 1955) and follows from a straightforward adaptation of the proof for the discrete case in Csóka and Herings (2016).

Theorem 5.3. *Let a financial network $(N, z, L) \in \mathcal{F}^*$ and division rules $(d^{*i})_{i \in N}$ be given. The set of clearing payment matrices is a complete lattice. In particular, there exists a least clearing payment matrix P^- and a greatest clearing payment matrix P^+.*

If all agents use proportional division rules, then it follows from Theorem 2 of Eisenberg and Noe (2001) that the clearing payment matrix is unique. Surprisingly, when using the constrained equal awards division rule for each agent, the clearing payment matrix need not be unique.

Example 5.4. We consider a financial network $(N, z, L) \in \mathcal{F}^*$ and division rules $(d^{*i})_{i \in N}$ with three agents $N = \{1, 2, 3\}$ where $d^{*1} = d^{*2} = d^{*3} = d^{*\mathrm{cea}}$. Table 6 presents the

endowments, the liabilities, the least clearing payment matrix P^- and the greatest clearing payment matrix P^+ and the induced asset values and equities.

z	L			P^-			$a(N,z,P^-)$	$e(N,z,P^-)$	P^+			$a(N,z,P^+)$	$e(N,z,P^+)$
1	0	2	1	0	1	1	2	0	0	2	1	3	0
1	2	0	1	1	0	1	2	0	2	0	1	3	0
1	0	0	0	0	0	0	3	3	0	0	0	3	3

Table 6: The clearing payment matrices P^- and P^+ and their induced asset values and equities using constrained equal awards division rules in Example 5.4 for the financial network $F = (N, z, L)$.

For financial networks, we select the greatest clearing payment matrix to define a bankruptcy rule that is based on division rules for claims problems.

Definition 5.5. The bankruptcy rule $b^* : \mathcal{F}^* \to \mathcal{M}^*$ is based on division rules $(d^{*i})_{i \in \mathbb{N}}$ if for every $F = (N, z, L) \in \mathcal{F}^*$ it holds that $b^*(F) = P^+$, where P^+ is the greatest clearing payment matrix for financial network F and division rules $(d^{*i})_{i \in N}$.

Using Definition 5.5 the *constrained equal awards rule* $cea^* : \mathcal{F}^* \to \mathcal{M}^*$ follows when all agents use the constrained equal awards division rule and the *constrained equal losses rule* $cel^* : \mathcal{F}^* \to \mathcal{M}^*$ when all agents use the constrained equal losses division rule. Not every bankruptcy rule for financial networks is based on division rules. An example is the pairwise netting proportional rule, where payments do not only depend on the asset value of an agent and his liabilities, but also on his claims towards other agents.

The greatest clearing payment matrix corresponding to a bankruptcy rule that is based on division rules for claims problems can be determined as the solution to a programming problem. Let some financial network $(N, z, L) \in \mathcal{F}^*$ and division rules $(d^{*i})_{i \in N}$ be given. The set of *feasible payment matrices* $\mathcal{P}^*$ is defined as the set of payment matrices where each row i belongs to the image of the division rule of agent i, that is

$$\mathcal{P}^* = \{P \in \mathcal{M}^*(N) \mid \forall i \in N, \ P_i \in d^{*i}(\mathbb{R}_+, L_i)\}.$$

Theorem 5.6. *Let $b^* : \mathcal{F}^* \to \mathcal{M}^*$ be a bankruptcy rule that is based on division rules $(d^{*i})_{i \in \mathbb{N}}$. Let a financial network $F = (N, z, L) \in \mathcal{F}^*$ be given. Then $b^*(F) = P^+$ if and only if P^+ solves the following programming problem:*

$$\max_{P \in \mathcal{P}^*} \sum_{i \in N} \sum_{j \in N} P_{ij},$$
subject to $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.1)
$$z + P^\top \mathbb{1} - P\mathbb{1} \geq 0.$$

Proof. Let P' be a solution to (5.1) and let some $i \in N$ be given. We show that $P'_i = d^{*i}(a_i(N, z, P'), L_i)$ from which it follows that P' is a clearing payment matrix.

If $P'_i = L_i$, then from the inequality in (5.1) we have that

$$a_i(N, z, P') = z_i + \sum_{j \in N} P'_{ji} \geq \sum_{j \in N} P'_{ij} = \sum_{j \in N} L_{ij}.$$

From the definition of a division rule, it now follows that $d^{*i}(a_i(N, z, P'), L_i) = L_i$.

Consider the case $P'_i < L_i$. We show that $e_i(N, z, P') = 0$. Suppose $e_i(N, z, P') > 0$. Since $P' \in \mathcal{P}^*$ there exists $E' \in \mathbb{R}_+$ such that $P'_i = d^{*i}(E', L_i)$. Since d^{*i} is continuous and $e_i(N, z, P') > 0$ there exists $\varepsilon > 0$ such that

$$z_i + \sum_{j \in N} P'_{ji} - \sum_{j \in N} d^{*i}_j(E' + \varepsilon, L_i) \geq 0.$$

The payment matrix P'' defined by

$$\begin{aligned} P''_i &= d^{*i}(E' + \varepsilon, L_i), \\ P''_j &= P'_j, \qquad\qquad j \neq i, \end{aligned}$$

satisfies the constraints in (5.1) and leads to a strictly higher value of the objective function than P', a contradiction. Consequently, it holds that $e_i(N, z, P') = 0$.

Since $P' \in \mathcal{P}^*$ there exists $E' \in \mathbb{R}_+$ such that $P'_i = d^{*i}(E', L_i)$ and from $P'_i < L_i$ and the definition of a division rule, we have $\sum_{j \in N} d^{*i}_j(E', L_i) = E'$. Since $e_i(N, z, P') = 0$, we therefore have that

$$E' = \sum_{j \in N} d^{*i}_j(E', L_i) = \sum_{j \in N} P'_{ij} = z_i + \sum_{j \in N} P'_{ji} = a_i(N, z, P').$$

We have shown that P' is a clearing payment matrix.

Let P^+ be the greatest clearing payment matrix, which exists by Theorem 5.3. Since P^+ satisfies feasibility and the constraint in (5.1), it follows that $P' = P^+$. $\qquad\square$

For the constrained equal award rule cea^* we can replace the requirement $P \in \mathcal{P}^*$ of the program in (5.1) by a set of simple constraints. Using Theorem 5.6, the following result follows in a straightforward way.

Theorem 5.7. *Let a financial network $F = (N, z, L) \in \mathcal{F}^*$ be given. Then $cea^*(F) = P^+$ if and only if there is $\lambda^+ \in \mathbb{R}^N_+$ such that (P^+, λ^+) solves the following programming problem:*

$$\begin{aligned} &\max_{P \in \mathbb{R}^{N \times N}_+, \lambda \in \mathbb{R}^N_+} \sum_{i \in N} \sum_{j \in N} P_{ij}, \\ &\text{subject to} \\ &P_{ij} = \min\{\lambda_i, L_{ij}\}, &&i, j \in N, \\ &\lambda_i \leq \max_{j \in N} L_{ij}, &&i \in N, \\ &z + P^\top \mathbb{1} - P\mathbb{1} \geq 0. \end{aligned} \qquad (5.2)$$

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The program in (5.2) maximizes the total payments as made by the agents subject to three conditions. The first condition expresses that agent i pays all of his claimants the amount λ_i, except when λ_i would exceed the value of the claim. This yields the feasibility condition of clearing payment matrices under the constrained equal awards rule. The second condition serves to pin down a unique value of λ_i in all circumstances. The third condition requires that no agent end up with negative equity. The maximization of the objective function guarantees that an agent only defaults if he has zero equity, since otherwise the objective function could be increased. It also guarantees that the greatest clearing payment matrix is selected.

The restriction $cea^*_{|\mathcal{F}}$ of the constrained equal awards rule cea^* to rational-valued financial networks in $\mathcal{F}$ is denoted by cea. The next result establishes that for financial networks $F \in \mathcal{F}$, the payment matrix $cea(F)$ belongs to the rational payment matrices $\mathcal{M}(N)$, so cea is well-defined.

Theorem 5.8. *Let a financial network $F = (N, z, L) \in \mathcal{F}$ be given. It holds that $cea(F) \in \mathcal{M}(N)$.*

Proof. Let (P^+, λ^+) be the solution to the programming problem (5.2), so $P^+ = cea^*(F)$. Let $A = \{(i,j) \in N \times N \mid P^+_{ij} = L_{ij}\}$ be the set of pairs of agents (i,j) such that the liability of i to j is fully settled under P^+. Let I be the set of agents such that $P^+_i = L_i$. Then (P^+, λ^+) is the unique solution to the linear system of equations

$$
\begin{aligned}
P_{ij} &= L_{ij}, \quad i,j \in A, \\
P_{ij} - \lambda_i &= 0, \quad (i,j) \in (N \times N) \setminus A, \\
\sum_{j \in N} P_{ij} - \sum_{j \in N} P_{ji} &= z_i, \quad i \in N \setminus I.
\end{aligned}
$$

Since all coefficients in this system of equations are rational and the system has a unique solution, the solution must be rational too. We have shown that $P^+ \in \mathcal{M}(N)$. □

Also for the constrained equal losses rule, we can replace the requirement $P \in \mathcal{P}^*$ of the program in (5.1) by a set of simple constraints. Using Theorem 5.3, we obtain the following result in a straightforward way.

Theorem 5.9. *Let a financial network $F = (N, z, L) \in \mathcal{F}^*$ be given. Then $cel^*(F) = P^+$ if and only if there is $\mu^- \in \mathbb{R}^N_+$ such that (P^+, μ^-) solves the following programming problem:*

$$
\begin{aligned}
&\max_{P \in \mathbb{R}^{N \times N}_+, \mu \in \mathbb{R}^N_+} \sum_{i \in N} \sum_{j \in N} P_{ij}, \\
&\text{subject to} \\
&P_{ij} = \max\{L_{ij} - \mu_i, 0\}, \qquad i,j \in N, \\
&z + P^\top \mathbb{1} - P \mathbb{1} \geq 0.
\end{aligned}
\tag{5.3}
$$

The program in (5.3) maximizes the total payments as made by the agents subject to three conditions. The first condition expresses that agent i pays all of his claimants the amount their claims minus μ_i, except when μ_i would exceed the value of the claim, corresponding to the feasibility condition of clearing payment matrices under the constrained equal losses rule. The second condition requires that no agent ends up with negative equity. The maximization of the objective function guarantees that agents only default if they have zero equity and that the greatest clearing payment matrix is selected. Since for every $i \in N$ it holds that $z_i > 0$, we have that $\mu_i < \max_{j \in N} L_{ij}$.

The restriction $cel^*_{|\mathcal{F}}$ of the constrained equal losses rule cel^* to financial networks in $\mathcal{F}$ is denoted by cel. The next result establishes that for financial networks in $F \in \mathcal{F}$, the payment matrix $cel(F)$ belongs to the rational payment matrices $\mathcal{M}(N)$, so cel is well-defined.

Theorem 5.10. *Let a financial network $F = (N, z, L) \in \mathcal{F}$ be given. It holds that $cel(F) \in \mathcal{M}(N)$.*

Proof. Let (P^+, μ^-) be the solution to the programming problem (5.3), so $P^+ = cel^*(F)$. Let $A = \{(i, j) \in N \times N \mid P^+_{ij} = L_{ij}\}$ be the set of pairs of agents (i, j) such that the liability of i to j is fully settled under P^+. Then (P^+, μ^-) is the unique solution to the linear system of equations

$$
\begin{aligned}
P_{ij} &= L_{ij}, \quad i, j \in A, \\
P_{ij} + \mu_i &= L_{ij}, \quad (i, j) \in (N \times N) \setminus A, \\
\sum_{j \in N} P_{ij} - \sum_{j \in N} P_{ji} &= z_i, \quad i \in N \setminus I.
\end{aligned}
$$

Since all coefficients in this system of equations are rational and the system has a unique solution, the solution must be rational too. We have shown that $P^+ \in \mathcal{M}(N)$. $\qquad\square$

Although clearing payment matrices are not always unique, the resulting equities are. For instance, in Example 5.4 it holds that the payment matrices P^- and P^+ lead to the same equities. The following result is a modest generalization of a result in Groote Schaarsberg, Reijnierse, and Borm (2013), who assume that all agents use the same division rule. It is straightforward to extend their proof to the case where agents do not necessarily use the same division rules.

Theorem 5.11. *Let a financial network $(N, z, L) \in \mathcal{F}^*$ and division rules $(d^{*i})_{i \in N}$ be given. Let P and P' be clearing payment matrices. Then it holds that, for every $i \in N$, $e_i(N, z, P) = e_i(N, z, P')$.*

6 Independence of the Axioms

In this section, we show the independence of the axioms B, L, P, I, and N on the rational domain by providing five examples of bankruptcy rules satisfying all the axioms except one.

Example 6.1 (All except B). Consider the following bankruptcy rule based on the proportional rule but pretending that the liabilities are twice the actual liabilities. Let $b^1 : \mathcal{F} \to \mathcal{M}$ be defined by setting $b^1(N, z, L) = p(N, z, 2L)$ for every $(N, z, L) \in \mathcal{F}$.

Then b^1 obviously does not satisfy B.

Recall that the proportional rule p satisfies L, P, I, and N by Theorem 4.1.

Since p satisfies L, P, and I, it follows almost immediately that b^1 satisfies L, P, and I.

Since merging identical agents and then doubling the liability matrix leads to the same liability matrix as doubling the liability matrix first and merging identical agents next, the axiom of N for b^1 follows from the axiom of N for p.

Example 6.2 (All except L). Consider the bankruptcy rule where all liabilities are paid. Let $b^2 : \mathcal{F} \to \mathcal{M}$ be defined by setting $b^2(N, z, L) = L$ for every $(N, z, L) \in \mathcal{F}$.

Then b^2 clearly does not satisfy L. Moreover, b^2 obviously satisfies B, P, I, and N.

Example 6.3 (All except P). Consider the bankruptcy rule where nothing is paid. Let $b^3 : \mathcal{F} \to \mathcal{M}$ be defined by setting $b^3(N, z, L) = 0^{N \times N}$.

Then b^3 clearly does not satisfy P. Moreover, b^3 obviously satisfies B, L, I, and N.

Example 6.4 (All except I). Consider the pairwise netting proportional rule pnp as defined in Definition 2.5.

As we have seen in Example 2.6, pnp does not satisfy Axiom I.

Axiom B is obviously satisfied by pnp.

To check that pnp satisfies L, consider a financial network $F = (N, z, L) \in \mathcal{F}$ and any agent $i \in N$. Then

$$
\begin{aligned}
e_i(N, z, pnp(F)) &= z_i + \sum_{j \in N} pnp_{ji}(F) - \sum_{j \in N} pnp_{ij}(F) \\
&= z_i + \sum_{j \in N} \min\{L_{ji}, L_{ij}\} + \sum_{j \in N} p_{ji}(N, z, L - \min\{L, L^\top\}) \\
&\quad - \sum_{j \in N} \min\{L_{ij}, L_{ji}\} - \sum_{j \in N} p_{ij}(N, z, L - \min\{L, L^\top\}) \\
&= e_i(N, z, p(N, z, L - \min\{L, L^\top\})) \geq 0,
\end{aligned}
\tag{6.1}
$$

since the proportional rule p satisfies L. Thus pnp satisfies L.

To verify that pnp satisfies P, consider a financial network $F = (N, z, L) \in \mathcal{F}$ and any agent $i \in N$ such that $pnp_i(F) < L_i$, implying that

$$
p_i(N, z, L - \min\{L, L^\top\}) < L_i - \min\{L_i, (L^\top)_i\}.
\tag{6.2}
$$

Since p satisfies P, (6.2) implies that

$$e_i(N, z, p(n, z, L - \min\{L, L^\top\})) = 0.$$

Using the same argument as in (6.1), it follows that $e_i(N, z, pnp(F)) = 0$, thus pnp satisfies P.

To verify that pnp satisfies N, we define the bankruptcy rules $b^4 : \mathcal{F} \to \mathcal{M}$ and $b^5 : \mathcal{F} \to \mathcal{M}$ by setting, for $F = (N, z, L) \in \mathcal{F}$, $b^4(F) = \min\{L, L^\top\}$ and $b^5(F) = p(N, z, L - \min\{L, L^\top\})$. It holds that $pnp(F) = b^4(F) + b^5(F)$. We show that both $b^4(F)$ and $b^5(F)$ satisfy N, from which it follows that pnp satisfies N.

We use Definition 3.8 of Axiom N to verify that b^4 satisfies N, since merging identical agents will not change what they pay or receive in total by pairwise netting, and the liabilities within pairs of unaffected agents are also not changed.

To show that b^5 satisfies N, observe that merging identical agents first and execute pairwise netting next leads to the same liability matrix as pairwise netting first and merging identical agents next. Since p satisfies Axiom N, it follows next that b^5 satisfies Axiom N.

Example 6.5 (All except N). Consider the constrained equal losses rule cel and its characterization as a programming problem (5.3). The rule cel clearly does not satisfy N.

Axioms B, L and I follow from the constraints in the programming problem (5.9).

If there is a financial network $F = (N, z, L) \in \mathcal{F}$ and an agent $i \in N$ such that $e_i(F) > 0$ and $cel_i(F) < L_i$, then the objective function of the programming problem (5.9) could be increased, a contradiction. Thus cel satisfies P.

7 The Axiomatization on the Real Domain

In this section we show that the axioms of claims boundedness (B), limited liability (L), priority of creditors (P), impartiality (I), non-manipulability by identical agents (N), and Continuity (C) characterize the proportional rule p^* for financial networks on the real domain. We also argue that these axioms are independent.

The definition of the axioms of B, L, P, I, and N is extended to the real domain in the straightforward way. Our last axiom is continuity. We endow $\mathcal{F}^*$ with the standard topology, based on the discrete topology for $\mathcal{N}$ and the Euclidean topology for endowments and liabilities. Let $(F^n)_{n \in \mathbb{N}} = (N^n, z^n, L^n)_{n \in \mathbb{N}}$ be a sequence of financial networks of $\mathcal{F}^*$. Notice that this sequence converges to the financial network $\overline{F} = (\overline{N}, \overline{z}, \overline{L})$ of $\mathcal{F}^*$ if and only if there is $n' \in \mathbb{N}$ such that for every $n \geq n'$ it holds that $N^n = \overline{N}$, $\lim_{n \to \infty} z^n = \overline{z}$, and $\lim_{n \to \infty} L^n = \overline{L}$.

Definition 7.1. A bankruptcy rule $b^* : \mathcal{F}^* \to \mathcal{M}^*$ satisfies the Axiom of *continuity* (C) if b^* is continuous.

The following example shows that the constrained equal awards rule cea^* does not satisfy Axiom C.

Example 7.2. Consider a financial network $(N, z, L) \in \mathcal{F}^*$ with three agents $N = \{1, 2, 3\}$. Table 7 presents the endowments, the liabilities, and the payment matrix P resulting from the constrained equal awards rule cea^* and the induced asset values and equities. Agents

z	L			P			$a(N, z, P)$	$e(N, z, P)$
1	0	2	1	0	2	1	3	0
1	2	0	1	2	0	1	3	0
1	0	0	0	0	0	0	3	3

Table 7: The payment matrix, asset values, and equities resulting from the constrained equal awards rule cea^* in Example 7.2 for the financial network $F = (N, z, L)$.

are all able to pay their liabilities, though agents 1 and 2 end up with zero equity.

Now, for $\varepsilon > 0$, consider the financial network $F^\varepsilon = (N, z, L^\varepsilon) \in \mathcal{F}^*$ as displayed in Table 8, where the liabilities of both agents 1 and 2 to agent 3 have gone up by ε.

z	L^ε			P^ε			$a(N, z, P^\varepsilon)$	$e(N, z, P^\varepsilon)$
1	0	2	$1+\varepsilon$	0	1	1	2	0
1	2	0	$1+\varepsilon$	1	0	1	2	0
1	0	0	0	0	0	0	3	3

Table 8: The payment matrix, asset values, and equities resulting from the constrained equal awards rule cea^* in Example 7.2 for the financial network $F^\varepsilon = (N, z, L^\varepsilon)$.

Since constrained equal awards requires the same payments from agent 1 to agents 2 and 3, up to their claims, agent 1 can pay at most one unit to both agents. The same is true for the payments of agent 2 to agents 1 and 3. Under these payments, agents 1 and 2 end up with zero equity and default partially on all their liabilities. We have that

$$\lim_{\varepsilon \downarrow 0} cea^*(F^\varepsilon) = \lim_{\varepsilon \downarrow 0} P^\varepsilon = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = P = cea^*(F),$$

so while the financial networks F^ε converge to F when ε tends to zero, the corresponding payment matrices do not converge.

The lack of continuity of cea^* in Example 7.2 is not resolved by making another selection from the set of clearing payment matrices. The matrix P^ε is the unique clearing payment matrix for the financial network F^ε when constrained equal award division rules are used.

The financial network F has many clearing payment matrices compatible with constrained equal award division rules. The greatest clearing payment matrix is equal to P and the least clearing payment matrix is P^ε. The following example shows that an alternative definition of the constrained equal awards rule that selects the least clearing payment matrix would not solve the lack of continuity.

Example 7.3. For $\varepsilon > 0$, consider the financial network $\widetilde{F}^\varepsilon = (N, z^\varepsilon, L) \in \mathcal{F}^*$ as displayed in Table 9.

z^ε	L			$\widetilde{P}^\varepsilon$			$a(N, z^\varepsilon, \widetilde{P}^\varepsilon)$	$e(N, z^\varepsilon, \widetilde{P}^\varepsilon)$
$1+\varepsilon$	0	2	1	0	2	1	$3 + \varepsilon$	ε
$1+\varepsilon$	2	0	1	2	0	1	$3 + \varepsilon$	ε
1	0	0	0	0	0	0	3	3

Table 9: The payment matrix, asset values, and equities resulting from the constrained equal awards rule cea^* in Example 7.3 for the financial network $\widetilde{F}^\varepsilon = (N, z^\varepsilon, L)$.

The payment matrix $\widetilde{P}^\varepsilon$ is the unique clearing payment matrix in the financial network $\widetilde{F}^\varepsilon$ under constrained equal award division rules. The financial networks $\widetilde{F}^\varepsilon$ tend to the financial network F of Example 7.2 as ε goes to zero. The payment matrices $\widetilde{P}^\varepsilon$ are all equal to $cea^*(F)$. Selecting the least clearing payment matrix for F under constrained equal awards division rules instead of the greatest clearing payment matrix $cea^*(F)$ would then lead to a violation of continuity in this example.

The next result shows that the proportional rule as defined on the real domain satisfies all our axioms.

Theorem 7.4. *The proportional rule p^* satisfies* B, L, P, I, N, *and* C.

Proof. The proof that p^* satisfies B, L, P, I, and N is analogous to the corresponding proof of Theorem 4.1.

We now prove that p^* satisfies C. Let $(F^n)_{n \in \mathbb{N}} = (N^n, z^n, L^n)_{n \in \mathbb{N}}$ be a sequence of financial networks of $\mathcal{F}^*$, which converges to the financial network $\overline{F} = (\overline{N}, \overline{z}, \overline{L})$ of $\mathcal{F}^*$. We have to show that the payment matrix defined by $P^n = p^*(F^n)$ converges to the payment matrix $p^*(\overline{F})$.

Without loss of generality, we can assume that, for every $n \in \mathbb{N}$, $N^n = \overline{N}$ and that, for every $i, j \in \overline{N}$, $L_{ij}^n > 0$ if $\overline{L}_{ij} > 0$. Also, using the boundedness of the sequence $(P^n)_{n \in \mathbb{N}}$, we can assume without loss of generality that it has a limit $\overline{P} \in \mathcal{M}^*(\overline{N})$. For every $n \in \mathbb{N}$, it holds by definition of p^* that

$$
P_{ij}^n = \begin{cases} 0, & \text{if } L_{ij}^n = 0, \\ \min\{\frac{L_{ij}^n}{\sum_{k \in N} L_{ik}^n} a_i(\overline{N}, z^n, P^n), L_{ij}^n\}, & \text{otherwise.} \end{cases}
$$

Let $i, j \in \overline{N}$ be such that $\overline{L}_{ij} = 0$. It holds that

$$\overline{P}_{ij} = \lim_{n \to \infty} P_{ij}^n \leq \lim_{n \to \infty} L_{ij}^n = \overline{L}_{ij} = 0.$$

Let $i, j \in \overline{N}$ be such that $\overline{L}_{ij} > 0$. Now it holds that

$$\overline{P}_{ij} = \lim_{n \to \infty} P_{ij}^n = \lim_{n \to \infty} \min\{\frac{L_{ij}^n}{\sum_{k \in N} L_{ik}^n} a_i(\overline{N}, z^n, P^n), L_{ij}^n\} = \min\{\frac{\overline{L}_{ij}}{\sum_{k \in N} \overline{L}_{ik}} a_i(\overline{N}, \overline{z}, \overline{P}), \overline{L}_{ij}\}.$$

We have shown that $\overline{P}$ is a solution to the system of equations (2.2) corresponding to the financial network $\overline{F}$. Since this solution is unique by Theorem 2 of Eisenberg and Noe (2001), it follows that $\overline{P} = p^*(\overline{F})$ as desired. □

We show next that if a bankruptcy rule satisfies the axioms of B, L, P, I, N, and C, then it must be the proportional rule.

Theorem 7.5. *If the bankruptcy rule b^* satisfies B, L, P, I, N, and C, then $b^* = p^*$.*

Proof. Let b^* be a bankruptcy rule satisfying B, L, P, I, N, and C. It follows from Theorem 4.4 that for every $F \in \mathcal{F}$ we have that $b^*(F) = p(F) = p^*(F)$.

Let $\overline{F} \in \mathcal{F}^*$ be a financial network and let $(F^n)_{n \in \mathbb{N}}$ be a sequence of financial networks in $\mathcal{F}$, so with rational endowments and liabilities, converging to $\overline{F}$. We have that

$$b^*(\overline{F}) = \lim_{n \to \infty} b^*(F^n) = \lim_{n \to \infty} p^*(F^n) = p^*(\overline{F}),$$

where the first equality follows from b^* satisfying C, the second follows since $F^n \in \mathcal{F}$, and the third since p^* satisfies C by Theorem 7.4. □

The following corollary follows immediately from Theorems 7.4 and 7.5.

Corollary 7.6. *The proportional rule p^* is the unique bankruptcy rule satisfying the axioms of B, L, P, I, N, and C.*

To show that the axioms are independent on the real domain, we make use of the rules in Examples 6.1-6.5. We also provide a rule which does not satisfy C but satisfies B, L, P, I, and N. It is easily verified that all the axioms satisfied by the rules in Examples 6.1-6.5 extend to the real domain. It is also easy to show that the extension of the rules in Examples 6.1-6.4 satisfy Axiom C. To show that the extension cel^* of cel as used in Example 6.5 satisfies Axiom C, we first show that the use of constrained equal losses division rules leads to a uniquely defined clearing payment matrix.

Theorem 7.7. *Let a financial network $(N, z, L) \in \mathcal{F}^*$ and constrained equal losses division rules $(d^{*\mathrm{cel}})_{i \in N}$ be given. Then there is a unique clearing payment matrix.*

Proof. By Theorem 5.3 there exists a least clearing payment matrix P^- and a greatest clearing payment matrix P^+. Let $\mu^+, \mu^- \in \mathbb{R}_+^N$ be such that

$$
\begin{aligned}
P_{ij}^- &= \max\{L_{ij} - \mu_i^+, 0\}, \quad i, j \in N, \\
P_{ij}^+ &= \max\{L_{ij} - \mu_i^-, 0\}, \quad i, j \in N.
\end{aligned}
$$

Notice that $P^- \leq P^+$ and $\mu^- \leq \mu^+$.

Suppose that the clearing payment matrix is not unique, that is,

$$
\mu^- < \mu^+. \tag{7.1}
$$

By Theorem 5.11 it holds that

$$
e_i(N, z, P^-) = e_i(N, z, P^+), \quad i \in N. \tag{7.2}
$$

Using the definition of a division rule, if $i \in N$ is such that $e_i(N, z, P^-) = e_i(N, z, P^+) > 0$, then $\mu_i^- = \mu_i^+ = 0$. The set of agents with zero equity is denoted by

$$
N^0 = \{i \in N \mid e_i(N, z, P^-) = e_i(N, z, P^+) = 0\}.
$$

The inequality in (7.1) implies that there is an agent $i_1 \in N^0$ such that

$$
\mu_{i_1}^- < \mu_{i_1}^+. \tag{7.3}
$$

Since the positive endowment z_{i_1} of agent i_1 must end up somewhere, there is a finite sequence of agents $(i_1, \ldots, i_m)$ such that

$$
P_{i_\ell i_{\ell+1}}^- = L_{i_\ell i_{\ell+1}} - \mu_{i_\ell}^+ > 0, \quad \ell = 1, \ldots, m-1, \tag{7.4}
$$

$$
e_{i_1}(N, z, P^-) = \cdots = e_{i_{m-1}}(N, z, P^-) = 0 \text{ and } e_{i_m}(N, z, P^-) > 0, \tag{7.5}
$$

so agent i_ℓ pays a positive amount to agent $i_{\ell+1}$ and agent i_m has positive equity. Using the fact that $P^- \leq P^+$ and (7.2), it also holds that

$$
P_{i_\ell i_{\ell+1}}^+ = L_{i_\ell i_{\ell+1}} - \mu_{i_\ell}^- > 0, \quad \ell = 1, \ldots, m-1, \tag{7.6}
$$

$$
e_{i_1}(N, z, P^+) = \cdots = e_{i_{m-1}}(N, z, P^+) = 0 \text{ and } e_{i_m}(N, z, P^+) > 0. \tag{7.7}
$$

We now show by induction that

$$
\mu_{i_\ell}^- < \mu_{i_\ell}^+, \quad \ell = 1, \ldots, m-1. \tag{7.8}
$$

For $\ell = 1$, (7.8) follows from (7.3).

Assume that (7.8) holds for some $\ell \leq m - 2$. We will show that it also holds for $\ell + 1$.

By (7.5) and (7.7), agent $i_{\ell+1}$ has zero equity in both P^- and P^+, thus

$$\sum_{j\in N}\max\{L_{i_{\ell+1}j}-\mu^+_{i_{\ell+1}},0\} = z_{i_{\ell+1}} + \sum_{j\in N}\max\{L_{ji_{\ell+1}}-\mu^+_j,0\},\qquad(7.9)$$

$$\sum_{j\in N}\max\{L_{i_{\ell+1}j}-\mu^-_{i_{\ell+1}},0\} = z_{i_{\ell+1}} + \sum_{j\in N}\max\{L_{ji_{\ell+1}}-\mu^-_j,0\}.\qquad(7.10)$$

We argue that the right-hand side of (7.10) is strictly greater than that of (7.9). Since $\mu^- \le \mu^+$, we have that

$$\max\{L_{ji_{\ell+1}}-\mu^-_j,0\} \ge \max\{L_{ji_{\ell+1}}-\mu^+_j,0\}, \quad j\in N,$$

It also holds that

$$\max\{L_{i_\ell i_{\ell+1}}-\mu^-_{i_\ell},0\} = L_{i_\ell i_{\ell+1}}-\mu^-_{i_\ell} > L_{i_\ell i_{\ell+1}}-\mu^+_{i_\ell} = \max\{L_{i_\ell i_{\ell+1}}-\mu^+_{i_\ell},0\},$$

where the first equality follows from 7.6, the inequality by the induction hypothesis, and the last equality from (7.4).

The left-hand side of (7.10) is then also strictly greater than that of (7.9), so

$$\sum_{j\in N}\max\{L_{i_{\ell+1}j}-\mu^-_{i_{\ell+1}},0\} > \sum_{j\in N}\max\{L_{i_{\ell+1}j}-\mu^+_{i_{\ell+1}},0\},\qquad(7.11)$$

implying that

$$\mu^-_{i_{\ell+1}} < \mu^+_{i_{\ell+1}}.$$

This completes the proof of (7.8). In particular, we have that

$$\mu^-_{i_{m-1}} < \mu^+_{i_{m-1}}.\qquad(7.12)$$

Finally, we have that

$$
\begin{aligned}
e_{i_m}(N,z,P^+) &- e_{i_m}(N,z,P^-) \\
&= \sum_{j\in N}\max\{L_{ji_m}-\mu^-_j,0\} - \sum_{j\in N}\max\{L_{i_mj}-\mu^-_{i_m},0\} \\
&\quad - \sum_{j\in N}\max\{L_{ji_m}-\mu^+_j,0\} + \sum_{j\in N}\max\{L_{i_mj}-\mu^+_{i_m},0\} \\
&= \sum_{j\in N}\max\{L_{ji_m}-\mu^-_j,0\} - \sum_{j\in N}\max\{L_{ji_m}-\mu^+_j,0\} \\
&> 0,
\end{aligned}
$$

where the second equality follows from $\mu^-_{i_m} = \mu^+_{i_m} = 0$ and the inequality follows from $\mu^- \le \mu^+$, (7.4) and (7.6) for $\ell = m-1$, and (7.12). We have obtained a contradiction to (7.2). Consequently, it follows that the clearing payment matrix is unique. $\qquad\square$

The result of Theorem 7.7 for constrained equal losses division rules is in stark contrast with the case of constrained equal awards division rules as demonstrated by Example 5.4. This is surprising since both division rules can be considered as each other's dual and

share many common features, see Thomson (2003). The proof of Theorem 7.7 cannot be adjusted to deal with the case of constrained equal awards division rules. As can be verified in Example 5.4, it is not true that agent 1 and agent 2 make a strictly higher payment to agent 3 when comparing P^+ to P^-. Therefore, the last step in the proof of Theorem 7.7 does not hold for the case of constrained equal awards division rules.

Theorem 7.8. *The constrained equal losses rule* cel^* *satisfies* C.

Proof. Let $(F^n)_{n \in \mathbb{N}} = (N^n, z^n, L^n)_{n \in \mathbb{N}}$ be a sequence of financial networks of $\mathcal{F}^*$ that converges to the financial network $\overline{F} = (\overline{N}, \overline{z}, \overline{L})$ of $\mathcal{F}^*$. We have to show that the payment matrix defined by $P^n = cel^*(F^n)$ converges to the payment matrix $cel^*(\overline{F})$.

Without loss of generality, we can assume that, for every $n \in \mathbb{N}$, $N^n = \overline{N}$. Also, using the boundedness of the sequence $(P^n)_{n \in \mathbb{N}}$, we can assume without loss of generality that it has a limit $\overline{P} \in \mathcal{M}^*(\overline{N})$. For every $n \in \mathbb{N}$, it holds by definition of cel^* that

$$P_{ij}^n = d_j^{*\mathrm{cel}}(a_i(\overline{N}, z^n, P^n), L_i), \quad i, j \in \overline{N}.$$

We have that

$$\overline{P}_{ij} = \lim_{n \to \infty} P_{ij}^n = \lim_{n \to \infty} d_j^{*\mathrm{cel}}(a_i(\overline{N}, z^n, P^n), L_i) = d_j^{*\mathrm{cel}}(a_i(\overline{N}, \overline{z}, \overline{P}), L_i), \quad i, j \in \overline{N},$$

where the third equality uses that $d_j^{*\mathrm{cel}}$ and a_i are continuous. It follows that $\overline{P}$ is a clearing payment matrix for the financial network $\overline{F}$ and constrained equal losses division rules $(d^{*\mathrm{cel}})_{i \in N}$. By Theorem 7.7, $\overline{P}$ is also the greatest clearing payment matrix and therefore equal to $cel^*(\overline{F})$ by definition of cel^*. $\square$

We complete the section by presenting a bankruptcy rule which does not satisfy C but satisfies B, L, P, I, and N.

Example 7.9 (All except C). Let $d^{*\mathrm{irr}}$ be the division rule that gives priority to claims that belong to $\mathbb{R} \setminus \mathbb{Q}$ over claims that belong to $\mathbb{Q}$ and makes proportional payments within each of the two priority classes. Let $b^{*6} : \mathcal{F}^* \to \mathcal{M}^*$ be the bankruptcy rule that is based on $(d^{*\mathrm{irr}})_{i \in \mathbb{N}}$. The bankruptcy rule b^{*6} obviously satisfies B, L, P, and I.

Axiom N for b^{*6} is satisfied, since the take-over of a set of agents K that are identical to agent j does not affect the priority class of the liabilities of agent j after the take-over. Technically, the addition of a finite number of rational numbers results in a rational number and the addition of a finite number of irrational numbers results in an irrational number. To verify that b^{*6} satisfies N then follows the reasoning for p^*.

The fact that b^{*6} does not satisfy C follows easily from the fact that $d^{*\mathrm{irr}}$ is not continuous in the vector of claims c. For instance, consider the case where one entry of c is a

positive rational number and another entry is a positive irrational number. For a sequence of claims vectors with only rational entries that converges to c, it is only at the limit that the irrational claim gets priority.

8   Conclusion

Many real-life bankruptcy problems are characterized by network aspects, meaning that the default of one agent can potentially snowball and lead to a chain of contagion defaults of other agents. As a consequence, the estates to be divided are endogenously determined, which makes the problem quite different from the typical case as studied in the axiomatic bankruptcy literature. The most important bankruptcy rule from a practical perspective is the proportional rule. This makes an axiomatic analysis of the proportional bankruptcy rule in financial networks imperative.

An important aspect of actual bankruptcy problems is that entities can merge or create spin-offs. When such activities do not generate added value, they should not influence the payments to and from other entities and the payments between other entities. Such a property is known as non-manipulability. We show that an unrestricted ability to form mergers or create spin-offs clashes with non-manipulability. Intuitively, an entity would have incentives to create a spin-off that contains all liabilities, while keeping all assets for itself. We consider a much less demanding non-manipulability property, called non-manipulability by identical agents. In this case, only mergers of identical entities or the split of an entity into a number of identical ones should not affect payments.

Our other main substantive axiom is impartiality, requiring that two agents with the same claim on an agent should receive the same payment from him. We show that the two main axioms lead to the proportional rule when complemented by the axioms of claims boundedness, limited liability, priority of creditors, and continuity. Continuity can be dropped when assuming that all endowments, assets, and liabilities occurring in financial networks are represented by rational numbers rather than reals. We show all axioms to be independent.

We believe that the widespread use of the proportional rule in bankruptcy situations across countries and over time, see Engle (2012) for a historical account of the popularity of the proportional rule, is intimately related to the attractiveness and simplicity of the axioms characterizing it.

Table 10 summarizes the bankruptcy rules used in this paper and the axioms that they satisfy.

	B	L	P	I	N	C
Proportional p^*	√	√	√	√	√	√
Constrained equal awards cea^*	√	√	√	√	¬	¬
Constrained equal losses cel^*	√	√	√	√	¬	√
Pairwise netting proportional pnp^*	√	√	√	¬	√	√
Extension of b^1 of Example 6.1, b^{*1}	¬	√	√	√	√	√
Extension of b^2 of Example 6.2, b^{*2}	√	¬	√	√	√	√
Extension of b^3 of Example 6.3, b^{*3}	√	√	¬	√	√	√
Example 7.9, b^{*6}	√	√	√	√	√	¬

Table 10: Bankruptcy rules and their properties

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