J. Group Theory **11** (2008), 63–74
DOI 10.1515/JGT.2008.004

Journal of Group Theory

Zassenhaus conjecture for central extensions of S_5

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(Communicated by S. Sidki)

Abstract. We confirm a conjecture of Zassenhaus about rational conjugacy of torsion units in integral group rings for a covering group of the symmetric group S_5 and for the general linear group $\mathrm{GL}(2,5)$. The first result, together with others from the literature, settles the conjugacy question for units of prime-power order in the integral group ring of a finite Frobenius group.

1 Introduction

The conjecture of the title states:

(ZC1) For a finite group G, every torsion unit in its integral group ring $\mathbb{Z}G$ is conjugate to an element of $\pm G$ by a unit of the rational group ring $\mathbb{Q}G$.

This conjecture remains not only open but also lacking in plausible means of finding either a proof or a counter-example, at least for non-solvable groups G. The purpose of this note is to add two further groups to the small list of non-solvable groups G for which conjecture (ZC1) has been verified (see [14], [17], [23], [24]).

Example 1. The conjecture (ZC1) holds for the covering group $\tilde{S}_5$ of the symmetric group S_5 which contains a unique conjugacy class of involutions.

Example 2. The conjecture (ZC1) holds for the general linear group $\mathrm{GL}(2,5)$.

We remark that $\mathrm{PGL}(2,5) \cong S_5$ (see [20, Kapitel II, 6.14 Satz]).

The covering group $\tilde{S}_5$ occurs as a Frobenius complement in Frobenius groups (for the classification of Frobenius complements see [27]). From already existing work in [13], [14], [21], it follows that Example 1 supplies the missing part for the proof of the following theorem.

Theorem 3. *Let G be a finite Frobenius group. Then each torsion unit in $\mathbb{Z}G$ which is of prime-power order is conjugate to an element of $\pm G$ by a unit of $\mathbb{Q}G$.*

* The first author's work was supported by OTKA T 037202, T 038059.

The proofs are obtained by applying a procedure, introduced in [23] and subsequently called the Luthar–Passi method [6], in an extended version developed in [17]. We shall use the validity of (ZC1) for S_5, established in [24] (see also [17, Section 5] for a proof using the Luthar–Passi method). Below, we briefly recall this method, which uses the character table and/or modular character tables in an automated process suited for being carried out on a computer, the result being that rational conjugacy of torsion units of a given order to group elements is either proven or not, and if not, at least some information about partial augmentations is obtained; cf. [6], [7], [8], [10]. We have tried to keep proofs free of useless ballast and to present them in a readable form, rather than producing systems of inequalities (as explained below) and their solutions which could reasonably be done on a computer. It is intended to provide routines for this in the GAP package LAGUNA [9].

2 Preliminaries

We provide the necessary background on torsion units in integral group rings. Let G be a finite group. Recall that for a group ring element $a = \sum_{g \in G} a_g g$ in $\mathbb{Z}G$ (with all a_g in $\mathbb{Z}$), the partial augmentation of a with respect to the conjugacy class x^G of an element x of G, denoted below by $\varepsilon_x(a)$ or $\varepsilon_{x^G}(a)$, is the sum $\sum_{g \in x^G} a_g$. The augmentation of a is the sum of all of its partial augmentations. It suffices to consider units of augmentation one, which form a group denoted by $\mathrm{V}(\mathbb{Z}G)$. So let u be a torsion unit in $\mathrm{V}(\mathbb{Z}G)$.

The familiar result of Berman and Higman (from [1] and [19, p. 27]) asserts that if $\varepsilon_z(u) \neq 0$ for some z in the center of G, then $u = z$.

A practical criterion for u to be conjugate to an element of G by a unit of $\mathbb{Q}G$ is that all but one of the partial augmentations of every power of u must vanish (see [25, Theorem 2.5]).

The next two remarks will be used repeatedly. While the first one is elementary and well known, the second is of a more recent nature, taken from [17] where an obvious generalization of [18, Proposition 3.1] is given.

Remark 4. Let N be a normal subgroup of G and set $\bar{G} = G/N$. We write $\bar{u}$ for the image of u under the natural map $\mathbb{Z}G \to \mathbb{Z}\bar{G}$. Since any conjugacy class of G maps onto a conjugacy class of $\bar{G}$, for any $x \in G$ the partial augmentation $\varepsilon_{\bar{x}}(\bar{u})$ is the sum of the partial augmentations $\varepsilon_{g^G}(u)$ with $g \in G$ such that $\bar{g}$ is conjugate to $\bar{x}$ in $\bar{G}$.

Now suppose that N is a central subgroup of G, and that $\bar{u} = 1$. Then $u \in N$. Indeed, $1 = \varepsilon_1(\bar{u}) = \sum_{n \in N} \varepsilon_n(u)$, and so u has a central group element in its support and the Berman–Higman result applies.

Remark 5. If $\varepsilon_g(u) \neq 0$ for some $g \in G$, then the order of g divides the order of u. Indeed, it is well known that then the prime divisors of the order of g divide the order of u (see [25, Theorem 2.7], as well as [18, Lemma 2.8] for an alternative proof). Further, it was observed in [17, Proposition 2.2] that the orders of the p-parts of g cannot exceed those of u.

On one occasion, we shall use the following remark.

Remark 6. Let p be a rational prime, and let $x \in G$. If y^G is a conjugacy class of G containing an element whose pth power is in x^G, we write $(y^G)^p = x^G$. A simple but powerful equation which leads to so many group ring consequences is

$$\varepsilon_{x^G}(a^p) \equiv \sum_{(y^G)^p = x^G} \varepsilon_{y^G}(a) \mod p$$

for all $a \in \mathbb{Z}G$. (This formula (in prime characteristic) can be traced back to work of Brauer. It obviously derives from a significant feature of the pth power map (which may be found in [28, Lemma 2.3.1]). The underlying basic idea was attributed to Landau by Zassenhaus in [30]. A generalization is given by Cliff's formula [11] (restated in [22, Lemma 2]). Applications can be found, for instance, in [15], [21], [22], [28, Section 2].)

We shall use the fact that if there is only one conjugacy class y^G with $(y^G)^p = x^G$, and $\varepsilon_{y^G}(u) \not\equiv 0 \mod p$, then $\varepsilon_{x^G}(u^p) \neq 0$.

Finally, we outline the Luthar–Passi method. Let u be of order n (say), and let ζ be a primitive complex nth root of unity. For a character χ afforded by a complex representation D of G, write $\mu(\xi, u, \chi)$ for the multiplicity of an nth root of unity ξ as an eigenvalue of the matrix $D(u)$. By [23],

$$\mu(\xi, u, \chi) = \frac{1}{n} \sum_{d \mid n} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\xi^{-d}).$$

When trying to show that u is rationally conjugate to an element of G, one may assume—by induction on the order of u—that the values of the summands for $d \neq 1$ are 'known'. The summand for $d = 1$ can be written as

$$\frac{1}{n} \sum_{g^G} \varepsilon_g(u)\, \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(g)\xi^{-1}),$$

a linear combination of the $\varepsilon_g(u)$ with 'known' coefficients. Note that the $\mu(\xi, u, \chi)$ are non-negative integers, bounded above by $\chi(1)$. Thus, in some sense, there are linear inequalities in the partial augmentations of u which impose constraints on them. Trying to make use of these inequalities is now understood as being the Luthar–Passi method.

A modular version of this method (see [17, Section 4] for details) can be derived from the following observation in the same way as the original (complex) version is derived from the (obvious) fact that $\chi(u) = \sum_{g^G} \varepsilon_g(u)\chi(g)$. Suppose that p is a rational prime which does not divide the order of u (i.e., u is a p-regular torsion unit).

Then for every Brauer character φ of G (relative to p) we have (see [17, Theorem 3.2]):

$$\varphi(u) = \sum_{\substack{g^G:g \text{ is} \\ p\text{-regular}}} \varepsilon_g(u)\varphi(g).$$

Thereby, the domain of φ is naturally extended to the set of p-regular torsion units in $\mathbb{Z}G$.

3 A covering group of S_5

A presentation of a covering group of S_n is given by

$$\tilde{S}_n = \langle g_1, \ldots, g_{n-1}, z \mid g_i^2 = (g_j g_{j+1})^3 = (g_k g_l)^2 = z, z^2 = [z, g_i] = 1$$
$$\text{for } 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n-2, k \leqslant l-2 \leqslant n-3\rangle.$$

Recent results in the representation theory of the covering groups of symmetric groups can be found in Bessenrodt's survey article [2]. We merely remark that the complex spin characters of $\tilde{S}_n$, i.e., those characters which are not characters of S_n, were determined by Schur [29].

The group $\tilde{S}_5$ has catalogue number 89 in the Small Group Library in GAP [16] (the other covering group of S_5 has number 90). The spin characters of $\tilde{S}_5$ as produced by GAP are shown in Table 1 (dots indicate zeros).

	1a	5a	4a	2a	10a	6a	3a	8a	8b	4b	12a	12b
χ_5	4	−1	.	−4	1	2	−2	.	.	.	.	.
χ_6	4	−1	.	−4	1	−1	1	.	.	.	β	$-\beta$
χ_7	4	−1	.	−4	1	−1	1	.	.	.	$-\beta$	β
χ_{11}	6	1	.	−6	−1	.	.	α	$-\alpha$	.	.	.
χ_{12}	6	1	.	−6	−1	.	.	$-\alpha$	α	.	.	.

Irrational entries: $\alpha = -\zeta_8 + \zeta_8^3 = -\sqrt{2}$ where $\zeta_8 = \exp(2\pi i/8)$,
$\beta = \zeta_{12}^7 - \zeta_{12}^{11} = -\sqrt{3}$ where $\zeta_{12} = \exp(2\pi i/12)$.

Table 1. Spin characters of $\tilde{S}_5$

We turn to the proof that conjecture (ZC1) holds for $\tilde{S}_5$. Let z be the central involution in $\tilde{S}_5$. Then we have a natural homomorphism

$$\pi : \mathbb{Z}\tilde{S}_5 \to \mathbb{Z}\tilde{S}_5/\langle z\rangle = \mathbb{Z}S_5.$$

Let u be a non-trivial torsion unit in $\mathrm{V}(\mathbb{Z}\tilde{S}_5)$. We shall show that all but one of its partial augmentations vanish. Since (ZC1) is true for S_5, the order of $\pi(u)$ agrees with the order of an element of S_5, and it follows that the order of u agrees with the order of an element of $\tilde{S}_5$ (see Remark 4). By the Berman–Higman result we can assume that $\varepsilon_1(u) = 0$ and $\varepsilon_z(u) = 0$. Further, we can assume that the order of u is even, since otherwise rational conjugacy of u to an element of G follows from the validity of (ZC1) for S_5 and [13, Theorem 2.2]. Denote the partial augmentations of u by $\varepsilon_{1a}, \varepsilon_{5a}, \ldots, \varepsilon_{12b}$ (so that ε_{5a}, for example, denotes the partial augmentation of u with respect to the conjugacy class of elements of order 5). So $\varepsilon_{1a} = \varepsilon_{2a} = 0$. Since all but one of the partial augmentations of $\pi(u)$, the image of u in $\mathbb{Z}S_5$, vanish, and a partial augmentation of $\pi(u)$ is the sum of the partial augmentations of u taken for classes which fuse in S_5, we have

$$\begin{aligned} &\varepsilon_{4a}, \varepsilon_{4b}, \varepsilon_{8a} + \varepsilon_{8b}, \varepsilon_{3a} + \varepsilon_{6a}, \varepsilon_{5a} + \varepsilon_{10a}, \varepsilon_{12a} + \varepsilon_{12b} \in \{0, 1\}, \\ &|\varepsilon_{4a}| + |\varepsilon_{4b}| + |\varepsilon_{8a} + \varepsilon_{8b}| + |\varepsilon_{3a} + \varepsilon_{6a}| + |\varepsilon_{5a} + \varepsilon_{10a}| + |\varepsilon_{12a} + \varepsilon_{12b}| = 1. \end{aligned} \tag{1}$$

Suppose that u has order 2 *or* 4. Then the partial augmentations of u which are possibly non-zero are ε_{4a}, ε_{4b}, ε_{8a} and ε_{8b} (by Remark 5). Thus

$$\chi_{11}(u) = \alpha(\varepsilon_{8a} - \varepsilon_{8b}) = -\sqrt{2}(\varepsilon_{8a} - \varepsilon_{8b}).$$

Also, $\chi_{11}(u)$ is a sum of fourth roots of unity. Since $\sqrt{2} \notin \mathbb{Q}(i)$ it follows that $\varepsilon_{8a} - \varepsilon_{8b} = 0$. Using $\varepsilon_{8a} + \varepsilon_{8b} \in \{0, 1\}$ from (1) we obtain $\varepsilon_{8a} = \varepsilon_{8b} = 0$. Now $|\varepsilon_{4a}| + |\varepsilon_{4b}| = 1$ by (1), and so all but one of the partial augmentations of u vanish, with either $\varepsilon_{4a} = 1$ or $\varepsilon_{4b} = 1$. It follows that u is rationally conjugate to an element of G (necessarily of order 4).

Suppose that u has order 6 *or* 10. Then $u^3 = z$ or $u^5 = z$, respectively (by Remark 4), i.e., zu is of order 3 or 5. Thus zu is, as already noted, rationally conjugate to an element of G, and hence the same holds for u itself.

Suppose that u has order 12. Then the partial augmentations of u which are possibly non-zero are ε_{4a}, ε_{4b}, ε_{8a}, ε_{8b}, ε_{3a}, ε_{6a}, ε_{12a} and ε_{12b}. The unit $\pi(u)$ has order 6 (by Remark 5), so that $\varepsilon_{12a} + \varepsilon_{12b} = 1$ and $\varepsilon_{4a} = \varepsilon_{4b} = \varepsilon_{8a} + \varepsilon_{8b} = \varepsilon_{3a} + \varepsilon_{6a} = 0$ by (1). Now $\chi_{11}(u) = -\sqrt{2}(\varepsilon_{8a} - \varepsilon_{8b})$ but $\sqrt{2} \notin \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \zeta_3)$, so that $\varepsilon_{8a} = \varepsilon_{8b}$ and consequently $\varepsilon_{8a} = \varepsilon_{8b} = 0$. Further $\chi_5(u) = 2(\varepsilon_{6a} - \varepsilon_{3a}) = 4\varepsilon_{6a} = \varepsilon_{6a}\chi_5(1)$, and so if $\varepsilon_{6a} \neq 0$ then u is mapped under a representation of G affording χ_5 to the identity matrix or the negative of the identity matrix, leading to the contradiction $\chi_5(1) = \chi_5(u^6) = \chi_5(z) = -4$. Thus $\varepsilon_{3a} = \varepsilon_{6a} = 0$. So far, we have shown that ε_{12a} and ε_{12b} are the only possibly non-vanishing partial augmentations of u. We continue with a formal application of the Luthar–Passi method. Let ξ be a 12th root of unity. Then

$$\mu(\xi, u, \chi_6) = \tfrac{1}{12}(\mathrm{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\xi^{-1}) + 6\mu(\xi^2, u^2, \chi_6) + \mathrm{Tr}_{\mathbb{Q}(\zeta_{12}^3)/\mathbb{Q}}(\chi_6(u^3)\xi^{-3})).$$

Since u^3 is rationally conjugate to an element of order 4 in G, we have $\chi_6(u^3) = 0$. Since $\chi_6(u) = \beta(\varepsilon_{12a} - \varepsilon_{12b}) = (\zeta_{12}^7 - \zeta_{12}^{11})(\varepsilon_{12a} - \varepsilon_{12b})$, we have

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\zeta_{12}^{-7}) = 6(\varepsilon_{12\mathrm{a}} - \varepsilon_{12\mathrm{b}}),$$
$$\mathrm{Tr}_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(\chi_6(u)\zeta_{12}^{-11}) = -6(\varepsilon_{12\mathrm{a}} - \varepsilon_{12\mathrm{b}}).$$

Next, $\chi_6(u^4) = 1$ since u^4 is rationally conjugate to an element of order 3 in G, and $\chi_6(u^6) = \chi_6(z) = -4$, from which it is easy to see that $\mu(\xi^2, u^2, \chi_6) = 1$ for a primitive 12th root of unity ξ. Thus

$$\mu(\zeta_{12}^{7}, u, \chi_6) = \tfrac{1}{2}((\varepsilon_{12\mathrm{a}} - \varepsilon_{12\mathrm{b}}) + 1) \geqslant 0,$$
$$\mu(\zeta_{12}^{11}, u, \chi_6) = \tfrac{1}{2}(-(\varepsilon_{12\mathrm{a}} - \varepsilon_{12\mathrm{b}}) + 1) \geqslant 0,$$

from which we obtain $|\varepsilon_{12\mathrm{a}} - \varepsilon_{12\mathrm{b}}| \leqslant 1$. Together with $\varepsilon_{12\mathrm{a}} + \varepsilon_{12\mathrm{b}} = 1$ this implies that $\varepsilon_{12\mathrm{a}} = 0$ or $\varepsilon_{12\mathrm{b}} = 0$. We have shown that all but one of the partial augmentations of u vanish.

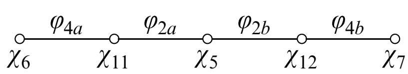
Suppose that u has order 8. Then the partial augmentations of u which are possibly non-zero are $\varepsilon_{4\mathrm{a}}$, $\varepsilon_{4\mathrm{b}}$, $\varepsilon_{8\mathrm{a}}$ and $\varepsilon_{8\mathrm{b}}$. Since $\pi(u)$ has order 4, its partial augmentations with respect to classes of elements of order 2 vanish and consequently $\varepsilon_{4\mathrm{a}} = \varepsilon_{4\mathrm{b}} = 0$. We have $\chi_{11}(u) = \alpha(\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}}) = (-\zeta_8 + \zeta_8^3)(\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}})$, and this time the Luthar–Passi method gives

$$\mu(\zeta_8^3, u, \chi_{11}) = \tfrac{1}{2}((\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}}) + 3) \geqslant 0,$$
$$\mu(\zeta_8, u, \chi_{11}) = \tfrac{1}{2}(-(\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}}) + 3) \geqslant 0,$$

from which we obtain $|\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}}| \leqslant 3$. Together with $\varepsilon_{8\mathrm{a}} + \varepsilon_{8\mathrm{b}} = 1$ this implies that $(\varepsilon_{8\mathrm{a}}, \varepsilon_{8\mathrm{b}}) \in \{(1,0),(0,1),(-1,2),(2,-1)\}$. At this point we are stuck when limiting attention to complex characters only.

However, we may resort to p-modular characters. Examining Brauer characters of small degree seems most promising. Thus it is natural to choose $p = 5$ since $\tilde{S}_5$ is a subgroup of $\mathrm{SL}(2,25)$. This can be seen as follows. The group $\mathrm{PSL}(2,25)$ contains $\mathrm{PGL}(2,5)$ as a subgroup (see [20, Kapitel II, 8.27 Hauptsatz]) which is isomorphic to S_5, and its pre-image in $\mathrm{SL}(2,25)$ is isomorphic to $\tilde{S}_5$ (since the Sylow 2-subgroups of $\mathrm{SL}(2,25)$ are generalized quaternion groups). Let φ be the Brauer character afforded by a faithful representation $D : \tilde{S}_5 \to \mathrm{SL}(2,25)$. The Brauer lift can be chosen such that $\varphi(x) = \alpha = -\zeta_8 + \zeta_8^3$ for an element x in the conjugacy class 8a of G (since $D(x)$ has determinant 1). Class 8b is represented by x^5, and so we obtain $\varphi(u) = \varepsilon_{8\mathrm{a}}\varphi(x) + \varepsilon_{8\mathrm{b}}\varphi(x^5) = (-\zeta_8 + \zeta_8^3)(\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}})$. Since $\varphi(u)$ is the sum of two 8th roots of unity, it follows that $|\varepsilon_{8\mathrm{a}} - \varepsilon_{8\mathrm{b}}| \leqslant 1$ and consequently $\varepsilon_{8\mathrm{a}} = 0$ or $\varepsilon_{8\mathrm{b}} = 0$. The proof is complete.

Remark 7. The spin characters of $\tilde{S}_5$ form a single 5-block of defect 1, with Brauer tree



(cf. [26, Theorem 4]). Above, we have chosen $\varphi = \varphi_{2a}$ (representations affording φ_{2a} and φ_{2b} are conjugate under the Frobenius homomorphism). Since we had to examine the character value $\chi_{11}(u)$, it was certainly a good choice to further consider a modular constituent of χ_{11}.

One may dare to ask whether the theory of cyclic blocks can provide additional insight into Zassenhaus' conjecture (ZC1). We have no opinion on this, but we digress into a brief discussion of another conjecture of Zassenhaus, (ZCAut), where this is actually the case. (ZCAut) asserts that the group $\mathrm{Aut}_{\mathrm{n}}(\mathbb{Z}G)$ of augmentation preserving automorphisms of $\mathbb{Z}G$ is generated by automorphisms of G and central automorphisms; though not valid in general, it may very well be valid for simple groups G. The main point here is that $\mathrm{Aut}_{\mathrm{n}}(\mathbb{Z}G)$ *acts* on various structures associated with $\mathbb{Z}G$. For one thing, ring automorphisms give rise to autoequivalences of module categories. In [5], rigidity of autoequivalences of the module category of a Brauer tree algebra was studied, with first applications to (ZCAut) for simple groups. For another thing, $\mathrm{Aut}_{\mathrm{n}}(\mathbb{Z}G)$ acts on the class sums of $\mathbb{Z}G$. This immediately shows that the action on characters—both ordinary and modular—is compatible with taking tensor products; see [3], and [4] for a thorough examination of the consequences resulting for (ZCAut).

4 The general linear group GL(2, 5)

We set $G = \mathrm{GL}(2,5)$. Let z be a generator of $\mathrm{Z}(G)$, which is a cyclic group of order 4. The quotient $G/\langle z\rangle$ is isomorphic to S_5, for which (ZC1) is known to hold. Let π denote the natural map $\mathbb{Z}G \to \mathbb{Z}G/\langle z\rangle$.

Let u be a non-trivial torsion unit in $\mathrm{V}(\mathbb{Z}G)$. We will show that all but one of its partial augmentations vanish. For this, we use part of the character table of G, shown in Table 2 in the form obtained by requiring CharacterTable("GL25") in GAP [16], together with the natural 2-dimensional representation of G in characteristic 5. In Table 2, the row labelled 'in S_5' indicates to which classes in the quotient S_5

class	1a	4c	2b	4d	4e	4f	4g	24a	12a	8a	6a	24b	3a	8b	24c	12b	24d
in S_5	1a	4a	2a	4a	4a	2a	4a	6a	3a	2b	3a	6a	3a	2b	6a	3a	6a
χ_2	1	i	-1	$-i$	$-i$	1	i	i	-1	$-i$	1	$-i$	1	i	i	-1	$-i$
χ_6	5	i	-1	$-i$	$-i$	1	i	$-i$	1	i	-1	i	-1	$-i$	$-i$	1	i
χ_{16}	4	.	.	.	.	.	.	$-i$	-1	$-2i$	1	i	1	$2i$	$-i$	-1	i
χ_9	6	α	.	$\bar{\alpha}$	$-\bar{\alpha}$	.	$-\alpha$	.	.	.	.	.	.	.	.	.	.
χ_{14}	6	$-\alpha$	.	$-\bar{\alpha}$	$\bar{\alpha}$	.	α	.	.	.	.	.	.	.	.	.	.
χ_{15}	4	.	.	.	.	.	.	β	$-i$	.	-1	$-\bar{\beta}$	1	.	$-\beta$	i	$\bar{\beta}$
χ_{21}	4	.	.	.	.	.	.	.	$2i$	.	2	.	-2	.	.	$-2i$	.
χ_{22}	4	.	.	.	.	.	.	$-\beta$	$-i$	.	-1	$\bar{\beta}$	1	.	β	i	$-\bar{\beta}$

Irrational entries: $\alpha = 1 + i$, $\beta = -\zeta + \zeta^{17}$ where $\zeta = \exp(2\pi i/24)$.

Table 2. Part of the character table of $\mathrm{GL}(2,5)$

the listed classes of G are mapped. The classes omitted are the classes 2a, 4a, 4b of central 2-elements, and the classes 5a, 20a, 10a, 20b of elements of order divisible by 5.

The characters χ_6 and χ_{16} have kernel $\langle z^2 \rangle$. The faithful characters χ_9, χ_{14}, χ_{15}, χ_{21} and χ_{22} of G form a 5-block of G, with Brauer tree

$$\begin{tikzcd} \underset{\chi_{15}}{\circ} \arrow[r, dash, "\varphi_{4a}"] & \underset{\chi_{9}}{\circ} \arrow[r, dash, "\varphi_{2a}"] & \underset{\chi_{21}}{\circ} \arrow[r, dash, "\varphi_{2b}"] & \underset{\chi_{14}}{\circ} \arrow[r, dash, "\varphi_{4b}"] & \underset{\chi_{22}}{\circ} \end{tikzcd}$$

(cf. the theory of blocks of cyclic defect). Set $\varphi = \varphi_{4a} = (\chi_{15} - \chi_9)|_{G_{5'}}$ (restriction to 5-regular elements). Then φ is an irreducible 5-modular Brauer character of G of degree 2 afforded by a natural representation $G \to \mathrm{GL}(2,5)$.

We remark that the remaining irreducible faithful characters of G form a 5-block of G which is algebraically conjugate to the one that we consider.

We write $\varepsilon_{4\mathrm{c}}, \varepsilon_{2\mathrm{b}}, \ldots, \varepsilon_{24\mathrm{d}}$ for the partial augmentations of u at the classes listed in Table 2. We assume that u is not a central unit, so that its partial augmentations at central group elements are zero. It follows from Remark 4, and the validity of (ZC1) for S_5, that the order of u agrees with the order of some group element of G.

Suppose that the order of u is divisible by 5. Then $\pi(u)$ has order 5, by Remark 4, and u is the product of a unit of order 5 and a central group element of G. Since there is only one class of elements of order 5 in G, the 5-part of u is rationally conjugate to an element of G (by Remark 5), and thus the same is valid for u.

Suppose that u has order 2. The group G has only one class of non-central elements of order 2, and so Remark 5 applies.

Suppose that u has order 4. Then $\varepsilon_g(u) = 0$ for a group element g which is not a non-central element of order 2 or 4 (by Remark 5). Evaluating the Brauer character φ at u gives

$$\varphi(u) = (\varepsilon_{4\mathrm{c}} - \varepsilon_{4\mathrm{g}})(1+i) + (\varepsilon_{4\mathrm{d}} - \varepsilon_{4\mathrm{e}})(1-i). \tag{2}$$

First, suppose that $\pi(u)$ has order 2. Then $u^2 = z^2$ (by Remark 4), and so $\varphi(u^2) = -2$ and $\varphi(u)$ is the sum of two primitive fourth roots of unity. These roots of unity are distinct since u is non-central in $\mathbb{Z}G$. Thus $\varphi(u) = i + (-i) = 0$ and (2) gives $\varepsilon_{4\mathrm{c}} = \varepsilon_{4\mathrm{g}}$ and $\varepsilon_{4\mathrm{d}} = \varepsilon_{4\mathrm{e}}$. Since (ZC1) holds for S_5 we have

$$\varepsilon_{4\mathrm{c}} + \varepsilon_{4\mathrm{g}} + \varepsilon_{4\mathrm{d}} + \varepsilon_{4\mathrm{e}} = 0, \quad \varepsilon_{2\mathrm{b}} + \varepsilon_{4\mathrm{f}} = 1.$$

From this we further obtain $\varepsilon_{4\mathrm{d}} = -\varepsilon_{4\mathrm{c}}$ and $\chi_2(u) = 1 - 2\varepsilon_{2\mathrm{b}} + 4\varepsilon_{4\mathrm{c}} i$. Since $|\chi_2(u)| = 1$ it follows that $\varepsilon_{4\mathrm{c}} = 0$ and $\varepsilon_{2\mathrm{b}} \in \{0,1\}$. Thus all but one of the partial augmentations of u vanish.

Secondly, suppose that $\pi(u)$ has order 4. Then $\varphi(u^2) \neq -2$. Since $\varphi(u)$ is the sum of two distinct fourth roots of unity we have $|\varphi(u)| < 2$. Thus $\varphi(u) \in \{\pm(1+i), \pm(1-i)\}$ by (2). Since (ZC1) holds for S_5 we have

$$\varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 1, \quad \varepsilon_{2b} + \varepsilon_{4f} = 0.$$

From this and (2) we further obtain that for some $a \in \mathbb{Z}$ and $\delta_i \in \{0, 1\}$, with exactly one δ_i non-zero, $\varepsilon_{4c} = a + \delta_1$, $\varepsilon_{4g} = a + \delta_2$, $\varepsilon_{4c} = a - \delta_3$ and $\varepsilon_{4g} = a - \delta_4$. Thus $\chi_2(u) = (\delta_1 + \delta_2 - \delta_3 - \delta_4)i - 2\varepsilon_{2b} + 4ai$, from which $\varepsilon_{2b} = 0$ and $a = 0$ follows. Thus all but one of the partial augmentations of u vanish.

Suppose that u has order 8. Then $\varepsilon_{8a} \neq 0$ or $\varepsilon_{8b} \neq 0$ by [12, Corollary 4.1] (an observation sometimes attributed to Zassenhaus; cf. [30, Lemma 3]).

Suppose that $\varepsilon_{8b} = -\varepsilon_{8a}$. Then $\chi_{16}(u) = -4\varepsilon_{8b}i$ (remember Remark 5). Since χ_{16} has degree 4 it follows that $|\varepsilon_{8a}| = |\varepsilon_{8b}| = 1$. The class 8a is the only class consisting of elements whose square is in 4a, the class consisting of one of the central elements of order 4. Also 8b is the only class consisting of elements whose square is in 4b. Thus $\varepsilon_{4a}(u^2) \neq 0$ and $\varepsilon_{4b}(u^2) \neq 0$ by Remark 6. But we already know that u^2 is rationally conjugate to a group element, and so we have reached a contradiction.

Hence $\varepsilon_{8a} + \varepsilon_{8b} \neq 0$, and since ε_{8a} and ε_{8b} are the classes of G which map onto class ε_{2b} in S_5, in fact $\varepsilon_{8a} + \varepsilon_{8b} = 1$. Now $\chi_{16}(u) = 2(1 - 2\varepsilon_{8a})i$, and $|\chi_{16}(u)| \leqslant 4$ implies that $\varepsilon_{8a} \in \{0, 1\}$, so that one of ε_{8a} and ε_{8b} vanishes.

Next, we show that $\chi_9(u) = 0$. Since S_5 has no elements of order 8 we have $u^4 = z^2$ by Remark 4. From $\chi_9(u^4) = \chi_9(z^2) = -\chi_9(1)$ we conclude that $\chi_9(u) \in \zeta_8\mathbb{Z}[i]$ for a primitive 8th root of unity ζ_8, and inspection of the character table shows that $\chi_9(u) \in \mathbb{Z}[i]$. But definitely $\zeta_8 \notin \mathbb{Z}[i]$, and so $\chi_9(u) = 0$.

Since

$$\chi_9(u) = (\varepsilon_{4c} - \varepsilon_{4g})(1 + i) + (\varepsilon_{4d} - \varepsilon_{4e})(1 - i) \tag{3}$$

and $\varepsilon_{4c} + \varepsilon_{4g} + \varepsilon_{4d} + \varepsilon_{4e} = 0$, it follows that $\varepsilon_{4c} = \varepsilon_{4g} = -\varepsilon_{4d} = -\varepsilon_{4e}$. Also we have $\varepsilon_{2b} + \varepsilon_{4f} = 0$. So $\chi_2(u) = (\pm 1 + 4\varepsilon_{4c})i - 2\varepsilon_{2b}$, which implies $\varepsilon_{2b} = 0$ and $\varepsilon_{4c} = 0$, and we are done.

Suppose that u has order 3. The group G has only one class of elements of order 3, and so Remark 5 applies.

Suppose that u has order 6. The only partial augmentations of u which are possibly non-zero are ε_{2b}, ε_{3a} and ε_{6a}. Since the class ε_{6a} maps in S_5 to the class of elements of order 3 it follows that $\pi(u)$ is rationally conjugate to a group element of order 3 in S_5. Hence u is the product of z^2 and a unit of order 3 (by Remark 4), and u is rationally conjugate to a group element.

Suppose that u has order 12. Then the only partial augmentations of u which are possibly non-zero are at classes of elements of order 2, 4, 3, 6 and 12. The classes of elements of order 3, 6 and 12 map in S_5 to the class of elements of order 3. Thus $\pi(u)$ is of order 3 and u is the product of z and a unit of order 3, so that u is rationally conjugate to a group element.

Suppose that u has order 24. Then $\pi(u)$ is rationally conjugate to an element of order 6 in S_5, and so

$$\begin{aligned} \varepsilon_{24a} + \varepsilon_{24b} + \varepsilon_{24c} + \varepsilon_{24d} &= 1, \\ \varepsilon_{12a} + \varepsilon_{6a} + \varepsilon_{3a} + \varepsilon_{12b} &= 0, \\ \varepsilon_{8a} + \varepsilon_{8b} &= 0, \\ \varepsilon_{4c} + \varepsilon_{4d} + \varepsilon_{4e} + \varepsilon_{4g} &= 0, \\ \varepsilon_{2b} + \varepsilon_{4f} &= 0. \end{aligned} \tag{4}$$

From $\chi_9(u^{12}) = -\chi_9(1)$ we conclude that $\chi_9(u) \in \zeta_8\mathbb{Z}[i,\zeta_3]$ for a primitive 8th root of unity ζ_8 and a primitive cube root of unity ζ_3. Inspection of the character table shows that $\chi_9(u) \in \mathbb{Z}[i]$, and so $\chi_9(u) = 0$ as $\zeta_8 \notin \mathbb{Z}[i,\zeta_3]$. In the same way we argue that $\chi_{21}(u) = 0$. Thus evaluation (3) of $\chi_9(u)$ is zero, and with (4) it follows that $\varepsilon_{4c} = \varepsilon_{4g} = -\varepsilon_{4d} = -\varepsilon_{4e}$. Now $(\chi_2 + \chi_6)(u) = -4\varepsilon_{2a} + 8\varepsilon_{4c}i$. Since $\chi_2 + \chi_6$ has degree 6 we conclude that $\varepsilon_{4c} = 0$.

We have $0 = \chi_{21}(u) = 2(\varepsilon_{6a} - \varepsilon_{3a}) + 2i(\varepsilon_{12a} - \varepsilon_{12b})$, and therefore $\varepsilon_{6a} = \varepsilon_{3a}$ and $\varepsilon_{12a} = \varepsilon_{12b}$. Further $\varepsilon_{6a} = -\varepsilon_{12a}$ from (4). Thus $\chi_{16}(u) \in -4\varepsilon_{12a} + i\mathbb{Z}$. From $\chi_{16}(u^6) = -\chi_{16}(1)$ we obtain $\chi_{16}(u) \in i\mathbb{Z}[\zeta_3]$. It follows that $-4\varepsilon_{12a}i \in \mathbb{Z}[\zeta_3]$ and $\varepsilon_{12a} = 0$. Now $\chi_2(u) \in -2\varepsilon_{2b} + i\mathbb{Z}$ and so $\varepsilon_{2b} = 0$. Also $(\chi_2 + \chi_{16})(u) = -2\varepsilon_{2b} - 6\varepsilon_{8a}i$ and since $\chi_2 + \chi_{16}$ has degree 5 we have $\varepsilon_{8a} = 0$.

Set $a = \varepsilon_{24a} + \varepsilon_{24c}$ and $b = \varepsilon_{24b} + \varepsilon_{24d}$. Then $\chi_2(u) = (a-b)i$ and thus $a - b = \pm 1$. Together with $a + b = 1$ this implies that $(a,b) = (1,0)$ or $(a,b) = (0,1)$. In the first case, $\chi_{15}(u) = (2\varepsilon_{24a} - 1)\beta - 2\varepsilon_{24b}\bar{\beta}$, and in the second $\chi_{15}(u) = 2\varepsilon_{24a}\beta + (1 - 2\varepsilon_{24b})\bar{\beta}$. Using the sum formula for sin with $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ it is easiest to calculate $\beta = -\sqrt{\frac{3}{2}}(1+i)$. In particular, $|\beta| = \sqrt{3}$. Since $\chi_{15}(u)$ is the sum of four roots of unity, it is readily seen that if $\chi_{15}(u)$ assumes the first value, then $\varepsilon_{24b} = 0$ and $\varepsilon_{24a} \in \{0,1\}$, and if $\chi_{15}(u)$ assumes the second value, then $\varepsilon_{24a} = 0$ and $\varepsilon_{24b} \in \{0,1\}$. It follows that exactly one of ε_{24a}, ε_{24c}, ε_{24b} and ε_{24d} is non-zero, and we are done.

The observant reader might have noticed that the last argument can be replaced by a simpler 'modular' argument: we already know that $\chi_{15}(u)$ agrees with the value of χ_{15} at a class of elements of order 24 since $\chi_9(u) = 0$ and $\varphi = \chi_9 - \chi_{15}$ on 5-regular elements.

Acknowledgement. We are grateful to the referee for pointing out to us a gap in one of our calculations in the original version of this paper.

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Received 14 September, 2006; revised 8 March, 2007

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