INTEGRAL GROUP RING OF THE FIRST MATHIEU SIMPLE GROUP

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Abstract

We investigate the classical Zassenhaus conjecture for the normalized unit group of the integral group ring of the simple Mathieu group M_{11}. As a consequence, for this group we confirm the conjecture by Kimmerle about prime graphs.

1 Introduction and main results

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. The following famous conjecture was formulated by H. Zassenhaus in [15]:

Conjecture 1 (ZC) Every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element of G.

This conjecture is already confirmed for several classes of groups but, in general, the problem remains open, and a counterexample is not known.

Various methods have been developed to deal with this conjecture. One of the original ones was suggested by I. S. Luthar and I. B. S. Passi [12, 13], and it was improved further by M. Hertweck [9]. Using this method, the conjecture was proved for several new classes of groups, in particular for S_5 and for some finite simple groups (see [4, 9, 10, 12, 13]).

The Zassenhaus conjecture appeared to be very hard, and several weakened variations of it were formulated (see, for example, [3]). One of the most interesting modifications was suggested by W. Kimmerle [11]. Let us briefly introduce it now.

Let G be a finite group. Denote by $\#(G)$ the set of all primes dividing the order of G. Then the *Gruenberg–Kegel graph* (or the *prime graph*) of G is a graph $\pi(G)$ with vertices labelled by primes from $\#(G)$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group G. Then the conjecture by Kimmerle can be formulated in the following way:

Conjecture 2 (KC) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.

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For Frobenius groups and solvable groups this conjecture was confirmed in [11]. In the present paper we continue the investigation of **(KC)**, and confirm it for the first simple Mathieu group M_{11}, using the Luthar–Passi method. Moreover, this allows us to give a partial solution of **(ZC)** for M_{11}.

Our main results are the following:

Theorem 1 *Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$, where G is the simple Mathieu group M_{11}. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. We have:*

(i) *if $|u| \neq 12$, then $|u|$ coincides with the order of some element $g \in G$;*

(ii) *if $|u| \in \{2, 3, 5, 11\}$, then u is rationally conjugate to some $g \in G$;*

(iii) *if $|u| = 4$, then the tuple of partial augmentations of u belongs to the set*

$$\{ (\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a},\ \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0,\\ kx \notin \{2a, 4a\},\ (\nu_{2a}, \nu_{4a}) \in \{ (0,1),\ (2,-1) \} \};$$

(iv) *if $|u| = 6$, then the tuple of partial augmentations of u belongs to the set*

$$\{ (\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0,\\ kx \notin \{2a, 3a, 6a\},\ (\nu_{2a}, \nu_{3a}, \nu_{6a}) \in \{ (-2,3,0), (0,0,1),\\ (0,3,-2), (2,-3,2), (2,0,-1) \} \};$$

(v) *if $|u| = 8$, then the tuple of partial augmentations of u belongs to the set*

$$\{ (\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0,\\ kx \notin \{4a, 8a, 8b\},\ (\nu_{4a}, \nu_{8a}, \nu_{8b}) \in \{ (0,0,1), (0,1,0),\\ (2,-1,0), (2,0,-1) \} \};$$

(vi) *if $|u| = 12$, then the tuple of partial augmentations of u cannot belong to the set*

$$\mathbb{Z}^9 \setminus \{ (\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{6a}, \nu_{5a}, \nu_{8a}, \nu_{8b}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^9 \mid \nu_{kx} = 0,\\ kx \notin \{2a, 4a, 6a\},\ (\nu_{2a}, \nu_{4a}, \nu_{6a}) \in \{ (-1,1,1), (1,1,-1) \} \}.$$

Corollary 1 *Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$, where G is the simple Mathieu group M_{11}. Then $\pi(G) = \pi(V(\mathbb{Z}G))$, where $\pi(G)$ and $\pi(V(\mathbb{Z}G))$ are prime graphs of G and $V(\mathbb{Z}G)$, respectively. Thus, for M_{11} the conjecture by Kimmerle is true.*

2 Notation and preliminaries

Let $u = \sum \alpha_g g$ be a normalized torsion unit of order k and let $\nu_i = \varepsilon_{C_i}(u)$ be a partial augmentation of u. By S. D. Berman's Theorem [2] we have that $\nu_1 = 0$ and

$$\nu_2 + \nu_3 + \cdots + \nu_m = 1. \tag{2.1}$$

For any character χ of G of degree n, we have that $\chi(u) = \sum_{i=2}^{m} \nu_i \chi(h_i)$, where h_i is a representative of the conjugacy class C_i.

We need the following results:

Proposition 1 (see [12]) *Suppose that u is an element of $\mathbb{Z}G$ of order k. Let z be a primitive k-th root of unity. Then for every integer l and any character χ of G, the number*

$$\mu_l(u, \chi) = \frac{1}{k} \sum_{d|k} \mathrm{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}} \{\chi(u^d) z^{-dl}\} \tag{2.2}$$

is a non-negative integer.

Proposition 2 (see [6]) *Let u be a torsion unit in $V(\mathbb{Z}G)$. Then the order of u divides the exponent of G.*

Proposition 3 (see [12] and Theorem 2.7 in [14]) *Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u) = 0$.*

M. Hertweck (see [10], Proposition 3.1; [9], Lemma 5.6) obtained the next result:

Proposition 4 *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

(i) *If u has order p^n, then $\varepsilon_x(u) = 0$ for every x of G whose p-part is of order strictly greater than p^n.*

(ii) *If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Note that the first part of Proposition 4 gives a partial answer to the conjecture by A. Bovdi (see [1]). Also M. Hertweck ([9], Lemma 5.5) gives a complete answer to the same conjecture in the case when $G = \mathrm{PSL}(2, \mathbb{F})$, where $\mathbb{F} = \mathrm{GF}(p^k)$.

In the rest of the paper, for the partial augmentation ν_i we shall also use the notation ν_{kx}, where k is the order of the representative of the i-th conjugacy class, and x is a distinguishing letter for this particular class with elements of order k.

Proposition 5 (see [12] and Theorem 2.5 in [14]) *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

Proposition 6 (see [6]) *Let p be a prime, and let u be a torsion unit of $V(\mathbb{Z}G)$ of order p^n. Then for $m \neq n$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p.*

The Brauer character table modulo p of the group M_{11} will be denoted by $\mathfrak{BCT}(p)$.

3 Proof of the Theorem

It is well known [7, 8] that $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $\exp(G) = 1320$. The character table of G, as well as the Brauer character tables $\mathfrak{BCT}(p)$, where $p \in \{2, 3, 5, 11\}$, can be found using the computational algebra system GAP [7].

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 8 and 11, first of all we shall investigate units with these orders. After this, by Proposition 2, the order of each torsion unit divides the exponent of G, so it will be enough to consider units of orders 10, 12, 15, 22, 24, 33 and 55, because if u will be a unit of another possible order, then there is $t \in \mathbb{N}$ such that u^t has an order from this list. We shall prove that units of all these orders except 12 do not appear in $V(\mathbb{Z}G)$. For units of order 12 we are not able to prove this, but we reduce this question to only two cases.

Let $u \in V(\mathbb{Z}G)$ have order k. By S. D. Berman's Theorem [2] and Proposition 3 we have $\nu_{1a} = 0$ and

$$\begin{array}{cll}
\nu_{3a} = \nu_{5a} = \nu_{6a} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 2, 4, 8; \\
\nu_{2a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 3; \\
\nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 5; \\
\nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 6; \\
\nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when} & k = 11; \\
\nu_{3a} = \nu_{6a} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 10; \\
\nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 12; \\
\nu_{2a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 15; \\
\nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = 0 & \text{when} & k = 22; \\
\nu_{5a} = \nu_{11a} = \nu_{11b} = 0 & \text{when} & k = 24; \\
\nu_{2a} = \nu_{4a} = \nu_{5a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when} & k = 33; \\
\nu_{2a} = \nu_{3a} = \nu_{4a} = \nu_{6a} = \nu_{8a} = \nu_{8b} = 0 & \text{when} & k = 55.
\end{array} \tag{3.1}$$

It follows immediately by Proposition 5 that the units of orders 3 and 5 are rationally conjugate to some element of G.

Now we consider each case separately:

- Let u be an involution. Then using (3.1) and Proposition 4 we obtain that $\nu_{4a} = \nu_{8a} = \nu_{8b} = 0$, so $\nu_{2a} = 1$.
- Let u be a unit of order 4. Then by (3.1) and Proposition 4 we have $\nu_{2a} + \nu_{4a} = 1$. By (2.2), $\mu_0(u, \chi_3) = \frac{1}{4}(-4\nu_{2a} + 8) \geq 0$ and $\mu_2(u, \chi_3) = \frac{1}{4}(4\nu_{2a} + 8) \geq 0$, so $\nu_{2a} \in \{-2, -1, 0, 1, 2\}$. Now using the inequalities

$$\mu_0(u, \chi_5) = \tfrac{1}{4}(6\nu_{2a} - 2\nu_{4a} + 14) \geq 0;$$
$$\mu_2(u, \chi_5) = \tfrac{1}{4}(-6\nu_{2a} + 2\nu_{4a} + 14) \geq 0,$$

  we get that there are only two integral solutions $(\nu_{2a}, \nu_{4a}) \in \{\, (0, 1),\ (2, -1) \,\}$ satisfying (2.1) and Proposition 6, such that all $\mu_i(u, \chi_j)$ are non-negative integers.
- Let u be a unit of order 6. Then by (2.1), (2.2) and Proposition 4 we obtain

$$\nu_{2a} + \nu_{3a} + \nu_{6a} = 1.$$

Now, using $\mathfrak{BCT}(11)$ from the system of inequalities $\mu_0(u,\chi_6) = \frac{1}{6}(-4\nu_{3a} + 12) \geq 0$ and $\mu_3(u,\chi_6) = \frac{1}{6}(4\nu_{3a} + 12) \geq 0$, we have that $\nu_{3a} \in \{-3, 0, 3\}$. Furthermore, from the system of inequalities

$$\begin{aligned} \mu_3(u,\chi_2) &= \tfrac{1}{6}(-2\nu_{2a} + 4\nu_{6a} + 8) \geq 0; \\ \mu_0(u,\chi_2) &= \tfrac{1}{6}(2\nu_{2a} - 4\nu_{6a} + 10) \geq 0; \\ \mu_1(u,\chi_2) &= \tfrac{1}{6}(\nu_{2a} - 2\nu_{6a} + 8) \geq 0, \end{aligned}$$

we get that $\nu_{2a} - 2\nu_{6a} \in \{-2, 4\}$, so $\nu_{6a} \in \{-2,-1,0,1,2\}$. Using the inequalities

$$\begin{aligned} \mu_0(u,\chi_3) &= \tfrac{1}{6}(-4\nu_{2a} + 2\nu_{3a} + 2\nu_{6a} + 10) \geq 0; \\ \mu_2(u,\chi_3) &= \tfrac{1}{6}(2\nu_{2a} - \nu_{3a} - \nu_{6a} + 7) \geq 0, \end{aligned}$$

we obtain only the following integral solutions $(\nu_{2a}, \nu_{3a}, \nu_{6a})$:

$$\{\,(-2,3,0),\ (0,0,1),\ (0,3,-2),\ (2,-3,2),\ (2,0,-1)\,\}, \tag{3.2}$$

such that all $\mu_i(u,\chi_j)$ are non-negative integers.
Using the GAP package LAGUNA [5], we tested all possible $\mu_i(u,\chi_j)$ for all tuples $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ from (3.2), and were not able to produce a contradiction. Thus, in this case, as well as in the case of elements of order 4, the Luthar–Passi method is not enough to prove the rational conjugacy.

- Let u be a unit of order 8. By (3.1) and Proposition 4 we have

$$\nu_{2a} + \nu_{4a} + \nu_{8a} + \nu_{8b} = 1.$$

Since $|u^2| = 4$, by (3.2) it yields that $\chi_j(u^2) = \overline{\nu}_{2a}\chi_j(2a) + \overline{\nu}_{4a}\chi_j(4a)$. Now using $\mathfrak{BCT}(3)$, by (2.2) in the case when $(\overline{\nu}_{2a}, \overline{\nu}_{4a}) = (0,1)$ we obtain

$$\begin{aligned} &\mu_0(u,\chi_5) = \tfrac{1}{8}(-8\nu_{2a} + 8) \geq 0; && \mu_4(u,\chi_5) = \tfrac{1}{8}(8\nu_{2a} + 8) \geq 0; \\ &\mu_0(u,\chi_4) = \tfrac{1}{8}(8\nu_{2a} + 8\nu_{4a} + 16) \geq 0; && \mu_4(u,\chi_4) = \tfrac{1}{8}(-8\nu_{2a} - 8\nu_{4a} + 16) \geq 0; \\ &\mu_1(u,\chi_2) = \tfrac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; && \mu_4(u,\chi_7) = \tfrac{1}{8}(-8\nu_{8a} - 8\nu_{8b} + 24) \geq 0; \\ &\mu_5(u,\chi_2) = \tfrac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 4) \geq 0; && \mu_0(u,\chi_7) = \tfrac{1}{8}(8\nu_{8a} + 8\nu_{8b} + 24) \geq 0. \end{aligned}$$

It follows that $-1 \leq \nu_{2a} \leq 1$, $-3 \leq \nu_{4a} \leq 3$, $-2 \leq \nu_{8a}, \nu_{8b} \leq 2$. Considering the additional inequality

$$\mu_0(u,\chi_2) = \tfrac{1}{8}(4\nu_{2a} - 4\nu_{4a} - 4\nu_{8a} - 4\nu_{8b} + 4) \geq 0,$$

and using Proposition 6, it is easy to check that this system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\begin{Bmatrix} (0,2,0,-1), & (0,2,-1,0), & (0,-2,1,2), \\ (0,-2,2,1), & (0,0,1,0), & (0,0,0,1) \end{Bmatrix}. \tag{3.3}$$

In the case when $(\overline{\nu}_{2a}, \overline{\nu}_{4a}) = (2, -1)$ using $\mathfrak{BCT}(3)$, by (2.2) we obtained that $\mu_0(u, \chi_5) = -\mu_4(u, \chi_5) = -\nu_{2a} = 0$ and

$$\begin{aligned} \mu_0(u, \chi_4) &= \tfrac{1}{8}(8\nu_{4a} + 16) \geq 0; & \mu_4(u, \chi_4) &= \tfrac{1}{8}(-8\nu_{4a} + 16) \geq 0; \\ \mu_1(u, \chi_2) &= \tfrac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 4) \geq 0; & \mu_5(u, \chi_2) &= \tfrac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 4) \geq 0. \end{aligned}$$

It is easy to check that $-2 \leq \nu_{4a}, \nu_{8a}, \nu_{8b} \leq 2$, and this system has the following integral solutions:

$$\begin{array}{llll} \{ & (0,2,-1,0), & (0,2,0,-1), & (0,0,0,1), \\ & (0,-2,2,1), & (0,0,1,0), & (0,-2,1,2) \quad \}. \end{array} \tag{3.4}$$

Now using $\mathfrak{BCT}(11)$ in the case when $(\overline{\nu}_{2a}, \overline{\nu}_{4a}) = (0, 1)$, by (2.2) we get

$$\begin{aligned} \mu_0(u, \chi_3) &= \tfrac{1}{8}(-8\nu_{2a} + 8) \geq 0; & \mu_4(u, \chi_3) &= \tfrac{1}{8}(8\nu_{2a} + 8) \geq 0; \\ \mu_1(u, \chi_3) &= \tfrac{1}{8}(4t + 12) \geq 0; & \mu_5(u, \chi_3) &= \tfrac{1}{8}(-4t + 12) \geq 0; \\ \mu_0(u, \chi_2) &= \tfrac{1}{8}(4v - 4w + 12) \geq 0; & \mu_4(u, \chi_2) &= \tfrac{1}{8}(-4v + 4w + 12) \geq 0; \\ \mu_0(u, \chi_5) &= \tfrac{1}{8}(4z - 4w + 12) \geq 0; & \mu_4(u, \chi_5) &= \tfrac{1}{8}(-4z + 4w + 12) \geq 0, \end{aligned}$$

where $t = \nu_{8a} - \nu_{8b}$, $z = 3\nu_{2a} - \nu_{4a}$, $v = \nu_{2a} + \nu_{4a}$ and $w = \nu_{8a} + \nu_{8b}$. From this it follows that $-1 \leq \nu_{2a} \leq 1$, $-8 \leq \nu_{2a} \leq 10$, $-4 \leq \nu_{8a}, \nu_{8b} \leq 4$, and, using Proposition 6, it is easy to check that this system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\begin{array}{lllll} \{ & (0,2,1,-2), & (0,2,-2,1), & (0,2,0,-1), & (0,2,-1,0), \\ & (0,0,1,0), & (0,0,2,-1), & (0,0,0,1), & (0,0,-1,2) \quad \}. \end{array} \tag{3.5}$$

In the case when $(\overline{\nu}_{2a}, \overline{\nu}_{4a}) = (2, -1)$, first using $\mathfrak{BCT}(11)$, by (2.2) we obtain $\mu_0(u, \chi_3) = -\mu_4(u, \chi_3) = -\nu_{2a} = 0$ and

$$\begin{aligned} \mu_1(u, \chi_3) &= \tfrac{1}{8}(4\nu_{8a} - 4\nu_{8b} + 12) \geq 0; \\ \mu_5(u, \chi_3) &= \tfrac{1}{8}(-4\nu_{8a} + 4\nu_{8b} + 12) \geq 0; \\ \mu_0(u, \chi_2) &= \tfrac{1}{8}(4\nu_{4a} - 4\nu_{8a} - 4\nu_{8b} + 12) \geq 0; \\ \mu_4(u, \chi_2) &= \tfrac{1}{8}(-4\nu_{4a} + 4\nu_{8a} + 4\nu_{8a} + 12) \geq 0; \\ \mu_0(u, \chi_5) &= \tfrac{1}{8}(-4\nu_{4a} - 4\nu_{8a} - 4\nu_{8a} + 28) \geq 0; \\ \mu_4(u, \chi_5) &= \tfrac{1}{8}(4\nu_{4a} + 4\nu_{8a} + 4\nu_{8a} + 28) \geq 0. \end{aligned}$$

It is easy to check that $-7 \leq \nu_{4a} \leq 9$, $-4 \leq \nu_{8a}, \nu_{8b} \leq 4$, and the system has the following integral solutions $(\nu_{2a}, \nu_{4a}, \nu_{8a}, \nu_{8b})$:

$$\begin{array}{lllll} \{ & (0,2,-1,0), & (0,0,-1,2), & (0,2,0,-1), & (0,0,0,1), \\ & (0,0,2,-1), & (0,2,-2,1), & (0,0,1,0), & (0,2,1,-2) \quad \}. \end{array} \tag{3.6}$$

It follows from (3.3)–(3.6) that the only four solutions which appear in both cases when $p = 3$ and $p = 11$ are the following ones: $\nu_{2a} = 0$ and

$$(\nu_{4a}, \nu_{8a}, \nu_{8b}) \in \{\, (0,0,1), (0,1,0), (2,-1,0), (2,0,-1) \,\}.$$

Again, we were not able to produce a contradiction computing all possible $\mu_i(u, \chi_j)$ for all above listed tuples $(\nu_{4a}, \nu_{8a}, \nu_{8b})$ for the ordinary character table of G as well as for $\mathfrak{BCT}(p)$, where $p \in \{3, 5, 11\}$.

- Let u be a unit of order 11. Then using $\mathfrak{BCT}(3)$, by (2.2) we have

$$\mu_1(u,\chi_2) = \tfrac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 5) \geq 0;$$
$$\mu_2(u,\chi_2) = \tfrac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 5) \geq 0,$$

which has only the following trivial solutions $(\nu_{11a}, \nu_{11b}) = \{(1,0),(0,1)\}$.

For all the above mentioned cases except elements of orders 4, 6 and 8 we see that there is precisely one conjugacy class with non-zero partial augmentation. Thus, by Proposition 5, part (ii) of the Theorem is proved.

It remains to prove parts (i) and (vi), considering units of $V(\mathbb{Z}G)$ of orders 10, 12, 15, 22, 24, 33 and 55. Now we treat each of these cases separately:

- Let u be a unit of order 10. Then by (2.1), (3.1) and Proposition 4 we get $\nu_{2a} + \nu_{5a} = 1$. Using $\mathfrak{BCT}(3)$, by (2.2) we have the system of inequalities

$$\mu_5(u,\chi_4) = \tfrac{1}{10}(-8\nu_{2a} + 8) \geq 0;$$
$$\mu_0(u,\chi_4) = \tfrac{1}{10}(8\nu_{2a} + 12) \geq 0;$$
$$\mu_2(u,\chi_2) = \tfrac{1}{10}(-\nu_{2a} + 6) \geq 0,$$

which has no integral solutions such that $\mu_5(u,\chi_4)$, $\mu_0(u,\chi_4)$, $\mu_2(u,\chi_2) \in \mathbb{Z}$.

- Let u be a unit of order 12. By (2.1), (3.1) and Proposition 4, we obtain that

$$\nu_{2a} + \nu_{3a} + \nu_{4a} + \nu_{6a} = 1.$$

Since $|u^2| = 6$ and $|u^3| = 4$, by (3.2) it yields that

$$\chi_j(u^2) = \overline{\nu}_{2a}\chi_j(2a) + \overline{\nu}_{3a}\chi_j(3a) + \overline{\nu}_{6a}\chi_j(6a)$$

and $\chi_j(u^3) = \widetilde{\nu}_{2a}\chi_j(2a) + \widetilde{\nu}_{4a}\chi_j(4a)$.

Consider the following four cases from parts (ii) and (iii) of the Theorem:

1. Let $(\overline{\nu}_{2a}, \overline{\nu}_{3a}, \overline{\nu}_{6a}) \in \{(0,0,1),(2,-3,2),(-2,3,0)\}$ and suppose that

$$(\widetilde{\nu}_{2a}, \widetilde{\nu}_{4a}) \in \{(0,1),(2,-1)\}.$$

Then by (2.2) we have $\mu_0(u,\chi_2) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction.

2. Let $(\overline{\nu}_{2a}, \overline{\nu}_{3a}, \overline{\nu}_{6a}) = (2,0,-1)$ and $(\widetilde{\nu}_{2a}, \widetilde{\nu}_{4a}) \in \{(0,1),(2,-1)\}$. Then by (2.2) we have $\mu_1(u,\chi_3) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction.

3. Let $(\overline{\nu}_{2a}, \overline{\nu}_{3a}, \overline{\nu}_{6a}) = (0,3,-2)$ and $(\widetilde{\nu}_{2a}, \widetilde{\nu}_{4a}) = (0,1)$. According to (2.2), $\mu_6(u,\chi_6) = -\mu_0(u,\chi_6) = \frac{2}{3}\nu_{3a} = 0$, so $\nu_{3a} = 0$, and we have the system

$$\mu_2(u,\chi_5) = \tfrac{1}{12}(6\nu_{2a} - 2\nu_{4a} + 8) \geq 0;$$
$$\mu_4(u,\chi_5) = \tfrac{1}{12}(-6\nu_{2a} + 2\nu_{4a} + 4) \geq 0;$$
$$\mu_2(u,\chi_3) = \tfrac{1}{12}(-4\nu_{2a} + 2\nu_{6a} + 6) \geq 0;$$
$$\mu_4(u,\chi_3) = \tfrac{1}{12}(4\nu_{2a} - 2\nu_{6a} + 6) \geq 0;$$
$$\mu_4(u,\chi_2) = \tfrac{1}{12}(-4\nu_{2a} - 4\nu_{4a} + 2\nu_{6a} + 10) \geq 0;$$
$$\mu_2(u,\chi_2) = \tfrac{1}{12}(4\nu_{2a} + 4\nu_{4a} - 2\nu_{6a} + 2) \geq 0,$$

that has only two solutions $\{\,(-1,0,1,1),\ (1,0,1,-1)\,\}$ with $\mu_i(u,\chi_j) \in \mathbb{Z}$.

4. Let $(\overline{\nu}_{2a}, \overline{\nu}_{3a}, \overline{\nu}_{6a}) = (0, 3, -2)$ and $(\widetilde{\nu}_{2a}, \widetilde{\nu}_{4a}) = (2, -1)$. Using (2.2), we get $\mu_6(u, \chi_6) = -\mu_0(u, \chi_6) = \frac{2}{3}\nu_{3a} = 0$, so $\nu_{3a} = 0$. Put $t = 2\nu_{2a} - \nu_{6a}$. Then by (2.2)

$$\mu_0(u, \chi_3) = \tfrac{1}{12}(-4t + 4) \geq 0; \qquad \mu_4(u, \chi_3) = \tfrac{1}{12}(2t - 2) \geq 0,$$

so $2\nu_{2a} - \nu_{6a} = 1$. Now by (2.2) we have

$$\mu_2(u, \chi_5) = \tfrac{1}{12}(2(3\nu_{2a} - \nu_{4a}) - 8) \geq 0,$$
$$\mu_0(u, \chi_9) = \tfrac{1}{12}(-4(3\nu_{2a} - \nu_{4a}) + 28) \geq 0,$$

and $3\nu_{2a} - \nu_{4a} = 4$. Using (2.1) we obtain that $\nu_{2a} = \nu_{4a} = -\nu_{6a} = 1$. Finally, $\mu_4(u, \chi_9) = \frac{1}{12}(6\nu_{2a} - 2\nu_{4a} + 28) = \frac{8}{3} \notin \mathbb{Z}$, a contradiction. Thus, part (vi) of the Theorem is proved.

- Let u be a unit of order 15. Then by (2.1) and (3.1) we have $\nu_{3a} + \nu_{5a} = 1$. Now using the character table of G, by (2.2) we get the system of inequalities

$$\mu_0(u, \chi_2) = \tfrac{1}{15}(8\nu_{3a} + 12) \geq 0; \quad \mu_5(u, \chi_2) = \tfrac{1}{15}(-4\nu_{3a} + 9) \geq 0,$$

that has no integral solutions such that $\mu_0(u, \chi_2),\ \mu_5(u, \chi_2) \in \mathbb{Z}$.

- Let u be a unit of order 22. Then by (2.1), (3.1) and Proposition 4 we obtain that

$$\nu_{2a} + \nu_{11a} + \nu_{11b} = 1.$$

In (2.2) we need to consider two cases: $\chi(u^2) = \chi(11a)$ and $\chi(u^2) = \chi(11b)$, but in both cases by (2.2) we have

$$\mu_0(u, \chi_2) = -\mu_{11}(u, \chi_2) = \tfrac{1}{22}(20\nu_{2a} - 10\nu_{11a} - 10\nu_{11b} + 2) = 0.$$

It yields $10\nu_{2a} - 5\nu_{11a} - 5\nu_{11b} = -1$, that has no integral solutions.

- Let u be a unit of order 24. Then by (2.1) and (3.1) we have

$$\nu_{2a} + \nu_{3a} + \nu_{4a} + \nu_{6a} + \nu_{8a} + \nu_{8b} = 1.$$

Since $|u^2| = 12$, $|u^4| = 6$, $|u^3| = 8$, $|u^6| = 4$, and G has two conjugacy classes of elements of order 8, we need to consider 40 various cases defined by parts (iii)–(vi) of the Theorem.

Let $(\overline{\nu}_{2a}, \overline{\nu}_{3a}, \overline{\nu}_{6a}) \in \{\, (0, 3, -2), (2, 0, -1)\,\}$, where

$$\chi_j(u^4) = \overline{\nu}_{2a}\chi_j(2a) + \overline{\nu}_{3a}\chi_j(3a) + \overline{\nu}_{6a}\chi_j(6a).$$

According to (2.2) we have that $\mu_1(u, \chi_2) = \frac{1}{2} \notin \mathbb{Z}$, a contradiction. In the remaining cases similarly we obtain that $\mu_1(u, \chi_2) = \frac{1}{4} \notin \mathbb{Z}$, which is also a contradiction.

- Let u be a unit of order 33. Then by (2.1) and (3.1) we have $\nu_{3a} + \nu_{11a} + \nu_{11b} = 1$. Again, in (2.2) we need to consider two cases: $\chi(u^3) = \chi(11a)$ and $\chi(u^3) = \chi(11b)$, but both cases lead us to the same system of inequalities

$$\mu_1(u, \chi_5) = \tfrac{1}{33}(2\nu_{3a} + 9) \geq 0; \quad \mu_{11}(u, \chi_5) = \tfrac{1}{33}(-20\nu_{3a} + 9) \geq 0,$$

which has no integral solutions with $\mu_1(u, \chi_5),\ \mu_{11}(u, \chi_5) \in \mathbb{Z}$.

- Let u be a unit of order 55. Then by (2.1) and (3.1) we have $\nu_{5a}+\nu_{11a}+\nu_{11b}=1$. Considering two cases when $\chi(u^5)=\chi(11a)$ and $\chi(u^5)=\chi(11b)$, we get the same systems of inequalities

$$\mu_5(u,\chi_8)=\tfrac{1}{55}(4\nu_{5a}+40)\geq 0;\quad \mu_5(u,\chi_5)=\tfrac{1}{55}(-4\nu_{5a}+15)\geq 0,$$

  which also has no integral solutions such that $\mu_5(u,\chi_5),\ \mu_5(u,\chi_8)\in\mathbb{Z}$.

Thus, the theorem is proved, and now the corollary follows immediately.

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