Zassenhaus conjecture for central extensions of $S_5$

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Abstract. We confirm a conjecture of Zassenhaus about rational conjugacy of torsion units in integral group rings for a covering group of the symmetric group $S_5$ and for the general linear group $GL(2, 5)$. The first result, together with others from the literature, settles the conjugacy question for units of prime-power order in the integral group ring of a finite Frobenius group.

1 Introduction

The conjecture of the title states:

(ZC1) For a finite group $G$, every torsion unit in its integral group ring $\mathbb{Z}G$ is conjugate to an element of $\pm G$ by a unit of the rational group ring $\mathbb{Q}G$.

This conjecture remains not only open but also lacking in plausible means of finding either a proof or a counter-example, at least for non-solvable groups $G$. The purpose of this note is to add two further groups to the small list of non-solvable groups $G$ for which conjecture (ZC1) has been verified (see [14], [17], [23], [24]).

Example 1. The conjecture (ZC1) holds for the covering group $\tilde{S}_5$ of the symmetric group $S_5$ which contains a unique conjugacy class of involutions.

Example 2. The conjecture (ZC1) holds for the general linear group $GL(2, 5)$.

We remark that $PGL(2, 5) \cong S_5$ (see [20, Kapitel II, 6.14 Satz]).

The covering group $\tilde{S}_5$ occurs as a Frobenius complement in Frobenius groups (for the classification of Frobenius complements see [27]). From already existing work in [13], [14], [21], it follows that Example 1 supplies the missing part for the proof of the following theorem.

Theorem 3. Let $G$ be a finite Frobenius group. Then each torsion unit in $\mathbb{Z}G$ which is of prime-power order is conjugate to an element of $\pm G$ by a unit of $\mathbb{Q}G$.

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The proofs are obtained by applying a procedure, introduced in [23] and subsequently called the Luthar–Passi method [6], in an extended version developed in [17]. We shall use the validity of (ZC1) for $S_5$, established in [24] (see also [17, Section 5] for a proof using the Luthar–Passi method). Below, we briefly recall this method, which uses the character table and/or modular character tables in an automated process suited for being carried out on a computer, the result being that rational conjugacy of torsion units of a given order to group elements is either proven or not, and if not, at least some information about partial augmentations is obtained; cf. [6], [7], [8], [10]. We have tried to keep proofs free of useless ballast and to present them in a readable form, rather than producing systems of inequalities (as explained below) and their solutions which could reasonably be done on a computer. It is intended to provide routines for this in the GAP package LAGUNA [9].

2 Preliminaries

We provide the necessary background on torsion units in integral group rings. Let $G$ be a finite group. Recall that for a group ring element $a = \sum_{g \in G} a_g g$ in $\mathbb{Z}G$ (with all $a_g$ in $\mathbb{Z}$), the partial augmentation of $a$ with respect to the conjugacy class $X$ of $G$ of an element $x$ of $G$, denoted below by $e_x(a)$ or $e_{X,x}(a)$, is the sum $\sum_{g \in X} e_{x}(a)$. The augmentation of $a$ is the sum of all of its partial augmentations. It suffices to consider units of augmentation one, which form a group denoted by $V(\mathbb{Z}G)$. So let $u$ be a torsion unit in $V(\mathbb{Z}G)$.

The familiar result of Berman and Higman (from [1] and [19, p. 27]) asserts that if $e_z(u) \neq 0$ for some $z$ in the center of $G$, then $u = z$.

A practical criterion for $u$ to be conjugate to an element of $G$ by a unit of $\mathbb{Q}G$ is that all but one of the partial augmentations of every power of $u$ must vanish (see [25, Theorem 2.5]).

The next two remarks will be used repeatedly. While the first one is elementary and well known, the second is of a more recent nature, taken from [17] where an obvious generalization of [18, Proposition 3.1] is given.

**Remark 4.** Let $N$ be a normal subgroup of $G$ and set $\bar{G} = G/N$. We write $\bar{a}$ for the image of $a$ under the natural map $\mathbb{Z}G \to \mathbb{Z}\bar{G}$. Since any conjugacy class of $G$ maps onto a conjugacy class of $\bar{G}$, for any $x \in G$ the partial augmentation $e_x(\bar{a})$ is the sum of the partial augmentations $e_{x^g}(u)$ with $g \in G$ such that $\bar{g}$ is conjugate to $x$ in $\bar{G}$.

Now suppose that $N$ is a central subgroup of $G$, and that $\bar{a} = 1$. Then $u \in N$. Indeed, $1 = e_1(\bar{a}) = \sum_{n \in N} e_n(u)$, and so $u$ has a central group element in its support and the Berman–Higman result applies.

**Remark 5.** If $e_g(u) \neq 0$ for some $g \in G$, then the order of $g$ divides the order of $u$. Indeed, it is well known that then the prime divisors of the order of $g$ divide the order of $u$ (see [25, Theorem 2.7], as well as [18, Lemma 2.8] for an alternative proof). Further, it was observed in [17, Proposition 2.2] that the orders of the $p$-parts of $g$ cannot exceed those of $u$. 


On one occasion, we shall use the following remark.

**Remark 6.** Let $p$ be a rational prime, and let $x \in G$. If $y^G$ is a conjugacy class of $G$ containing an element whose $p$th power is in $x^G$, we write $(y^G)^p = x^G$. A simple but powerful equation which leads to so many group ring consequences is

$$e_{x^G}(a^p) \equiv \sum_{(y^G)^p = x^G} e_{y^G}(a) \mod p$$

for all $a \in \mathbb{Z}G$. (This formula (in prime characteristic) can be traced back to work of Brauer. It obviously derives from a significant feature of the $p$th power map (which may be found in [28, Lemma 2.3.1]). The underlying basic idea was attributed to Landau by Zassenhaus in [30]. A generalization is given by Cliff’s formula [11] (re-stated in [22, Lemma 2]). Applications can be found, for instance, in [15, 21, 22, 28, Section 2].)

We shall use the fact that if there is only one conjugacy class $y^G$ with $(y^G)^p = x^G$, and $e_{y^G}(a) \neq 0 \mod p$, then $e_{x^G}(a^p) \neq 0$.

Finally, we outline the Luthar–Passi method. Let $u$ be of order $n$ (say), and let $\zeta$ be a primitive complex $n$th root of unity. For a character $\chi$ afforded by a complex representation $D$ of $G$, write $\mu(\zeta, u, \chi)$ for the multiplicity of an $n$th root of unity $\zeta$ as an eigenvalue of the matrix $D(u)$. By [23],

$$\mu(\zeta, u, \chi) = \frac{1}{n} \sum_{d \mid n} \text{Tr}_{Q(\zeta^d)/Q}(\chi(u^d)\zeta^{-d}).$$

When trying to show that $u$ is rationally conjugate to an element of $G$, one may assume—by induction on the order of $u$—that the values of the summands for $d \neq 1$ are ‘known’. The summand for $d = 1$ can be written as

$$\frac{1}{n} \sum_{u^d} e_{\chi}(u) \text{Tr}_{Q(\zeta)/Q}(\chi(g)\zeta^{-1}),$$

a linear combination of the $e_{\chi}(u)$ with ‘known’ coefficients. Note that the $\mu(\zeta, u, \chi)$ are non-negative integers, bounded above by $\chi(1)$. Thus, in some sense, there are linear inequalities in the partial augmentations of $u$ which impose constraints on them. Trying to make use of these inequalities is now understood as being the Luthar–Passi method.

A modular version of this method (see [17, Section 4] for details) can be derived from the following observation in the same way as the original (complex) version is derived from the (obvious) fact that $\chi(u) = \sum_{g \in G} e_{\chi}(u)\chi(g)$. Suppose that $p$ is a rational prime which does not divide the order of $u$ (i.e., $u$ is a $p$-regular torsion unit).
Then for every Brauer character \( \varphi \) of \( G \) (relative to \( p \)) we have (see [17, Theorem 3.2]):

\[
\varphi(u) = \sum_{g'' \in G} \epsilon_{g}(u)\varphi(g).
\]

Thereby, the domain of \( \varphi \) is naturally extended to the set of \( p \)-regular torsion units in \( ZG \).

### 3 A covering group of \( \tilde{S}_5 \)

A presentation of a covering group of \( S_5 \) is given by

\[
\tilde{S}_5 = \langle g_1, \ldots, g_{n-1}, z | g_i^2 = (g_i g_{i+1})^3 = (g_k g_l)^2 = z, z^2 = [z, g_1] = 1 \rangle
\]

for \( 1 \leq i \leq n - 1, 1 \leq j \leq n - 2, 1 \leq k \leq n - 2, k \leq l \leq n - 3 \).

Recent results in the representation theory of the covering groups of symmetric groups can be found in Bessenrodt’s survey article [2]. We merely remark that the complex spin characters of \( \tilde{S}_5 \), i.e., those characters which are not characters of \( S_5 \), were determined by Schur [29].

The group \( \tilde{S}_5 \) has catalogue number 89 in the Small Group Library in GAP [16] (the other covering group of \( S_5 \) has number 90). The spin characters of \( \tilde{S}_5 \) as produced by GAP are shown in Table 1 (dots indicate zeros).

<table>
<thead>
<tr>
<th></th>
<th>1a</th>
<th>5a</th>
<th>4a</th>
<th>2a</th>
<th>10a</th>
<th>6a</th>
<th>3a</th>
<th>8a</th>
<th>8b</th>
<th>4b</th>
<th>12a</th>
<th>12b</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_5 )</td>
<td>4</td>
<td>-1</td>
<td>.</td>
<td>-4</td>
<td>1</td>
<td>2</td>
<td>-2</td>
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<tr>
<td>( Z_6 )</td>
<td>4</td>
<td>-1</td>
<td>.</td>
<td>-4</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>( \beta )</td>
<td>-( \beta )</td>
<td>.</td>
</tr>
<tr>
<td>( Z_7 )</td>
<td>4</td>
<td>-1</td>
<td>.</td>
<td>-4</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>-( \beta )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( Z_{11} )</td>
<td>6</td>
<td>1</td>
<td>.</td>
<td>-6</td>
<td>-1</td>
<td>.</td>
<td>.</td>
<td>( z )</td>
<td>-( z )</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( Z_{12} )</td>
<td>6</td>
<td>1</td>
<td>.</td>
<td>-6</td>
<td>-1</td>
<td>.</td>
<td>.</td>
<td>-( z )</td>
<td>( z )</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

Irrational entries:

\[
x = -\zeta_8 + \zeta_8^2 = -\sqrt{2} \text{ where } \zeta_8 = \exp(2\pi i/8),
\]

\[
\beta = \zeta_{12}^7 - \zeta_{12}^{11} = -\sqrt{3} \text{ where } \zeta_{12} = \exp(2\pi i/12).
\]

Table 1. Spin characters of \( \tilde{S}_5 \)

We turn to the proof that conjecture (ZC1) holds for \( \tilde{S}_5 \). Let \( z \) be the central involution in \( \tilde{S}_5 \). Then we have a natural homomorphism

\[
\pi : \mathbb{Z}\tilde{S}_5 \rightarrow \mathbb{Z}\tilde{S}_5/\langle z \rangle = \mathbb{Z}S_5.
\]

Let \( u \) be a non-trivial torsion unit in \( V(\mathbb{Z}\tilde{S}_5) \). We shall show that all but one of its partial augmentations vanish. Since \( (ZC1) \) is true for \( S_5 \), the order of \( \pi(u) \) agrees with
the order of an element of $S_5$, and it follows that the order of $u$ agrees with the order of an element of $S_5$ (see Remark 4). By the Berman–Higman result we can assume that $e_1(u) = 0$ and $e_3(u) = 0$. Further, we can assume that the order of $u$ is even, since otherwise rational conjugacy of $u$ to an element of $G$ follows from the validity of (ZC1) for $S_5$ and [13, Theorem 2.2]. Denote the partial augmentations of $u$ by $e_{1a}, e_{5a}, \ldots, e_{12b}$ (so that $e_{5a}$, for example, denotes the partial augmentation of $u$ with respect to the conjugacy class of elements of order 5). So $e_{1a} = e_{2a} = 0$. Since all but one of the partial augmentations of $\pi(u)$, the image of $u$ in $ZS_5$, vanish, and a partial augmentation of $\pi(u)$ is the sum of the partial augmentations of $u$ taken for classes which fuse in $S_5$, we have

$$
\begin{align*}
&\{e_{4a}, e_{4b}, e_{8a} + e_{8b}, e_{3a} + e_{5a} + e_{10a}, e_{12a} + e_{12b} \in \{0, 1\}, \\
&e_{4a} + |e_{4b}| + |e_{5a} + e_{8a}| + |e_{3a} + e_{6a}| + |e_{5a} + e_{10a}| + |e_{12a} + e_{12b}| = 1.\
\end{align*}
$$

(1)

**Suppose that $u$ has order 2 or 4.** Then the partial augmentations of $u$ which are possibly non-zero are $e_{4a}$, $e_{4b}$, $e_{8a}$ and $e_{8b}$ (by Remark 5). Thus

$$
\chi_{11}(u) = \pi(e_{8a} - e_{8b}) = -\sqrt{2}(e_{8a} - e_{8b}).
$$

Also, $\chi_{11}(u)$ is a sum of fourth roots of unity. Since $\sqrt{2} \notin \mathbb{Q}(i)$ it follows that $e_{8a} - e_{8b} = 0$. Using $e_{8a} + e_{8b} \in \{0, 1\}$ from (1) we obtain $e_{8a} = e_{8b} = 0$. Now $|e_{4a}| + |e_{4b}| = 1$ by (1), and so all but one of the partial augmentations of $u$ vanish, with either $e_{4a} = 1$ or $e_{4b} = 1$. It follows that $u$ is rationally conjugate to an element of $G$ (necessarily of order 4).

**Suppose that $u$ has order 6 or 10.** Then $u^3 = z$ or $u^5 = z$, respectively (by Remark 4), i.e., $zu$ is of order 3 or 5. Thus $zu$ is, as already noted, rationally conjugate to an element of $G$, and hence the same holds for $u$ itself.

**Suppose that $u$ has order 12.** Then the partial augmentations of $u$ which are possibly non-zero are $e_{4a}$, $e_{4b}$, $e_{8a}$, $e_{8b}$, $e_{3a}$, $e_{5a}$, $e_{12a}$ and $e_{12b}$. The unit $\pi(u)$ has order 6 (by Remark 5), so that $e_{12a} = e_{12b} = 1$ and $e_{4a} = e_{4b} = e_{8a} + e_{8b} = e_{3a} + e_{5a} = 0$ by (1). Now $\chi_{11}(u) = -\sqrt{2}(e_{8a} - e_{8b})$ but $\sqrt{2} \notin \mathbb{Q}(\xi_{12}) = \mathbb{Q}(i, \xi_5)$, so that $e_{8a} = e_{8b}$ and consequently $e_{8a} = e_{8b} = 0$. Further $\chi_5(u) = 2(e_{6a} - e_{3a}) = 4e_{6a} = e_{6a}\xi_5(1)$, and so if $e_{6a} \neq 0$ then $u$ is mapped under a representation of $G$ affording $\chi_5$ to the identity matrix or the negative of the identity matrix, leading to the contradiction $\chi_5(1) = \chi_5(u^6) = \chi_5(z) = -4$. Thus $e_{3a} = e_{6a} = 0$. So far, we have shown that $e_{12a}$ and $e_{12b}$ are the only possibly non-vanishing partial augmentations of $u$. We continue with a formal application of the Luther–Passi method. Let $\xi$ be a 12th root of unity. Then

$$
\mu(\xi, u, \chi_6) = \frac{1}{12}(\text{Tr}_{\mathbb{Q}(\xi_{12})/\mathbb{Q}}(\chi_6(u)^{\xi^{-1}}) + 6\mu(\xi^2, u^2, \chi_6) + \text{Tr}_{\mathbb{Q}(\xi_{12})/\mathbb{Q}}(\chi_6(u^3)^{\xi^{-3}})).
$$

Since $u^3$ is rationally conjugate to an element of order 4 in $G$, we have $\chi_6(u^3) = 0$. Since $\chi_6(u) = \beta(e_{12a} - e_{12b}) = (\xi_1^2 - \xi_{12}^2)(e_{12a} - e_{12b})$, we have
\[
\begin{align*}
\text{Tr}_{Q(\xi_{12})/Q}(\chi_8(u)\xi_{12}^{-2}) &= 6(e_{12a} - e_{12b}), \\
\text{Tr}_{Q(\xi_{12})/Q}(\chi_8(u)\xi_{12}^{-11}) &= -6(e_{12a} - e_{12b}).
\end{align*}
\]

Next, \(\chi_8(u^4) = 1\) since \(u^4\) is rationally conjugate to an element of order 3 in \(G\), and \(\chi_8(u^6) = \chi_8(z) = -4\), from which it is easy to see that \(\mu(\xi^2, u^2, \chi_8) = 1\) for a primitive 12th root of unity \(\xi\). Thus

\[
\begin{align*}
\mu(\xi_{12}, u, \chi_8) &= \frac{1}{4}((e_{12a} - e_{12b}) + 1) \geq 0, \\
\mu(\xi_{12}, u, \chi_6) &= \frac{1}{4}((e_{12a} - e_{12b}) + 1) \geq 0,
\end{align*}
\]

from which we obtain \(|e_{12a} - e_{12b}| \leq 1\). Together with \(e_{12a} + e_{12b} = 1\) this implies that \(e_{12a} = 0\) or \(e_{12b} = 0\). We have shown that all but one of the partial augmentations of \(u\) vanish.

Suppose that \(u\) has order 8. Then the partial augmentations of \(u\) which are possibly non-zero are \(e_{8a}, e_{8b}, e_{8a} + e_{8b}\). Since \(\pi(u)\) has order 4, its partial augmentations with respect to classes of elements of order 2 vanish and consequently \(e_{8a} = e_{8b} = 0\).

We have \(\chi_{11}(u) = \pi(e_{8a} - e_{8b}) = (e_{8a} + e_{8b})\chi_{11}(e_{8a} - e_{8b})\), and this time the Luther–Passi method gives

\[
\begin{align*}
\mu(\xi_{8}, u, \chi_{11}) &= \frac{1}{4}((e_{8a} - e_{8b}) + 3) \geq 0, \\
\mu(\xi_{8}, u, \chi_{11}) &= \frac{1}{4}((e_{8a} - e_{8b}) + 3) \geq 0,
\end{align*}
\]

from which we obtain \(|e_{8a} - e_{8b}| \leq 3\). Together with \(e_{8a} + e_{8b} = 1\) this implies that \((e_{8a}, e_{8b}) \in \{(1, 0), (0, 1), (-1, 2), (2, -1)\}\). At this point we are stuck when limiting attention to complex characters only.

However, we may resort to \(p\)-modular characters. Examining Brauer characters of small degree seems most promising. Thus it is natural to choose \(p = 5\) since \(\tilde{S}_5\) is a subgroup of \(\text{SL}(2, 25)\). This can be seen as follows. The group \(\text{PSL}(2, 25)\) contains \(\text{PGL}(2, 5)\) as a subgroup (see [20, Kapitel II, 8.27 Hauptsatz]) which is isomorphic to \(S_5\), and so its 2-pre-image in \(\text{SL}(2, 25)\) is isomorphic to \(S_5\) (since the Sylow 2-subgroups of \(\text{SL}(2, 25)\) are generalized quaternion groups). Let \(\varphi\) be the Brauer character afforded by a faithful representation \(D: \tilde{S}_5 \rightarrow \text{SL}(2, 25)\). The Brauer lift can be chosen such that \(\varphi(x) = z = -\xi_8 + \xi_8^3\) for an element \(x\) in the conjugacy class 8a of \(G\) (since \(D(x)\) has determinant 1). Class 8b is represented by \(x^3\), and so we obtain \(\varphi(u) = e_{8a}\varphi(x) + e_{8b}\varphi(x^3) = (e_{8a} - e_{8b})(e_{8a} - e_{8b})\). Since \(\varphi(u)\) is the sum of two 8th roots of unity, it follows that \(|e_{8a} - e_{8b}| \leq 1\) and consequently \(e_{8a} = 0\) or \(e_{8b} = 0\). The proof is complete.

**Remark 7.** The spin characters of \(\tilde{S}_5\) form a single 5-block of defect 1, with Brauer tree

\[
\begin{array}{cccccc}
\varphi_{4a} & \varphi_{4b} & \varphi_{2b} & \varphi_{2a} & \varphi_{11} & \varphi_6 \\
\chi_6 & \chi_{11} & \chi_5 & \chi_{12} & \chi_7
\end{array}
\]
The entries: 4aAutomorphisms of ZG are conjugate under the Frobenius homomorphism. Since we had to examine the character value $\chi_{11}(u)$, it was certainly a good choice to further consider a modular constituent of $\chi_{11}$.

One may dare to ask whether the theory of cyclic blocks can provide additional insight into Zassenhaus’ conjecture (ZC1). We have no opinion on this, but we digress into a brief discussion of another conjecture of Zassenhaus, (ZCAut), where this is actually the case. (ZCAut) asserts that the group $\text{Aut}_n(\mathbb{Z}G)$ of augmentation preserving automorphisms of $\mathbb{Z}G$ is generated by automorphisms of $G$ and central automorphisms; though not valid in general, it may very well be valid for simple groups $G$. The main point here is that $\text{Aut}_n(\mathbb{Z}G)$ acts on various structures associated with $\mathbb{Z}G$. For one thing, ring automorphisms give rise to autoequivalences of module categories. In [5], rigidity of autoequivalences of the module category of a Brauer tree algebra was studied, with first applications to (ZCAut) for simple groups. For another thing, $\text{Aut}_n(\mathbb{Z}G)$ acts on the class sums of $\mathbb{Z}G$. This immediately shows that the action on characters—both ordinary and modular—is compatible with taking tensor products; see [3], and [4] for a thorough examination of the consequences resulting for (ZCAut).

4 The general linear group GL(2, 5)

We set $G = \text{GL}(2, 5)$. Let $z$ be a generator of $\mathbb{Z}(G)$, which is a cyclic group of order 4. The quotient $G/\langle z \rangle$ is isomorphic to $S_5$, for which (ZC1) is known to hold. Let $\pi$ denote the natural map $\mathbb{Z}G \rightarrow \mathbb{Z}G/\langle z \rangle$.

Let $u$ be a non-trivial torsion unit in $V(\mathbb{Z}G)$. We will show that all but one of its partial augmentations vanish. For this, we use part of the character table of $G$, shown in Table 2 in the form obtained by requiring CharacterTable(“GL25”) in GAP [16], together with the natural 2-dimensional representation of $G$ in characteristic 5. In Table 2, the row labelled ‘in $S_5$‘ indicates to which classes in the quotient $S_5$

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{class in } S_5 & 1 & 4a & 2b & 4d & 4e & 4f & 4g & 24a & 12a & 8a & 6a & 24b & 8b & 24c & 12b & 24d \\
\hline
\chi_2 & 1 & i & -i & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 & i & -1 & -i & 1 \\
\chi_6 & 5 & i & -1 & -i & i & -i & 1 & i & -1 & i & -1 & -i & 1 & i & -1 & -i & 1 \\
\chi_{16} & 4 & . & . & . & . & -i & -1 & -2i & 1 & i & 1 & 2i & -i & -1 & i \\
\chi_9 & 6 & \alpha & \bar{\alpha} & \bar{\alpha} & \alpha & . & . & . & . & . & . & . & . & . & . & . & . \\
\chi_{14} & 6 & -\alpha & -\bar{\alpha} & \alpha & \bar{\alpha} & . & . & . & . & . & . & . & . & . & . & . & . \\
\chi_{15} & 4 & . & . & . & . & \beta & -i & . & -1 & -\beta & 1 & . & -\beta & i & \beta \\
\chi_{21} & 4 & . & . & . & . & 2i & . & 2 & -2 & . & -2i & . & 2 & -i & 2 & -i \\
\chi_{22} & 4 & . & . & . & . & -\beta & -i & . & -1 & \beta & 1 & . & \beta & i & -\beta & i & -\beta \\
\hline
\end{array}
$$

Irrational entries: $\alpha = 1 + i$, $\beta = -\zeta + \zeta^{17}$ where $\zeta = \exp(2\pi i/24)$.

Table 2. Part of the character table of GL(2, 5)
the listed classes of $G$ are mapped. The classes omitted are the classes $2a$, $4a$, $4b$ of central $2$-elements, and the classes $5a$, $20a$, $10a$, $20b$ of elements of order divisible by $5$.

The characters $\chi_6$ and $\chi_{16}$ have kernel $\langle z^2 \rangle$. The faithful characters $\chi_9$, $\chi_{14}$, $\chi_{15}$, $\chi_{21}$ and $\chi_{22}$ of $G$ form a $5$-block of $G$, with Brauer tree

\[
\begin{array}{cccc}
\chi_{15} & \chi_9 & \chi_{21} & \chi_{14} & \chi_{22} \\
\varphi_{4a} & \varphi_{2a} & \varphi_{2b} & \varphi_{4b} \\
\end{array}
\]

(cf. the theory of blocks of cyclic defect). Set $\varphi = \varphi_{4a} = (\chi_{15} - \chi_{21})|_G$, (restriction to $5$-regular elements). Then $\varphi$ is an irreducible $5$-modular Brauer character of $G$ of degree $2$ afforded by a natural representation $G \to \text{GL}(2, 5)$.

We remark that the remaining irreducible faithful characters of $G$ form a $5$-block of $G$ which is algebraically conjugate to the one that we consider.

We write $e_{4c}, e_{2b}, \ldots, e_{24d}$ for the partial augmentations of $u$ at the classes listed in Table 2. We assume that $u$ is not a central unit, so that its partial augmentations at central group elements are zero. It follows from Remark 4, and the validity of (ZC1) for $S_5$, that the order of $u$ agrees with the order of some group element of $G$.

**Suppose that the order of $u$ is divisible by $5$.** Then $\pi(u)$ has order $5$, by Remark 4, and $u$ is the product of a unit of order $5$ and a central group element of $G$. Since there is only one class of elements of order $5$ in $G$, the $5$-part of $u$ is rationally conjugate to an element of $G$ (by Remark 5), and thus the same is valid for $u$.

**Suppose that $u$ has order $2$.** The group $G$ has only one class of non-central elements of order $2$, and so Remark 5 applies.

**Suppose that $u$ has order $4$.** Then $\varphi_{4}(u) = 0$ for a group element $g$ which is not a non-central element of order $2$ or $4$ (by Remark 5). Evaluating the Brauer character $\varphi$ at $u$ gives

\[
\varphi(u) = (e_{4c} - e_{4g})(1 + i) + (e_{4d} - e_{4c})(1 - i).
\]  

(2)

First, suppose that $\pi(u)$ has order $2$. Then $u^2 = z^2$ (by Remark 4), and so $\varphi(u^2) = -2$ and $\varphi(u)$ is the sum of two primitive fourth roots of unity. These roots of unity are distinct since $u$ is non-central in $\mathbb{Z}G$. Thus $\varphi(u) = i + (-i) = 0$ and (2) gives $e_{4c} = e_{4g}$ and $e_{4d} = e_{4c}$. Since (ZC1) holds for $S_5$ we have

$$e_{4c} + e_{4g} + e_{4d} + e_{4c} = 0, \quad e_{2b} + e_{4d} = 1.$$  

From this we further obtain $e_{4d} = -e_{4c}$ and $\chi_2(u) = 1 - 2e_{2b} + 4e_{4c}i$. Since $|\chi_2(u)| = 1$ it follows that $e_{4c} = 0$ and $e_{2b} \in \{0, 1\}$. Thus all but one of the partial augmentations of $u$ vanish.

Secondly, suppose that $\pi(u)$ has order $4$. Then $\varphi(u^2) \neq -2$. Since $\varphi(u)$ is the sum of two distinct fourth roots of unity we have $|\varphi(u)| < 2$. Thus $\varphi(u) \in \{\pm (1 + i), \pm (1 - i)\}$ by (2). Since (ZC1) holds for $S_5$ we have
\( \varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 1, \quad \varepsilon_{2b} + \varepsilon_{4f} = 0. \)

From this and (2) we further obtain that for some \( a \in \mathbb{Z} \) and \( \delta_i \in \{0, 1\} \), with exactly one \( \delta_i \) non-zero, \( \varepsilon_{4c} = a + \delta_1, \quad \varepsilon_{4g} = a + \delta_2, \quad \varepsilon_{4c} = a - \delta_3 \) and \( \varepsilon_{4g} = a - \delta_4 \). Thus \( \chi_5(u) = (\delta_1 + \delta_2 - \delta_3 - \delta_4)i - 2\varepsilon_{2b} + 4ai \), from which \( \varepsilon_{2b} = 0 \) and \( a = 0 \) follows. Thus all but one of the partial augmentations of \( u \) vanish.

**Suppose that \( u \) has order 8.** Then \( \varepsilon_{8a} \neq 0 \) or \( \varepsilon_{8b} \neq 0 \) by [12, Corollary 4.1] (an observation sometimes attributed to Zassenhaus; cf. [30, Lemma 3]).

Suppose that \( \varepsilon_{8b} = -\varepsilon_{8a} \). Then \( \chi_{16}(u) = -4\varepsilon_{8a}i \) (remember Remark 5). Since \( \chi_{16} \) has degree 4 it follows that \( |\varepsilon_{8a}| = |\varepsilon_{8b}| = 1 \). The class \( \varepsilon_{8a} \) is the only class consisting of elements whose square is in \( 4a \), the class consisting of one of the central elements of order 4. Also \( \varepsilon_{8b} \) is the only class consisting of elements whose square is in \( 4b \). Thus \( \varepsilon_{8a}(u^2) \neq 0 \) and \( \varepsilon_{8b}(u^2) \neq 0 \) by Remark 6. But we already know that \( u^2 \) is rationally conjugate to a group element, and so we have reached a contradiction.

Hence \( \varepsilon_{8a} + \varepsilon_{8b} \neq 0 \), and since \( \varepsilon_{8a} \) and \( \varepsilon_{8b} \) are the classes of \( G \) which map onto class \( e_{2b} \) in \( S_5 \), in fact \( \varepsilon_{8a} + \varepsilon_{8b} = 1 \). Now \( \chi_{16}(u) = 2(1 - 2\varepsilon_{8a})i \), and \( |\chi_{16}(u)| \leq 4 \) implies that \( \varepsilon_{8a} \in \{0, 1\} \), so that one of \( \varepsilon_{8a} \) and \( \varepsilon_{8b} \) vanishes.

Next, we show that \( \chi_9(u) = 0 \). Since \( S_5 \) has no elements of order 8 we have \( u^4 = z^2 \) by Remark 4. From \( \chi_9(u^4) = \chi_9(z^2) = -\chi_9(1) \) we conclude that \( \chi_9(u) \in \zeta_9 \mathbb{Z}[i] \) for a primitive 8th root of unity \( \zeta_9 \), and inspection of the character table shows that \( \chi_9(u) \in \mathbb{Z}[i] \). But definitely \( \zeta_9 \notin \mathbb{Z}[i] \), and so \( \chi_9(u) = 0 \).

Since

\[
\chi_9(u) = (\varepsilon_{4c} - \varepsilon_{4g})(1 + i) + (\varepsilon_{4d} + \varepsilon_{4e})(1 - i)
\]

(3)

and \( \varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 0 \), it follows that \( \varepsilon_{4c} = \varepsilon_{4g} = -\varepsilon_{4d} = -\varepsilon_{4e} \). Also we have \( \varepsilon_{2b} + \varepsilon_{4f} = 0 \). So \( \chi_9(u) = (\pm 1 + 4\varepsilon_{4e})i - 2\varepsilon_{2b} \), which implies \( \varepsilon_{2b} = 0 \) and \( \varepsilon_{4e} = 0 \), and we are done.

**Suppose that \( u \) has order 3.** The group \( G \) has only one class of elements of order 3, and so Remark 5 applies.

**Suppose that \( u \) has order 6.** The only partial augmentations of \( u \) which are possibly non-zero are \( \varepsilon_{2b}, \varepsilon_{3a} \) and \( \varepsilon_{0a} \). Since the class \( \varepsilon_{0a} \) maps in \( S_5 \) to the class of elements of order 3 it follows that \( \pi(u) \) is rationally conjugate to a group element of order 3 in \( S_5 \). Hence \( u \) is the product of \( z^2 \) and a unit of order 3 (by Remark 4), and \( u \) is rationally conjugate to a group element.

**Suppose that \( u \) has order 12.** Then the only partial augmentations of \( u \) which are possibly non-zero are at classes of elements of order 2, 4, 3, 6 and 12. The classes of elements of order 3, 6 and 12 map in \( S_5 \) to the class of elements of order 3. Thus \( \pi(u) \) is of order 3 and \( u \) is the product of \( z \) and a unit of order 3, so that \( u \) is rationally conjugate to a group element.
Suppose that \( u \) has order 24. Then \( \pi(u) \) is rationally conjugate to an element of order 6 in \( S_5 \), and so

\[
\begin{align*}
\varepsilon_{24a} + \varepsilon_{24b} + \varepsilon_{24c} + \varepsilon_{24d} &= 1, \\
\varepsilon_{12a} + \varepsilon_{6a} + \varepsilon_{3a} + \varepsilon_{12b} &= 0, \\
\varepsilon_{3a} + \varepsilon_{3b} &= 0, \\
\varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} &= 0, \\
\varepsilon_{2b} + \varepsilon_{4f} &= 0.
\end{align*}
\]

From \( \chi_9(u^{12}) = -\chi_9(1) \) we conclude that \( \chi_9(u) \in \zeta_8 \mathbb{Z}[i, \zeta_3] \) for a primitive 8th root of unity \( \zeta_8 \) and a primitive cube root of unity \( \zeta_3 \). Inspection of the character table shows that \( \chi_9(u) \in \mathbb{Z}[i] \), and so \( \chi_9(u) = 0 \) as \( \zeta_8 \notin \mathbb{Z}[i, \zeta_3] \). In the same way we argue that \( \chi_{21}(u) = 0 \). Thus evaluation (3) of \( \chi_9(u) \) is zero, and with (4) it follows that \( \varepsilon_{4c} = \varepsilon_{4e} = -\varepsilon_{4d} = -\varepsilon_{4c} \). Now \( (\chi_2 + \chi_6)(u) = -4\varepsilon_{2a} + 8\varepsilon_{3c} \). Since \( \chi_2 + \chi_6 \) has degree 6 we conclude that \( \varepsilon_{2c} = 0 \).

We have \( 0 = \chi_{33}(u) = 2(\varepsilon_{6a} - \varepsilon_{3a}) + 2i(\varepsilon_{12a} - \varepsilon_{12b}) \), and therefore \( \varepsilon_{6a} = \varepsilon_{3a} \) and \( \varepsilon_{12a} = \varepsilon_{12b} \). Further \( \varepsilon_{6b} = -\varepsilon_{12a} \) from (4). Thus \( \chi_{16}(u) = -4\varepsilon_{12a} + i\mathbb{Z} \). From \( \chi_{16}(u^k) = -\chi_{16}(1) \) we obtain \( \chi_{16}(u) \in i\mathbb{Z}[\zeta_3] \). It follows that \(-4\varepsilon_{12a}i \in \mathbb{Z}[\zeta_3] \) and \( \varepsilon_{12a} = 0 \). Now \( \chi_{21}(u) = -2\varepsilon_{2b} + i\mathbb{Z} \) and so \( \varepsilon_{2b} = 0 \). Also \( (\chi_2 + \chi_{16})(u) = -2\varepsilon_{2b} - 6\varepsilon_{3a}i \) and since \( \chi_2 + \chi_{16} \) has degree 5 we have \( \varepsilon_{3a} = 0 \).

Set \( a = \varepsilon_{24a} + \varepsilon_{24b} \) and \( b = \varepsilon_{24b} + \varepsilon_{24d} \). Then \( \chi_2(u) = (a - b)i \) and thus \( a - b = \pm 1 \). Together with \( a + b = 1 \) this implies that \( (a, b) = (1, 0) \) or \( (a, b) = (0, 1) \). In the first case, \( \chi_{15}(u) = (2\varepsilon_{24a} - 1)i - 2\varepsilon_{24b}i \), and in the second \( \chi_{15}(u) = 2\varepsilon_{24a} - 1 + 2\varepsilon_{24b}i \). Using the sum formula for \( \sin \frac{\pi}{12} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \) it is easiest to calculate \( \beta = -\sqrt{1 + i} \). In particular, \( |\beta| = \sqrt{3} \). Since \( \chi_{15}(u) \) is the sum of four roots of unity, it is readily seen that if \( \chi_{15}(u) \) assumes the first value, then \( \varepsilon_{24b} = 0 \) and \( \varepsilon_{24a} \in \{0, 1\} \), and if \( \chi_{15}(u) \) assumes the second value, then \( \varepsilon_{24b} = 0 \) and \( \varepsilon_{24a} \in \{0, 1\} \). It follows that exactly one of \( \varepsilon_{24a}, \varepsilon_{24c}, \varepsilon_{24b} \) and \( \varepsilon_{24d} \) is non-zero, and we are done.

The observant reader might have noticed that the last argument can be replaced by a simpler ‘modular’ argument: we already know that \( \chi_{15}(u) \) agrees with the value of \( \chi_{15}(u) \) at a class of elements of order 24 since \( \chi_9(u) = 0 \) and \( \varphi = \chi_9 - \chi_{15} \) on 5-regular elements.

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References

Zassenhaus conjecture for central extensions of $S_5$


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