

TORSION UNITS IN INTEGRAL GROUP RING OF THE MATHIEU SIMPLE GROUP M_{22}

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Abstract

We investigate the possible character values of torsion units of the normalized unit group of the integral group ring of the Mathieu sporadic group M_{22} . We confirm the Kimmerle conjecture on prime graphs for this group and specify the partial augmentations for possible counterexamples to the stronger Zassenhaus conjecture.

1. *Introduction, conjectures and main results*

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G . A long-standing conjecture of H. Zassenhaus (**ZC**) says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [25]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [20] for the case of A_5 and then applied in [21] for the case of S_5 . Later M. Hertweck in [15] extended the method and applied it to $\text{PSL}(2, p^n)$. The same method has also proved to be useful for some groups containing non-trivial normal subgroups. For some recent results we refer to [5, 7, 14, 16, 15, 17]. Some related properties and weakened variations of the Zassenhaus conjecture can be found in [1, 3, 19].

To define the conjectures we will investigate, and describe the methods we will use, we introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G . The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by $\#(G)$ and an edge from p to q (for $p \neq q$) if there is an element of order pq in G . In [19], W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

It is easy to see that (**ZC**) implies **KC** since it implies that the set of orders of torsion units of $V(\mathbb{Z}G)$ is the same the set of orders of elements of G .

In ([19], §4) it is shown that (**KC**) holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p -version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 8, 10], (**KC**) was confirmed for the sporadic simple groups M_{11} , M_{12} , M_{23} and some Janko simple groups.

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Here we continue these investigations for the Mathieu simple group M_{22} . Although we cannot prove the rational conjugacy of torsion units of $V(\mathbb{Z}M_{22})$ with elements of M_{22} , our main result gives a lot of information on the orders and partial augmentations of these units. In particular, we confirm Kimmerle's conjecture for this group.

Let $G = M_{22}$. It is well known (see [13, 24]) that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $\exp(G) = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The group G has 12 irreducible characters of the following degrees: 1, 21, 45, 45, 55, 99, 154, 210, 231, 280, 280 and 385. Let

$$\mathcal{C} = \{C_1, C_{2a}, C_{3a}, C_{4a}, C_{4b}, C_{5a}, C_{6a}, C_{7a}, C_{7b}, C_{8a}, C_{11a}, C_{11b}\}$$

be the collection of all conjugacy classes of M_{22} , where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order $k > 1$. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt} . From the Berman–Higman Theorem (see [2] and [23], Ch. 5, p. 102) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \quad (1)$$

Hence, for any character χ of G , we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt} .

Our main result is the following theorem.

THEOREM 1. *Let G denote the Mathieu simple group M_{22} . Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the tuple*

$$(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{7a}, \nu_{7b}, \nu_{8a}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{11}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. The following properties hold.

- (i) *There are no elements of order 10, 14, 15, 21, 22, 33, 35, 55 or 77 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{12, 24\}$, then $|u|$ is the order of some element $g \in G$.*
- (ii) *If $|u| \in \{2, 3, 5\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 4$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{4a} and ν_{4b} , and the triple $(\nu_{2a}, \nu_{4a}, \nu_{4b})$ is one of*

$$\{(-2, -1, 4), (2, -1, 0), (0, -1, 2), (0, 6, -5), (-2, 6, -3), (0, 5, -4),$$

$$(-2, 5, -2), (-2, 4, -1), (2, 4, -5), (0, 4, -3), (-2, 3, 0), (2, 3, -4), (0, 3, -2),$$

$$(2, 0, -1), (-2, 0, 3), (0, 0, 1), (0, -6, 7), (2, -6, 5), (0, 2, -1), (2, 2, -3),$$

$$(-2, 2, 1), (0, -5, 6), (2, -5, 4), (0, -4, 5), (2, -4, 3), (-2, -3, 6), (0, -3, 4),$$

$$(2, -3, 2), (-2, -2, 5), (0, -2, 3), (2, -2, 1), (0, 1, 0), (-2, 1, 2), (2, 1, -2)\}.$$
- (iv) *If $|u| = 6$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{3a} and ν_{6a} , and the triple $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ is one of*

$$\{(-4, 6, -1), (-2, 6, -3), (4, -9, 6), (-4, 9, -4), (-2, 3, 0),$$

$$(-4, 3, 2), (0, 3, -2), (2, 0, -1), (-2, 0, 3), (0, 0, 1),$$

$$(2, -6, 5), (4, -6, 3), (0, -3, 4), (4, -3, 0), (2, -3, 2)\}.$$

- (v) *If $|u| = 7$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{7a} and ν_{7b}*

and the pair (ν_{7a}, ν_{7b}) is one of

$$\{ (0, 1), (2, -1), (1, 0), (-1, 2) \}.$$

(vi) If $|u| = 11$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{11a} and ν_{11b} and the pair (ν_{11a}, ν_{11b}) is one of

$$\begin{aligned} & \{ (5, -4), (0, 1), (-2, 3), (2, -1), (-3, 4), \\ & (-4, 5), (1, 0), (3, -2), (-1, 2), (4, -3) \}. \end{aligned}$$

Note that using our implementation of the Luthar–Passi method (including Hertweck’s extension), which we intend to make available in the GAP package LAGUNA [9], we are able to compute the set of 76 tuples containing (likely as a proper subset) possible tuples of partial augmentations for units of order 8, listed in Appendix 1. For the case of order 12 in Appendix 2 we listed 1166 tuples which can not be eliminated using the Luthar–Passi method.

As an immediate consequence of part (i) of the Theorem we obtain the corollary.

COROLLARY 1. *If $G \cong M_{22}$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. Preliminaries

The following result allows a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

PROPOSITION 1 (see [20] and Theorem 2.5 in [22]). *Let $u \in V(\mathbb{Z}G)$ be of order k . Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

The next results now serve to restrict the possible values of the partial augmentations of torsion units.

PROPOSITION 2 (see [14], Proposition 3.1; [15], Proposition 2.2). *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p -part, for some prime p , has order strictly greater than the order of the p -part of u , then $\varepsilon_x(u) = 0$.*

The next result is explained in detail in [20] and [4, 15].

PROPOSITION 3 (see [15, 20]). *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k th root of unity and χ is either a classical character or a p -Brauer character of G , then for every integer l the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} \text{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}} \{ \chi(u^d) z^{-dl} \} \tag{2}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

PROPOSITION 4 (see [11]). *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G .*

In case of units of prime power order, the following Proposition may also be useful to eliminate some tuples of partial augmentations.

PROPOSITION 5 (see [11]). *Let p be a prime, and let u be a torsion unit of $V(\mathbb{Z}G)$ of order p^n . Then for $m \neq n$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p .*

3. Proof of the Theorem

Throughout this section we denote M_{22} by G . The ordinary and p -Brauer character tables of G , which will be denoted by $\mathfrak{BC}\mathfrak{T}(p)$ where $p \in \{2, 3, 5, 7, 11\}$, can be found using the computational algebra system GAP [13], which derives its data from [12, 18]. For the characters and conjugacy classes we will use throughout the paper the same notation, including indexation, as used in the GAP Character Table Library.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8 and 11, we first investigate units of all of these orders except 8. After this, since by Proposition 4, the order of each torsion unit divides the exponent of G , it remains to consider units of orders 10, 12, 14, 15, 21, 22, 33, 35, 55 and 77. We prove that $V(\mathbb{Z}G)$ contains no units of any of these orders, except possibly for order 12 and 24.

Now we consider each case separately.

- Let u be a unit of order 2, 3 or 5. Using Proposition 2 we immediately obtain that all partial augmentations except one are zero. Thus by Proposition 1 part (ii) of Theorem 1 is proved.
- Let u be a unit of order 4. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{4a} + \nu_{4b} = 1$. Now using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{4}(10\nu_{2a} + 2\nu_{4a} + 2\nu_{4b} + 26) \geq 0; \\ \mu_2(u, \chi_2, *) &= \frac{1}{4}(-10\nu_{2a} - 2\nu_{4a} - 2\nu_{4b} + 26) \geq 0; \\ \mu_0(u, \chi_5, *) &= \frac{1}{4}(14\nu_{2a} + 6\nu_{4a} - 2\nu_{4b} + 62) \geq 0; \\ \mu_2(u, \chi_5, *) &= \frac{1}{4}(-14\nu_{2a} - 6\nu_{4a} + 2\nu_{4b} + 62) \geq 0. \end{aligned}$$

Put $t_1 = 5\nu_{2a} + \nu_{4a} + \nu_{4b}$ and $t_2 = 7\nu_{2a} + 3\nu_{4a} - \nu_{4b}$, then $t_1 \in \{2r + 1 \mid -7 \leq r \leq 6\}$ and $t_2 \in \{2s + 1 \mid -16 \leq s \leq 15\}$. Thus, we obtain the system of linear equations $\nu_{2a} + \nu_{4a} + \nu_{4b} = 1$, $5\nu_{2a} + \nu_{4a} + \nu_{4b} = t_1$, $7\nu_{2a} + 3\nu_{4a} - \nu_{4b} = t_2$. Solving such systems for all possible combinations of values of t_1 and t_2 , and considering additional inequalities

$$\begin{aligned} \mu_0(u, \chi_5, 3) &= \frac{1}{4}(2\nu_{2a} - 6\nu_{4a} + 2\nu_{4b} + 50) \geq 0; \\ \mu_2(u, \chi_5, 3) &= \frac{1}{4}(-2\nu_{2a} + 6\nu_{4a} - 2\nu_{4b} + 50) \geq 0, \end{aligned}$$

and also restrictions given by Proposition 5, we get only the 34 integer solutions $(\nu_{2a}, \nu_{4a}, \nu_{4b})$ listed in part (iii) of Theorem 1.

- Let u be a unit of order 6. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{3a} + \nu_{6a} = 1$.

By Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_1(u, \chi_2, *) &= \frac{1}{6}(5\nu_{2a} + 3\nu_{3a} - \nu_{6a} + 13) \geq 0; \\ \mu_3(u, \chi_2, *) &= \frac{1}{6}(-10\nu_{2a} - 6\nu_{3a} + 2\nu_{6a} + 22) \geq 0; \\ \mu_0(u, \chi_4, 7) &= \frac{1}{6}(12\nu_{2a} + 60) \geq 0; \quad \mu_3(u, \chi_4, 7) = \frac{1}{6}(-12\nu_{2a} + 48) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{6}(-3\nu_{2a} + 48) \geq 0.\end{aligned}$$

Using calculations similar to the previous case, we get only the 15 integer solutions $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ listed in part (iv) of Theorem 1.

• Let u be a unit of order 7. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{7b} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_1(u, \chi_3, *) &= \frac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 45) \geq 0; \quad \mu_3(u, \chi_3, *) = \frac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 45) \geq 0; \\ \mu_1(u, \chi_2, 2) &= \frac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 10) \geq 0; \quad \mu_3(u, \chi_2, 2) = \frac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 10) \geq 0.\end{aligned}$$

Using that $\nu_{7a} + \nu_{7b} = 1$, we get that $-1 \leq \nu_{7a} \leq 2$, and after this it is easy to check that we have only the four integer solutions (ν_{7a}, ν_{7b}) listed in part (v) of Theorem 1.

• Let u be a unit of order 11. By (1) and Proposition 2 we get $\nu_{11a} + \nu_{11b} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_1(u, \chi_{10}, *) &= \frac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 280) \geq 0; \\ \mu_2(u, \chi_{10}, *) &= \frac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 280) \geq 0; \\ \mu_1(u, \chi_5, 2) &= \frac{1}{11}(7\nu_{11a} - 4\nu_{11b} + 70) \geq 0; \\ \mu_2(u, \chi_5, 2) &= \frac{1}{11}(-4\nu_{11a} + 7\nu_{11b} + 70) \geq 0; \\ \mu_1(u, \chi_5, 3) &= \frac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 49) \geq 0; \\ \mu_2(u, \chi_5, 3) &= \frac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 49) \geq 0.\end{aligned}$$

Using calculations similar to the previous case, we get only the ten integer solutions for (ν_{11a}, ν_{11b}) listed in part (vi) of Theorem 1.

It remains to prove part (i) of Theorem 1, considering units of $V(\mathbb{Z}G)$ of orders 10, 14, 15, 21, 22, 33, 35, 55 and 77.

• Let u be a unit of order 10. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{5a} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{10}(20\nu_{2a} + 4\nu_{5a} + 30) \geq 0; \\ \mu_5(u, \chi_2, *) &= \frac{1}{10}(-20\nu_{2a} - 4\nu_{5a} + 20) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{10}(-3\nu_{2a} + 48) \geq 0,\end{aligned}$$

that has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 14. Then by (1) and Proposition 2 we get $\nu_{2a} + \nu_{7a} + \nu_{7b} = 1$. We need to consider four cases defined by part (v) of Theorem 1, but in all of them using Proposition 3 we obtain the same system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{14}(30\nu_{2a} + 26) \geq 0; \quad \mu_7(u, \chi_2, *) = \frac{1}{14}(-30\nu_{2a} + 16) \geq 0,$$

which has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 15. By (1) and Proposition 2 we get $\nu_{3a} + \nu_{5a} = 1$. Using

Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{15}(24\nu_{3a} + 8\nu_{5a} + 31) \geq 0; \\ \mu_5(u, \chi_2, *) &= \frac{1}{15}(-12\nu_{3a} - 4\nu_{5a} + 22) \geq 0.\end{aligned}$$

From this follows that $3\nu_{3a} + \nu_{5a} = -2$, and all conditions together leave us no integer solutions.

• Let u be a unit of order 21. Then by (1) and Proposition 2 we have

$$\nu_{3a} + \nu_{7a} + \nu_{7b} = 1.$$

We need to consider four cases determined by part (v) of Theorem 1. Using Proposition 3 we obtain the following systems of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{21}(36\nu_{3a} + 27) \geq 0; & \mu_7(u, \chi_2, *) &= \frac{1}{21}(-18\nu_{3a} + 18) \geq 0; \\ \mu_1(u, \chi_3, *) &= \frac{1}{21}(3\nu_{7a} - 4\nu_{7b} + \alpha) \geq 0; & \mu_9(u, \chi_3, *) &= \frac{1}{21}(-6\nu_{7a} + 8\nu_{7b} + \alpha) \geq 0; \\ \mu_3(u, \chi_3, *) &= \frac{1}{21}(8\nu_{7a} - 6\nu_{7b} + \beta) \geq 0,\end{aligned}$$

$$\text{where } (\alpha, \beta) = \begin{cases} (49, 42), & \text{when } \chi(u^3) = \chi(7a); \\ (42, 49), & \text{when } \chi(u^3) = \chi(7b); \\ (56, 35), & \text{when } \chi(u^3) = 2\chi(7a) - \chi(7b); \\ (35, 56), & \text{when } \chi(u^3) = -\chi(7a) + 2\chi(7b), \end{cases}$$

which have no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 22. Then by (1) and Proposition 2 we get

$$\nu_{2a} + \nu_{11a} + \nu_{11b} = 1.$$

We need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{22}(50\nu_{2a} - 10\nu_{11a} - 10\nu_{11b} + 16) \geq 0; \\ \mu_{11}(u, \chi_2, *) &= \frac{1}{22}(-50\nu_{2a} + 10\nu_{11a} + 10\nu_{11b} + 6) \geq 0,\end{aligned}$$

that has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 33. Then by (1) and Proposition 2 we get

$$\nu_{3a} + \nu_{11a} + \nu_{11b} = 1.$$

As in the previous case, we need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the same system:

$$\begin{aligned}\mu_0(u, \chi_2, *) &= \frac{1}{33}(60\nu_{3a} - 20\nu_{11a} - 20\nu_{11b} + 17) \geq 0; \\ \mu_{11}(u, \chi_2, *) &= \frac{1}{33}(-30\nu_{3a} + 10\nu_{11a} + 10\nu_{11b} + 8) \geq 0,\end{aligned}$$

that has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 35. Then by (1) and Proposition 2 we get $\nu_{5a} + \nu_{7a} + \nu_{7b} = 1$.

We need to consider four cases defined by part (v) of Theorem 1, but in all of them using Proposition 3 we obtain the same system of inequalities:

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24\nu_{5a} + 25) \geq 0; \quad \mu_0(u, \chi_7, 2) = \frac{1}{35}(-48\nu_{5a} + 90) \geq 0,$$

which has no solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

• Let u be a unit of order 55. Then by (1) and Proposition 2 we get

$$\nu_{5a} + \nu_{11a} + \nu_{11b} = 1.$$

We need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \frac{1}{55}(40\nu_{5a} - 40\nu_{11a} - 40\nu_{11b} + 15) \geq 0; \\ \mu_{11}(u, \chi_2, *) &= \frac{1}{55}(-10\nu_{5a} + 10\nu_{11a} + 10\nu_{11b} + 10) \geq 0; \\ \mu_1(u, \chi_{10}, *) &= \frac{1}{55}(-6\nu_{11a} + 5\nu_{11b} + \alpha) \geq 0; \\ \mu_5(u, \chi_{10}, *) &= \frac{1}{55}(24\nu_{11a} - 20\nu_{11b} + \alpha) \geq 0; \\ \mu_1(u, \chi_2, *) &= \frac{1}{55}(\nu_{5a} - \nu_{11a} - \nu_{11b} + 21) \geq 0, \end{aligned}$$

$$\text{where } \alpha = \begin{cases} 286, & \text{when } \chi(u^5) = \chi(11a); \\ 275, & \text{when } \chi(u^5) = \chi(11b); \\ 330, & \text{when } \chi(u^5) = 5\chi(11a) - 4\chi(11b); \\ 253, & \text{when } \chi(u^5) = -2\chi(11a) + 3\chi(11b); \\ 297, & \text{when } \chi(u^5) = 2\chi(11a) - \chi(11b); \\ 242, & \text{when } \chi(u^5) = -3\chi(11a) + 4\chi(11b); \\ 231, & \text{when } \chi(u^5) = -4\chi(11a) + 5\chi(11b); \\ 308, & \text{when } \chi(u^5) = 3\chi(11a) - 2\chi(11b); \\ 264, & \text{when } \chi(u^5) = -\chi(11a) + 2\chi(11b); \\ 319, & \text{when } \chi(u^5) = 4\chi(11a) - 3\chi(11b), \end{cases}$$

that has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u be a unit of order 77. Then by (1) and Proposition 2 we have

$$\nu_{7a} + \nu_{7b} + \nu_{11a} + \nu_{11b} = 1.$$

We must consider 40 cases determined by parts (v) and (vi) of Theorem 1, but luckily in all of them using Proposition 3 we obtain the same system of inequalities:

$$\begin{aligned} \mu_{11}(u, \chi_2, *) &= \frac{1}{77}(10\nu_{11a} + 10\nu_{11b} + 11) \geq 0; \\ \mu_0(u, \chi_2, *) &= \frac{1}{77}(-60\nu_{11a} - 60\nu_{11b} + 11) \geq 0, \end{aligned}$$

which has no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers. This finishes the proof of Theorem 1.

Appendix 1.

Possible partial augmentations $(\nu_{2a}, \nu_{4a}, \nu_{4b}, \nu_{8a})$ for units of order 8:

(-2, -2, 4, 1),	(-2, -1, 3, 1),	(-2, 0, 2, 1),	(-2, 0, 4, -1),	(-2, 1, 1, 1),
(-2, 1, 3, -1),	(-2, 2, 0, 1),	(-2, 2, 2, -1),	(-2, 2, 4, -3),	(-2, 3, -1, 1),
(-2, 3, 1, -1),	(-2, 3, 3, -3),	(-2, 4, -2, 1),	(-2, 4, 0, -1),	(-2, 4, 2, -3),
(-2, 5, -3, 1),	(-2, 5, -1, -1),	(-2, 5, 1, -3),	(-2, 6, -2, -1),	(-2, 6, 0, -3),
(0, -5, 3, 3),	(0, -4, 2, 3),	(0, -4, 4, 1),	(0, -3, 1, 3),	(0, -3, 3, 1),
(0, -3, 5, -1),	(0, -2, 0, 3),	(0, -2, 2, 1),	(0, -2, 4, -1),	(0, -2, 6, -3),
(0, -1, -1, 3),	(0, -1, 1, 1),	(0, -1, 3, -1),	(0, -1, 5, -3),	(0, 0, -2, 3),
(0, 0, 0, 1),	(0, 0, 2, -1),	(0, 0, 4, -3),	(0, 1, -3, 3),	(0, 1, -1, 1),
(0, 1, 1, -1),	(0, 1, 3, -3),	(0, 2, -4, 3),	(0, 2, -2, 1),	(0, 2, 0, -1),
(0, 2, 2, -3),	(0, 3, -5, 3),	(0, 3, -3, 1),	(0, 3, -1, -1),	(0, 3, 1, -3),
(0, 4, -4, 1),	(0, 4, -2, -1),	(0, 4, 0, -3),	(0, 5, -3, -1),	(0, 5, -1, -3),
(0, 6, -2, -3),	(2, -5, 1, 3),	(2, -5, 3, 1),	(2, -4, 0, 3),	(2, -4, 2, 1),
(2, -4, 4, -1),	(2, -3, -1, 3),	(2, -3, 1, 1),	(2, -3, 3, -1),	(2, -2, -2, 3),
(2, -2, 0, 1),	(2, -2, 2, -1),	(2, -1, -3, 3),	(2, -1, -1, 1),	(2, -1, 1, -1),
(2, 0, -2, 1),	(2, 0, 0, -1),	(2, 1, -3, 1),	(2, 1, -1, -1),	(2, 2, -2, -1),
(2, 3, -3, -1)				

Appendix 2.

Possible partial augmentations $(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{4b}, \nu_{6a})$ for units of order 12:

(-4, 9, -4, -4, 4),	(-4, 9, -3, -5, 4),	(-4, 9, -2, -6, 4),	(-4, 9, -1, -7, 4),
(-4, 9, -1, -3, 0),	(-4, 9, 0, -8, 4),	(-4, 9, 0, -4, 0),	(-4, 9, 1, -9, 4),
(-4, 9, 2, -10, 4),	(-4, 9, 3, -11, 4),	(-4, 12, -5, -5, 3),	(-4, 12, -4, -6, 3),
(-4, 12, -4, -4, 1),	(-4, 12, -3, -7, 3),	(-4, 12, -3, -5, 1),	(-4, 12, -2, -8, 3),
(-4, 12, -2, -6, 1),	(-4, 12, -1, -9, 3),	(-4, 12, -1, -7, 1),	(-4, 12, 0, -10, 3),
(-4, 12, 0, -8, 1),	(-3, 3, 0, 1, 0),	(-3, 3, 1, 0, 0),	(-3, 3, 2, -1, 0),
(-3, 3, 2, 3, -4),	(-3, 3, 3, -2, 0),	(-3, 3, 3, 2, -4),	(-3, 3, 4, -3, 0),
(-3, 3, 4, 1, -4),	(-3, 3, 5, -4, 0),	(-3, 3, 5, 0, -4),	(-3, 3, 6, -5, 0),
(-3, 6, -4, -1, 3),	(-3, 6, -3, -2, 3),	(-3, 6, -2, -3, 3),	(-3, 6, -2, 1, -1),
(-3, 6, -1, -4, 3),	(-3, 6, -1, 0, -1),	(-3, 6, 0, -5, 3),	(-3, 6, 0, -1, -1),

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(-3, 6, 1, -6, 3),	(-3, 6, 1, -2, -1),	(-3, 6, 2, -7, 3),	(-3, 6, 2, -3, -1),
(-3, 6, 3, -8, 3),	(-3, 6, 3, -4, -1),	(-3, 6, 4, -9, 3),	(-3, 6, 4, -5, -1),
(-3, 6, 5, -10, 3),	(-3, 6, 5, -6, -1),	(-3, 6, 6, -11, 3),	(-3, 9, -7, -4, 6),
(-3, 9, -6, -5, 6),	(-3, 9, -6, -3, 4),	(-3, 9, -5, -6, 6),	(-3, 9, -5, -4, 4),
(-3, 9, -5, -2, 2),	(-3, 9, -4, -7, 6),	(-3, 9, -4, -5, 4),	(-3, 9, -4, -3, 2),
(-3, 9, -4, -1, 0),	(-3, 9, -3, -8, 6),	(-3, 9, -3, -6, 4),	(-3, 9, -3, -4, 2),
(-3, 9, -3, -2, 0),	(-3, 9, -2, -9, 6),	(-3, 9, -2, -7, 4),	(-3, 9, -2, -5, 2),
(-3, 9, -2, -3, 0),	(-3, 9, -1, -10, 6),	(-3, 9, -1, -8, 4),	(-3, 9, -1, -6, 2),
(-3, 9, -1, -4, 0),	(-3, 9, 0, -11, 6),	(-3, 9, 0, -9, 4),	(-3, 9, 0, -7, 2),
(-3, 9, 0, -5, 0),	(-3, 9, 1, -12, 6),	(-3, 9, 1, -10, 4),	(-3, 9, 1, -8, 2),
(-3, 9, 1, -6, 0),	(-3, 9, 2, -13, 6),	(-3, 9, 2, -11, 4),	(-3, 9, 2, -9, 2),
(-3, 9, 2, -7, 0),	(-3, 9, 3, -14, 6),	(-3, 9, 3, -10, 2),	(-3, 9, 3, -8, 0),
(-3, 9, 4, -15, 6),	(-3, 12, -7, -6, 5),	(-3, 12, -7, -4, 3),	(-3, 12, -6, -7, 5),
(-3, 12, -6, -5, 3),	(-3, 12, -5, -8, 5),	(-3, 12, -5, -6, 3),	(-3, 12, -4, -9, 5),
(-3, 12, -4, -7, 3),	(-3, 12, -3, -10, 5),	(-3, 12, -3, -8, 3),	(-3, 12, -2, -11, 5),
(-3, 12, -2, -9, 3),	(-3, 12, -1, -12, 5),	(-3, 12, -1, -10, 3),	(-3, 12, 0, -13, 5),
(-3, 12, 0, -11, 3),	(-3, 12, 1, -12, 3),	(-2, 0, 0, 4, -1),	(-2, 0, 1, 3, -1),
(-2, 0, 2, 2, -1),	(-2, 0, 2, 6, -5),	(-2, 0, 3, 1, -1),	(-2, 0, 3, 5, -5),
(-2, 0, 4, 0, -1),	(-2, 0, 4, 4, -5),	(-2, 0, 5, -1, -1),	(-2, 0, 5, 3, -5),
(-2, 0, 6, -2, -1),	(-2, 0, 6, 2, -5),	(-2, 0, 7, -3, -1),	(-2, 0, 7, 1, -5),
(-2, 0, 8, -4, -1),	(-2, 0, 8, 0, -5),	(-2, 3, -4, 2, 2),	(-2, 3, -3, 1, 2),
(-2, 3, -2, 0, 2),	(-2, 3, -2, 4, -2),	(-2, 3, -1, -1, 2),	(-2, 3, -1, 1, 0),
(-2, 3, -1, 3, -2),	(-2, 3, 0, 0, -2),	(-2, 3, 0, 0, 0),	(-2, 3, 0, 2, -2),
(-2, 3, 0, 4, -4),	(-2, 3, 1, -3, 2),	(-2, 3, 1, -1, 0),	(-2, 3, 1, 1, -2),
(-2, 3, 1, 3, -4),	(-2, 3, 2, -4, 2),	(-2, 3, 2, -2, 0),	(-2, 3, 2, 0, -2),
(-2, 3, 2, 2, -4),	(-2, 3, 3, -5, 2),	(-2, 3, 3, -3, 0),	(-2, 3, 3, -1, -2),
(-2, 3, 3, 1, -4),	(-2, 3, 4, -6, 2),	(-2, 3, 4, -4, 0),	(-2, 3, 4, -2, -2),
(-2, 3, 4, 0, -4),	(-2, 3, 5, -7, 2),	(-2, 3, 5, -3, -2),	(-2, 3, 5, -1, -4),
(-2, 3, 6, -8, 2),	(-2, 3, 6, -4, -2),	(-2, 3, 6, -2, 4),	(-2, 3, 7, -9, 2),
(-2, 3, 7, -5, -2),	(-2, 3, 8, -10, 2),	(-2, 3, 8, -6, -2),	(-2, 6, -7, -1, 5),
(-2, 6, -6, -2, 5),	(-2, 6, -6, 0, 3),	(-2, 6, -5, -3, 5),	(-2, 6, -5, -1, 3),
(-2, 6, -5, 1, 1),	(-2, 6, -4, -4, 5),	(-2, 6, -4, -2, 3),	(-2, 6, -4, 0, 1),
(-2, 6, -4, 2, -1),	(-2, 6, -3, -5, 5),	(-2, 6, -3, -3, 3),	(-2, 6, -3, -1, 1),
(-2, 6, -3, 1, -1),	(-2, 6, -2, -6, 5),	(-2, 6, -2, -4, 3),	(-2, 6, -2, -2, 1),
(-2, 6, -2, 0, -1),	(-2, 6, -1, -7, 5),	(-2, 6, -1, -5, 3),	(-2, 6, -1, -3, 1),
(-2, 6, -1, -1, -1),	(-2, 6, 0, -8, 5),	(-2, 6, 0, -6, 3),	(-2, 6, 0, -4, 1),
(-2, 6, 0, -2, -1),	(-2, 6, 1, -9, 5),	(-2, 6, 1, -7, 3),	(-2, 6, 1, -5, 1),
(-2, 6, 1, -3, -1),	(-2, 6, 2, -10, 5),	(-2, 6, 2, -8, 3),	(-2, 6, 2, -6, 1),
(-2, 6, 2, -4, -1),	(-2, 6, 3, -11, 5),	(-2, 6, 3, -9, 3),	(-2, 6, 3, -7, 1),
(-2, 6, 3, -5, -1),	(-2, 6, 4, -12, 5),	(-2, 6, 4, -10, 3),	(-2, 6, 4, -8, 1),
(-2, 6, 4, -6, -1),	(-2, 6, 5, -13, 5),	(-2, 6, 5, -9, 1),	(-2, 6, 5, -7, -1),
(-2, 6, 6, -10, 1),	(-2, 6, 6, -8, -1),	(-2, 9, -9, -5, 8),	(-2, 9, -9, -3, 6),
(-2, 9, -8, -6, 8),	(-2, 9, -8, -4, 6),	(-2, 9, -7, -7, 8),	(-2, 9, -7, -5, 6),
(-2, 9, -7, -3, 4),	(-2, 9, -7, -1, 2),	(-2, 9, -6, -6, 8),	(-2, 9, -6, -6, 6),
(-2, 9, -6, -4, 4),	(-2, 9, -6, -2, 2),	(-2, 9, -5, -9, 8),	(-2, 9, -5, -7, 6),
(-2, 9, -5, -5, 4),	(-2, 9, -5, -3, 2),	(-2, 9, -4, -10, 8),	(-2, 9, -4, -8, 6),
(-2, 9, -4, -6, 4),	(-2, 9, -4, -4, 2),	(-2, 9, -3, -11, 8),	(-2, 9, -3, -9, 6),
(-2, 9, -3, -7, 4),	(-2, 9, -3, -5, 2),	(-2, 9, -2, -12, 8),	(-2, 9, -2, -10, 6),
(-2, 9, -2, -8, 4),	(-2, 9, -2, -6, 2),	(-2, 9, -1, -13, 8),	(-2, 9, -1, -11, 6),
(-2, 9, -1, -9, 4),	(-2, 9, -1, -7, 2),	(-2, 9, 0, -14, 8),	(-2, 9, 0, -12, 6),
(-2, 9, 0, -10, 4),	(-2, 9, 0, -8, 2),	(-2, 9, 1, -15, 8),	(-2, 9, 1, -13, 6),
(-2, 9, 1, -11, 4),	(-2, 9, 1, -9, 2),	(-2, 9, 2, -14, 6),	(-2, 9, 2, -12, 4),
(-2, 9, 2, -10, 2),	(-2, 9, 3, -13, 4),	(-2, 9, 3, -11, 2),	(-2, 9, 4, -12, 2),
(-2, 12, -9, -7, 7),	(-2, 12, -9, -5, 5),	(-2, 12, -8, -8, 7),	(-2, 12, -8, -6, 5),
(-2, 12, -7, -9, 7),	(-2, 12, -7, -7, 5),	(-2, 12, -6, -10, 7),	(-2, 12, -6, -8, 5),
(-2, 12, -5, -11, 7),	(-2, 12, -5, -9, 5),	(-2, 12, -4, -12, 7),	(-2, 12, -4, -10, 5),
(-2, 12, -3, -13, 7),	(-2, 12, -3, -11, 5),	(-2, 12, -2, -12, 5),	(-2, 12, -1, -13, 5),
(-2, 12, 0, -14, 5),	(-1, -3, 0, 7, -2),	(-1, -3, 1, 6, -2),	(-1, -3, 2, 5, -2),
(-1, -3, 2, 9, -6),	(-1, -3, 3, 4, -2),	(-1, -3, 3, 8, -6),	(-1, -3, 4, 3, -2),
(-1, -3, 4, 7, -6),	(-1, -3, 5, 2, -2),	(-1, -3, 5, 6, -6),	(-1, -3, 6, 1, -2),
(-1, -3, 6, 5, -6),	(-1, -3, 7, 0, -2),	(-1, -3, 7, 4, -6),	(-1, -3, 8, -1, -2),
(-1, -3, 8, 3, -6),	(-1, -3, 9, -2, -2),	(-1, -3, 9, 2, -6),	(-1, -3, 10, -3, -2),
(-1, -3, 10, 1, -6),	(-1, -3, 11, 0, -6),	(-1, 0, -4, 5, 1),	(-1, 0, -3, 4, 1),
(-1, 0, -2, 3, 1),	(-1, 0, -2, 7, -3),	(-1, 0, -1, 2, 1),	(-1, 0, -1, 4, -1),
(-1, 0, -1, 6, -3),	(-1, 0, 0, 1, 1),	(-1, 0, 0, 3, -1),	(-1, 0, 0, 5, -3),
(-1, 0, 0, 7, -5),	(-1, 0, 1, 0, 1),	(-1, 0, 1, 2, -1),	(-1, 0, 1, 4, -3),
(-1, 0, 1, 6, -5),	(-1, 0, 2, -1, 1),	(-1, 0, 2, 1, -1),	(-1, 0, 2, 3, -3),
(-1, 0, 2, 5, -5),	(-1, 0, 3, -2, 1),	(-1, 0, 3, 0, -1),	(-1, 0, 3, 2, -3),
(-1, 0, 3, 4, -5),	(-1, 0, 4, -3, 1),	(-1, 0, 4, -1, -1),	(-1, 0, 4, 1, -3),
(-1, 0, 4, 3, -5),	(-1, 0, 5, -4, 1),	(-1, 0, 5, -2, -1),	(-1, 0, 5, 0, -3),
(-1, 0, 5, 2, -5),	(-1, 0, 6, -5, 1),	(-1, 0, 6, -3, -1),	(-1, 0, 6, -1, -3),
(-1, 0, 6, 1, -5),	(-1, 0, 7, -6, 1),	(-1, 0, 7, -2, -3),	(-1, 0, 7, 0, -5),
(-1, 0, 8, -7, 1),	(-1, 0, 8, -3, -3),	(-1, 0, 8, -1, -5),	(-1, 0, 9, -4, -3),
(-1, 3, -7, 2, 4),	(-1, 3, -6, 1, 4),	(-1, 3, -6, 3, 2),	(-1, 3, -5, 0, 4),
(-1, 3, -5, 2, 2),	(-1, 3, -5, 4, 0),	(-1, 3, -4, -1, 4),	(-1, 3, -4, 1, 2),
(-1, 3, -4, 3, 0),	(-1, 3, -4, 5, -2),	(-1, 3, -3, -2, 4),	(-1, 3, -3, 0, 2),
(-1, 6, -9, -2, 7),	(-1, 6, -9, 0, 5),	(-1, 6, -8, -3, 7),	(-1, 6, -8, -1, 5),
(-1, 6, -7, -4, 7),	(-1, 6, -7, -2, 5),	(-1, 6, -7, 0, 3),	(-1, 6, -7, 2, 1),
(-1, 6, -6, -5, 7),	(-1, 6, -6, -3, 5),	(-1, 6, -6, -1, 3),	(-1, 6, -6, 1, 1),
(-1, 6, -5, -6, 7),	(-1, 6, -5, -4, 5),	(-1, 6, -5, -2, 3),	(-1, 6, -5, 0, 1),
(-1, 6, -4, -7, 7),	(-1, 6, -4, -5, 5),	(-1, 6, -4, -3, 3),	(-1, 6, -4, -1, 1),
(-1, 6, -3, -8, 7),	(-1, 6, -3, -6, 5),	(-1, 6, -3, -4, 3),	(-1, 6, -3, -2, 1),
(-1, 6, -2, -9, 7),	(-1, 6, -2, -7, 5),	(-1, 6, -2, -5, 3),	(-1, 6, -2, -3, 1),
(-1, 6, -1, -10, 7),	(-1, 6, -1, -8, 5),	(-1, 6, -1, -6, 3),	(-1, 6, -1, -4, 1),
(-1, 6, 0, -11, 7),	(-1, 6, 0, -9, 5),	(-1, 6, 0, -7, 3),	(-1, 6, 0, -5, 1),
(-1, 6, 1, -12, 7),	(-1, 6, 1, -10, 5),	(-1, 6, 1, -8, 3),	(-1, 6, 1, -6, 1),
(-1, 6, 2, -11, 5),	(-1, 6, 2, -9, 3),	(-1, 6, 2, -7, 1),	(-1, 6, 3, -12, 5),
(-1, 6, 3, -10, 3),	(-1, 6, 3, -8, 1),	(-1, 6, 4, -9, 1),	(-1, 6, 5, -10, 1),
(-1, 9, -10, -7, 10),	(-1, 9, -10, -5, 8),	(-1, 9, -9, -8, 10),	(-1, 9, -9, -6, 8),
(-1, 9, -9, -4, 6),	(-1, 9, -9, -2, 4),	(-1, 9, -8, -9, 10),	(-1, 9, -8, -7, 8),
(-1, 9, -8, -5, 6),	(-1, 9, -8, -3, 4),	(-1, 9, -7, -10, 10),	(-1, 9, -7, -8, 8),

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(-1, 9, -7, -6, 6),	(-1, 9, -7, -4, 4),	(-1, 9, -6, -11, 10),	(-1, 9, -6, -9, 8),
(-1, 9, -6, -7, 6),	(-1, 9, -6, -5, 4),	(-1, 9, -5, -12, 10),	(-1, 9, -5, -10, 8),
(-1, 9, -5, -8, 6),	(-1, 9, -5, -6, 4),	(-1, 9, -4, -13, 10),	(-1, 9, -4, -11, 8),
(-1, 9, -4, -9, 6),	(-1, 9, -4, -7, 4),	(-1, 9, -3, -14, 10),	(-1, 9, -3, -12, 8),
(-1, 9, -3, -10, 6),	(-1, 9, -3, -8, 4),	(-1, 9, -2, -13, 8),	(-1, 9, -2, -9, 4),
(-1, 9, -1, -14, 8),	(-1, 9, -1, -10, 4),	(-1, 9, 0, -11, 4),	(-1, 9, 1, -12, 4),
(-1, 12, -10, -7, 7),	(-1, 12, -9, -8, 7),	(-1, 12, -8, -9, 7),	(-1, 12, -7, -10, 7),
(-1, 12, -6, -11, 7),	(-1, 12, -5, -12, 7),	(-1, 12, -4, -13, 7),	(-1, 12, -3, -14, 7),
(0, -9, 5, 11, -6),	(0, -9, 6, 10, -6),	(0, -9, 7, 9, -6),	(0, -9, 8, 8, -6),
(0, -9, 9, 7, -6),	(0, -9, 10, 6, -6),	(0, -9, 11, 5, -6),	(0, -9, 12, 4, -6),
(0, -6, 0, 10, -3),	(0, -6, 1, 9, -3),	(0, -6, 2, 8, -3),	(0, -6, 2, 12, -7),
(0, -6, 3, 7, -3),	(0, -6, 3, 11, -7),	(0, -6, 4, 6, -3),	(0, -6, 4, 10, -7),
(0, -6, 5, 5, -3),	(0, -6, 5, 9, -7),	(0, -6, 6, 4, -3),	(0, -6, 6, 8, -7),
(0, -6, 7, 3, -3),	(0, -6, 7, 7, -7),	(0, -6, 8, 2, -3),	(0, -6, 8, 6, -7),
(0, -6, 9, 1, -3),	(0, -6, 9, 5, -7),	(0, -6, 10, 0, -3),	(0, -6, 10, 4, -7),
(0, -6, 11, 3, -7),	(0, -6, 12, 2, -7),	(0, -3, -4, 8, 0),	(0, -3, -3, 7, 0),
(0, -3, -2, 6, 0),	(0, -3, -2, 10, -4),	(0, -3, -1, 5, 0),	(0, -3, -1, 7, -2),
(0, -3, -1, 9, -4),	(0, -3, 0, 4, 0),	(0, -3, 0, 6, -2),	(0, -3, 0, 8, -4),
(0, -3, 0, 10, -6),	(0, -3, 1, 3, 0),	(0, -3, 1, 5, -2),	(0, -3, 1, 7, -4),
(0, -3, 1, 9, -6),	(0, -3, 2, 2, 0),	(0, -3, 2, 4, -2),	(0, -3, 2, 6, -4),
(0, -3, 2, 8, -6),	(0, -3, 3, 1, 0),	(0, -3, 3, 3, -2),	(0, -3, 3, 5, -4),
(0, -3, 3, 7, -6),	(0, -3, 4, 0, 0),	(0, -3, 4, 2, -2),	(0, -3, 4, 4, -4),
(0, -3, 4, 6, -6),	(0, -3, 5, -1, 0),	(0, -3, 5, 1, -2),	(0, -3, 5, 3, -4),
(0, -3, 5, 5, -6),	(0, -3, 6, -2, 0),	(0, -3, 6, 0, -2),	(0, -3, 6, 2, -4),
(0, -3, 6, 4, -6),	(0, -3, 7, -3, 0),	(0, -3, 7, -1, -2),	(0, -3, 7, 1, -4),
(0, -3, 7, 3, -6),	(0, -3, 8, -4, 0),	(0, -3, 8, -2, -2),	(0, -3, 8, 0, -4),
(0, -3, 8, 2, -6),	(0, -3, 9, -1, -4),	(0, -3, 9, 1, -6),	(0, -3, 10, -2, -4),
(0, -3, 10, 0, -6),	(0, 0, -7, 5, 3),	(0, 0, -6, 4, 3),	(0, 0, -6, 6, 1),
(0, 0, -5, 3, 3),	(0, 0, -5, 5, 1),	(0, 0, -5, 7, -1),	(0, 0, -4, 2, 3),
(0, 0, -4, 4, 1),	(0, 0, -4, 6, -1),	(0, 0, -4, 8, -3),	(0, 0, -3, 1, 3),
(0, 0, -3, 3, 1),	(0, 0, -3, 5, -1),	(0, 0, -3, 7, -3),	(0, 0, -2, 0, 3),
(0, 0, -2, 2, 1),	(0, 0, -2, 4, -1),	(0, 0, -2, 6, -3),	(0, 0, -1, -1, 3),
(0, 0, -1, 1, 1),	(0, 0, -1, 3, -1),	(0, 0, -1, 5, -3),	(0, 0, 0, -2, 3),
(0, 0, 0, 0, 1),	(0, 0, 0, 2, -1),	(0, 0, 0, 4, -3),	(0, 0, 1, -3, 3),
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 (4, -9, 2, 8, -4),
 (4, -6, -3, 7, -1),
 (4, -6, -1, 5, -1),

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