MAGIC MIRRORS, DENSE DIAMETERS, BAIRE CATEGORY

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Abstract. An old result of Zamfirescu says that for most convex curves C in the plane most points in $\mathbb{R}^2$ lie on infinitely many normals to C, where most is meant in Baire category sense. We strengthen this result by showing that 'infinitely many' can be replaced by 'contiunuum many' in the statement. We present further theorems in the same spirit.

1. Introduction

In a 1982 paper [5] Tudor Zamfirescu proved a remarkable result saying that 'most mirrors are magic'. For the mathematical formulation let $\mathcal{C}$ be the set of all closed convex curves in the plane $\mathbb{R}^2$. Fix some $C \in \mathcal{C}$ and $z \in C$ so that the tangent line, $T(z)$, to C at z is unique, then so is the normal line $N(z)$ to C at z. A point $u \in \mathbb{R}^2$ *sees an image* of another point $v \in \mathbb{R}^2$ via z if u and v and C lie on the same side of $T(z)$ and the line $N(z)$ halves the angle $\angle uzv$. In particular, u sees an image of itself via z if $u \in N(z)$ and u and C are on the same side of $T(z)$.

With the Haussdorf metric $\mathcal{C}$ becomes a complete metric space. It is well-known that the normal $N(z)$ is unique at every point $z \in C$ for most convex curves $C \in \mathcal{C}$ in the Baire category sense, that is, for the elements of a comeagre set of curves in $\mathcal{C}$. Now the 'most mirrors are magic' statement is, precisely, that for most convex curves, most points in $\mathbb{R}^2$ (again in Baire category sense) see infinitely many images of themselves. Another theorem from [5] says that for most convex curves, most points in $\mathbb{R}^2$ see infinitely many images of any given point $v \in \mathbb{R}^2$. Zamfirescu actually proves the existence of countably many images and self-images.

The purpose of this paper is to show that most mirrors are even more magic.

Theorem 1.1. *For most convex curves, most points in $\mathbb{R}^2$ see continuum many images of themselves.*

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Theorem 1.2. *For most convex curves C and for every point $v \in \mathbb{R}^2 \setminus C$, most points in $\mathbb{R}^2$ see continuum many images of v.*

The condition $v \notin C$ in the last theorem is used to avoid some trivial complications in the proof. The statement holds even for $v \in C$.

Remark. Let C^o denote the closed convex set whose boundary is C. The above definition of 'u sees an image of v via $z \in C$' means that the mirror side of C is the interior one, that is, the segment uz intersects the interior of C^o. Theorem 1.1 does not hold when the mirror is on the other side of C because every point in $\mathbb{R}^2 \setminus C^o$ lies on exactly one outer normal halfline to C.

A statement, analogous to Theorem 1.2 about affine diameters was proved in [2] in 1990, for typical d-dimensional convex bodies for every $d \geq 2$. The segment $[a, b]$ is an *affine diameter* of $C \in \mathcal{C}$ if there are distinct and parallel tangent lines to $a, b \in C$. The result in [2] says that for most convex curves $C \in \mathcal{C}$, most points on a fixed affine diameter of C are contained in infinitely many affine diameters of C. In this case again we show the existence of continuum many diameters passing through most points in C^o.

Theorem 1.3. *For most convex curves $C \in \mathcal{C}$, most points in C^o lie in continuum many diameters of C.*

Note that every point outside C^o lies on the line of at most one affine diameter as any two affine diameters have a point in common. It is not hard to see, actually, that every point outside C lies on a unique affine diameter.

2. Plan of proof

For $C \in \mathcal{C}$ let $\rho(z)$ denote the radius of curvature of C at $z \in C$. Let $\mathcal{D}$ denote the family of all convex curves $C \in \mathcal{C}$ such that

(1) there is a unique tangent line to C at every $z \in C$,
(2) $\{z \in C : \rho(z) = 0\}$ is dense in C,
(3) $\{z \in C : \rho(z) = \infty\}$ is dense in C.

It is well-known, see for instance [6], that $\mathcal{D}$ is comeagre in $\mathcal{C}$. We are going to show that every $C \in \mathcal{D}$ has the property required in Theorem 1.1. We will need slightly different conditions for Theorems 1.2 and 1.3. But the basic steps of the proofs are the same. We explain them in this section in the case of Theorem 1.1.

Let $C \in \mathcal{D}$ and define, for $z \in C$, the halfline $N^+(z) \subset N(z)$ that starts at z and intersect the interior of C^o. Note that every $u \in \mathbb{R}^2$ lies on some $N^+(z)$: namely when the farthest point from u on C is z. Set $L(u) = \{z \in C : u \in N^+(z)\}$ and define

$$H = \{u \in \mathbb{R}^2 : L(u) \text{ is not perfect}\}.$$

Lemma 2.1. *H is a Borel set.*

Write now $u = (u_1, u_2) \in \mathbb{R}^2$ and define $H^{u_2} = \{u_1 \in \mathbb{R} : (u_1, u_2) \in H\}$. This is just the section of H on the horizontal line $\ell(u_2) = \{(x, y) \in \mathbb{R}^2 : y = u_2\}$. There are two points $z \in C$ with $N(z)$ horizontal, so there are at most two exceptional values for u_2 where $\ell(u_2)$ coincides with some $N(z)$.

Lemma 2.2. *Apart from those exceptional values, H^{u_2} is meagre.*

These two lemmas imply Theorem 1.1. Indeed, deleting the (one or two) exceptional lines from H gives a Borel set H'. According to a theorem of Kuratowski (see [3] page 53), if all horizontal sections of the Borel set H' are meagre, then so is H', and then H itself is meagre. So its complement is comeagre, so $L(u)$ is perfect and non-empty for most $u \in \mathbb{R}^2$. The theorem follows now from the fact that a non-empty and perfect set has continuum many points. The proofs of Theorems 1.2 and 1.3 will use the same argument.

For the proof of Lemma 2.2 we need another lemma that appeared first as Lemma 2 in [4]. A function $g : [0, 1] \to \mathbb{R}^2$ is increasing on an interval $I \subset [0, 1]$ (resp. decreasing on I) if every $x, y \in I$ with $x \le y$ satisfy $g(x) \le g(y)$ (resp. $g(x) \ge g(y)$), and g is monotone in I if it is either increasing or decreasing there. For the sake of completeness we present the short proof.

Lemma 2.3. *Assume $g : [0, 1] \to \mathbb{R}^2$ is continuous and is not monotone in any subinterval of $[0, 1]$. Then the set*

$$B = \{b \in \mathbb{R} : \{x : g(x) = b\} \text{ is not perfect}\}$$

is meagre.

Proof of Lemma 2.3. For each $b \in B$ the level set $\{x : g(x) = b\}$ has an isolated point, and so there is an open interval $I_b \subset [0, 1]$ with rational endpoints in which $g(x) = b$ has a unique solution. For a given rational interval (p, q) define

$$B(p, q) = \{b \in B : I_b = (p, q)\}.$$

If every $B(p, q)$ is nowhere dense, then we are done since B, as a countable union of nowhere dense sets, is meagre. If some $B(p, q)$ is not nowhere dense, then there is a non-empty open interval I in which $B(p, q)$ is dense. The line $y = b$, for a dense subset of I, intersects the graph of g restricted to (p, q) in a single point. This implies easily that g is strictly monotone in a subinterval (p, q), contrary to our assumption. □

3. Proof of the lemmas

Fix $C \in \mathcal{D}$ and let $z(\alpha)$ denote the point $z \in C$ where the halfline $N^+(z)$ spans angle $\alpha \in [0, 2\pi)$ with a fixed unit vector in $\mathbb{R}^2$. This is a parametrization of C with $\alpha \in [0, 2\pi]$ and $z(0) = z(2\pi)$. We write

$C_{\alpha,\beta}$ for the arc $\{z(\gamma) : \alpha < \gamma < \beta\}$ when $0 \le \alpha < \beta \le 2\pi$, and the definition is extended, naturally, to the case when $\alpha < 2\pi < \beta$. We always assume that α, β are rational and $\beta - \alpha$ is small, smaller than 0.1, say.

Proof of Lemma 2.1 . Note first that the set

$$K = \{(u, z) \in \mathbb{R}^2 \times C : u \in N^+(z)\}$$

is closed. Further, $L(u)$ is not perfect if and only if there is a short arc $C_{\alpha,\beta}$ such that $u \in N^+(z)$ for a unique $z \in C_{\alpha,\beta}$. Thus

$$H = \bigcup_{\text{all } C_{\alpha,\beta}} \{u \in \mathbb{R}^2 : u \in N^+(z) \text{ for a unique } z \in C_{\alpha,\beta}\}.$$

Let $p : K \to \mathbb{R}^2$ be the projection $p(u, z) = u$. Let $P_{\alpha,\beta}$ be the set of points $u \in \mathbb{R}^2$ such that there are more than one $z \in C_{\alpha,\beta}$ with $u \in N^+(z)$. Then

$$P_{\alpha,\beta} = \bigcup_{\gamma} p(K \cap (\mathbb{R}^2 \times C_{\alpha,\gamma})) \cap p(K \cap (\mathbb{R}^2 \times C_{\gamma,\beta}))$$

where the union is taken over all rational γ with $\alpha < \gamma < \beta$. Consequently

$$H = \bigcup_{\text{all } C_{\alpha,\beta}} p(K \cap (\mathbb{R}^2 \times C_{\alpha,\beta})) \setminus P_{\alpha,\beta}.$$

Since $p(K \cap (\mathbb{R}^2 \times C_{\alpha,\beta}))$ is F_σ for every $\alpha < \beta$, it follows that H is indeed Borel. □

Proof of Lemma 2.2. The set $z \in C$ where $N^+(z)$ intersects $\ell(u_2)$ in a single point consists of one or two open subarcs of C, as one can check easily. Let C_1 be such an arc. It suffices to see that

$$E = H^{u_2} \cap \{u_1 \in \mathbb{R} : (u_1, u_2) = \ell(u_2) \cap N(z) \text{ for some } z \in C_1\}$$

is meagre, as H^{u_2} either coincides with this set, or is the union of two such sets.

We may assume that C_1 is the graph of a convex function $F : J \to \mathbb{R}$ and $u_2 > F(x)$ on J where J is an open interval. (This position can be reached after a suitable reflection about a horizontal line.) With this notation, E is the set of real numbers $u_1 \in \mathbb{R}$ such that the set of points $x \in J$ for which $(u_1, u_2) \in N^+(x, F(x))$ is not perfect.

Then $F'(x) = f(x)$ is continuous and increasing on J. Each $z \in C_1$ is a point $(x, F(x))$ on the graph of F. As $\rho(z) = (1+f(x))^{3/2}/f'(x)$, f' equals zero resp. infinity on a dense set in J. The normal $N(z)$ to $z = (x, F(x))$ has equation $(u_2 - F(x))f(x) = x - u_1$, as one checks readily. With $g(x) = (u_2 - F(x))f(x) - x$, $g'(x) = -f(x)^2 + (u_2 - F(x))f'(x) - 1$ and so on a dense set in J the value of $g'(x)$ is positive, and on another dense set in J it is negative. So g is not monotone in any subinterval of J. Lemma 2.3 implies now that E is meagre. □

4. Proof of Theorem 1.2

It is known [6] that for most $C \in \mathcal{D}$ there is a dense set $E \subset C$ such that at each point $z \in E$ the lower curvatures of radii in both directions $\rho_i^+(z), \rho_i^-(z)$ vanish and the upper curvatures of radii in both directions $\rho_s^+(z), \rho_s^-(z)$ are infinite. We let $\mathcal{D}_1$ denote the set of all $C \in \mathcal{D}$ possessing such a dense set E. We are going to show that for each $C \in \mathcal{D}_1$, most points see continuum many images of any given point $v \in \mathbb{R}^2$, $v \notin C$.

For $z \in C$ we define the line $R(z)$ as the reflected copy (with respect to $N(z)$) of the line through v and z. Note that $R(z)$ depends continuously from z. Here we need $v \notin C$.

If u sees an image of v via z, then $u \in R(z)$. More precisely, u sees an image of v via z iff u, v and C are on the same side of $T(z)$ and $u \in R(z)$. Let $R^+(z) \subset R(z)$ be the halfline that starts at z and is on the same side of $T(z)$ as C. Also, $R^+(z)$ is well defined for all $z \in C$.

As before, $\ell(u_2)$ is the horizontal line in $\mathbb{R}^2$ whose points have second coordinate equal to u_2. Define, for fixed $u_2 \in \mathbb{R}$, $H^{u_2} = \{u_1 \in \mathbb{R} : (u_1, u_2) \in H\}$. This is the same as the set of first coordinates of all $u \in H \cap \ell(u_2)$.

In the generic case $R(z)$ is not horizontal and so $R(z) \cap \ell(u_2)$ is a single point. But we have to deal with non-generic situations, that is, when $R(z)$ is horizontal and so coincides with $\ell(u_2)$ for some $u_2 \in \mathbb{R}$. Define $Z = \{z \in C : R(z) \text{ is horizontal}\}$ and $U_2 = \{u_2 \in \mathbb{R} : \ell(u_2) = R(z) \text{ for some } z \in Z\}$. Both Z and U_2 are closed sets and there is a one-to-one correspondence between them given by $z \leftrightarrow u_2$ iff $R(z) = \ell(u_2)$.

From now on we assume that Z is nowhere dense. We will justify this assumption at the end of the proof. Then U_2 is also nowhere dense. $C \setminus Z$ is open in C and so its connected components $C_1, C_2, \ldots$ are open arcs in C, and there are at most countably many of them.

This time we define $L(u, C_i)$ as the set of $z \in C_i$ via which u sees an image of v. Formally, $L(u, C_i) = \{z \in C_i : u \in R^+(z)\}$, and define again, for fixed $u_2 \in \mathbb{R}$,

$$H_i^{u_2} = \{u_1 \in \mathbb{R} : L((u_1, u_2), C_i) \text{ is not perfect}\}.$$

A very similar proof shows that $H_i^{u_2}$ is Borel. We omit the details, but mention that the condition $v \notin C$ is needed to show that the corresponding $K = \{(u, z) : \ldots\}$ is closed.

Lemma 4.1. *For $u_2 \notin U_2$ the set $H_i^{u_2}$ is meagre.*

Proof. With every $u_1 \in H_i^{u_2}$ we associate a (rational) open arc $C_{\alpha,\beta}$ of C_i such that $u = (u_1, u_2) \in R(z)$ for a unique $z \in C_{\alpha,\beta}$, namely for z_u. If the set of $u \in H_i^{u_2}$ that are associated with $C_{\alpha,\beta}$ is nowhere dense for every rational arc $C_{\alpha,\beta}$, then we are done as $H_i^{u_2}$ is the countable union of nowhere dense sets. So suppose that it is not nowhere dense

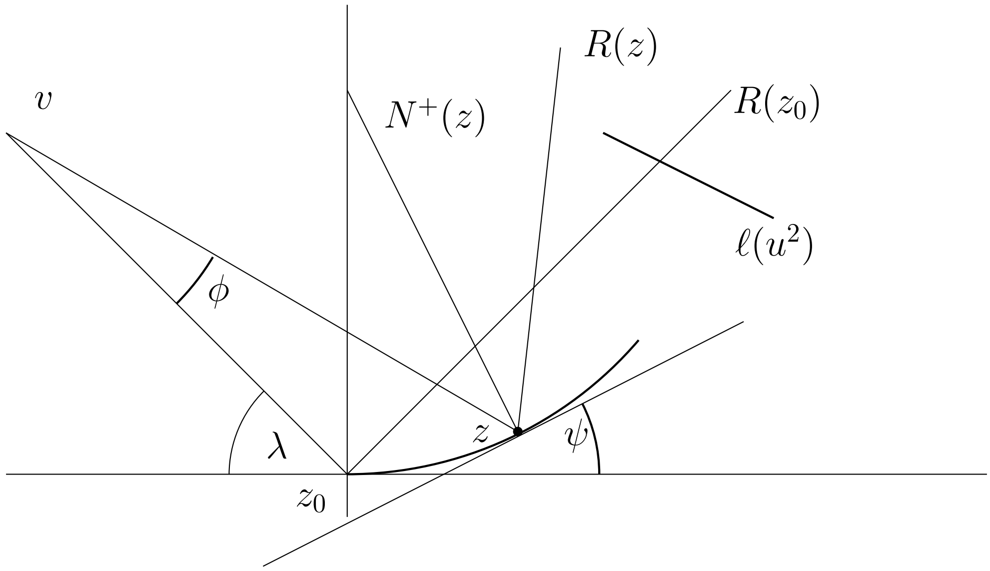

FIGURE 1. Theorem 1.2

for some $C_{\alpha,\beta}$. Then there is an open interval $I \in \mathbb{R}$ such that $H_i^{u_2}$ is dense in I.

Choose two distinct points w^-, w^+ from $I \cap H_i^{u_2}$. Then $z_{(w^-,u_2)}$ and $z_{(w^+,u_2)}$ are distinct points and so they are the endpoints of an open subarc $C_{\gamma,\delta}$ of $C_{\alpha,\beta}$. Define the map $h : C_{\gamma,\delta} \to I$ by $h(z) = u_1$ when $(u_1, u_2) = \ell(u_2) \cap R(z)$; h is clearly continuous. It is also monotone because its inverse is well-defined on a dense subset I.

We show next that this is impossible. Choose $z_0 \in C_{\gamma,\delta} \cap E$ (recall that E is dense in C).

We fix a new coordinate system in $\mathbb{R}^2$: the origin coincides with z_0, the x axis with $T(z_0)$, the tangent line to C at z_0, and the y axis is $N(z_0)$; see the figure. We assume w.l.o.g. that $v_1 < 0$ and $v_2 > 0$ where $v = (v_1, v_2)$. A subarc of $C_{\gamma,\delta}$ is the graph of a non-negative convex function $F : [0, \Delta) \to \mathbb{R}$ such that $F(0) = 0$ and $z = z(x) = (x, F(x))$ and $f(x) = F'(x)$ is an increasing function with $f(0) = 0$. If the lines $R(z(x))$ and $R(z(0))$ intersect, then they intersect in a single point whose y component is denoted by $y(x)$.

Claim 4.2. *For every $\varepsilon > 0$ there are $x_1, x_2 \in (0, \varepsilon)$ so that $y(x_1) < 0$ and $0 < y(x_2) < \varepsilon$.*

Proof. We use the notation of the figure. The sine theorem for the triangle with vertices $v, 0, z(x)$ implies that $\phi(x) = x \sin\lambda/|v|(1+o(1))$ where $o(1)$ is understood when $x \to 0$. The slope of the line $R(z(x))$ is $\tan(\lambda - \phi + 2\psi)$, and

$$\tan\psi(x) = f(x) = x \cdot \frac{f(x) - 0}{x - 0}.$$

The liminf and limsup of the last fraction (when $x \to 0$) is the curvature $\rho_i^+(z_0) = 0$ and $\rho_s^+(z_0) = \infty$ of C at z_0 as $z_0 \in E$. Consequently for every integer $n > 1$ there is $x \in (0, 1/n)$ with $\tan \psi(x) < x/n$ and also with $\tan \psi(x) > nx$. Then there is $x_1 < 1/n$ such that $\lambda/2 < \lambda - \phi(x_1) + 2\psi(x_1) < \lambda$ which implies, after a simple checking, that $y(x_1) < 0$. Also, there is $x_2 < 1/n$ such that $\lambda - \phi(x_2) + 2\psi(x_2) > \lambda + nx_2/2$. A direct computation shows then that $0 < y(x_2) < \varepsilon$ if n is chosen large enough. □

We return to the proof of Lemma 4.1. The claim shows that there are $x_1, x_2, x_3 \in (0, \Delta)$ with $x_1 < x_2 < x_3$ such that the line $R(z(x_1))$ and $R(z(x_3))$ strictly separate the origin and the point $R(z_0) \cap \ell(u_2)$ while $R(z(x_2))$ does not. Writing $z_i = z(x_i), i = 1, 2, 3$ this implies that z_2 is between z_1 and z_3 while $h(z_2)$ is not on the segment $(h(z_1), h(z_3))$. So h is not monotone. □

It is evident that U_2, and consequently U, is closed and nowhere dense, so U is meagre. The lemma implies, by Kuratowski's theorem, that $H_i \setminus U$ is meagre. It follows that H_i is meagre and then so is $H = \bigcup_i H_i$. Thus every point in the complement of H sees an image of v via a perfect set in C, except possibly for the points of the meagre set U. This perfect set is nonempty, because every point sees an image of v via some $z \in C$ (for instance by Zamfirescu's result [5, Theorem 1]). So most points see continuum many images of v.

Finally we justify the assumption that Z is nowhere dense. This is done by choosing the horizontal direction (which is at our liberty) suitably. So for a given direction $(\cos \theta, \sin \theta)$ write $Z(\theta)$ for the set of $z \in C$ such that $R(z)$ is parallel with this direction. Every $Z(\theta)$ is closed and so there is one (actually, many) among them that contains no non-empty open arc of C. Choose the corresponding θ for the horizontal direction, then $Z = Z(\theta)$ is nowhere dense. □

5. Proof of Theorem 1.3

Write $\mathcal{C}_1$ for the set of all convex curves C that have a unique tangent at every $z \in C$. Assume $C \in \mathcal{C}_1$ and use the parametrization $z : [0, 2\pi) \to C$ as before. For $z \in C$ with $z = z(\alpha)$ let $z^* \in C$ be the opposite point, that is $z^* = z(\alpha+\pi)$. It is evident that $z^{**} = z$. Further, $[z, z^*]$ is always an affine diameter of C and all affine diameters of C are of this form. We need a geometric lemma.

Lemma 5.1. *Most convex curves $C \in \mathcal{C}_1$ have the following property: for every $\varepsilon > 0$ every subarc C_0 of C contains points x, y such that*

$$\frac{|x - y|}{|x^* - y^*|} < \varepsilon.$$

The lemma follows from a result in [1], we give a separate proof in the next section. From now on we assume that $C \in \mathcal{C}_1$ has the property in the lemma.

We use again the same proof scheme: for $u \in C^o$ define $L(u) = \{z \in C : u \in [z, z^*]\}$; this set is nonempty as one can check easily that every point $u \in C^o$ lies on at least one affine diameter. (This holds for every convex curve, not only for the ones in $\mathcal{C}_1$.) We set next $H = \{u \in C^o : L(u) \text{ is not perfect}\}$, and, for fixed $u_2 \in \mathbb{R}^2$, $H^{u_2} = H \cap \ell(u_2)$. The same proof as in Section 3 shows that H is Borel. We claim that H is meagre which implies Theorem 1.3.

C has a horizontal affine diameter and we assume w.l.o.g. that it lies on the line $\ell(0)$. To see that H is meagre it suffices to show (by Kuratowski's theorem) that H^{u_2} is meagre as a subset of $\ell(u_2)$ for $u_2 \neq 0$. We only consider $u_2 \in \mathbb{R}$, $u_2 \neq 0$ with $\ell(u_2) \cap C \neq \emptyset$. With each $u \in H^{u_2}$ we associate an isolated point $z_u \in C$ and a short rational arc $C_{\alpha,\beta}$ such that z_u is the unique $z \in C_{\alpha,\beta}$ with $u \in [z, z^*]$. We are done if, for each short rational arc $C_{\alpha,\beta}$, the set of $u \in H^{u_2}$ that are associated with $C_{\alpha,\beta}$ is nowhere dense. So suppose that this fails for some $C_{\alpha,\beta}$. Then there is an open interval $I \subset \ell(u_2)$ on which H^{u_2} is dense. Choose distinct points u^- and u^+ from $I \cap H^{u_2}$ and let z^-, z^+ be the corresponding isolated points on $C_{\alpha,\beta}$. We suppose (by symmetry) that $C_{\alpha,\beta}$ is below the line $\ell(u_2)$.

From now on we consider the subarc $C_0 \subset C_{\alpha,\beta}$ whose endpoints are z^- and z^+ and its opposite arc C_0^*. We note here that the map $z \to z^*$ is order preserving on C_0, meaning that if $v \in C_0$ is between $v_1, v_2 \in C_0$, then v^* lies between v_1^* and v_2^* on C_0^*.

Define a map $m : C_0 \to \ell(u_2)$ via $m(z) = \ell(u_2) \cap [z, z^*]$; m is continuous. It is one-to-one on a dense subset of C_0 which implies that m is order-preserving in the sense that if $v \in C_0$ is between $v_1, v_2 \in C_0$, then $m(v)$ lies between $m(v_1)$ and $m(v_2)$ on $\ell(u_2)$. We show that this is impossible.

Using Lemma 5.1 choose two points v_1, v_2 on C_0 very close to each other so that $|v_1 - v_2|$ is much shorter than $|v_1^* - v_2^*|$. Then the segment $[v_1, v_2]$ is almost parallel with $[v_1^*, v_2^*]$, and the diameters $[v_1, v_1^*]$ and $[v_2, v_2^*]$ intersect in a point very close to $[v_1, v_2]$, so this point is below $\ell(u_2)$. Now apply Lemma 5.1 on the arc between v_1^* and v_2^*. We get points w_1 and w_2 very close to each other on this arc so that $|w_1 - w_2|$ is much shorter than $|w_1^* - w_2^*|$. This time the diameters $[w_1, w_1^*]$ and $[w_2, w_2^*]$ intersect above $\ell(u_2)$. We assume (by choosing the names w_1, w_2 properly) that v_1^*, w_1, w_2, v_2^* come in this order on C_0^* and so v_1, w_1^*, w_2^*, v_2 come in this order on C_0. The order of their m-images on $\ell(u_2)$ is $m(v_1), m(w_2^*), m(w_1^*), m(v_2)$. Thus indeed, m is not order preserving. $\square$

6. Proof of Lemma 5.1

Given $C \in \mathcal{C}_1$ define $A_{k,n}$ as the short arc between $z_k = z(2\pi k/2n)$ and $z_{k+1} = z(2\pi(k+1)/2n)$ where $k = 0, 1, \ldots, 2n-1$. For positive integers n, m let $\mathcal{F}_{n,m}$ be the set of all $C \in \mathcal{C}_1$ for which there is $A_{k,n}$ such that for all $x, y \in A_{k,n}$ $(x \neq y)$

$$\frac{|x-y|}{|x^*-y^*|} \geq \frac{1}{m}.$$

It is easy to see that $\mathcal{F}_{n,m}$ is closed in $\mathcal{C}_1$, we omit the details. We show next that it is nowhere dense.

Fix a $C \in \mathcal{C}_1$ and $\varepsilon > 0$ and let $U(C)$ denote the ε-neighbourhood of C. We construct another convex curve $\Gamma \in \mathcal{C}_1$ that is contained in $U(C)$ but is not an element of $\mathcal{F}_{n,m}$. Fix $k \in \{0, 1, \ldots, n-1\}$ and consider a fixed arc $A_{k,n}$ and its opposite arc $A^*_{k,n} = A_{k+n,n}$. Let T_k be the tangent line to C at $z((k+\frac{1}{2})\pi/n)$ and T^*_k be the parallel tangent line at $z((k+n+\frac{1}{2})\pi/n)$. Translate T^*_k a little so that the translated copy intersects C in two points x_1, y_1 and the segment $[x_1, y_1]$ lies in $U(C)$ and is much shorter than $[z_{k+n}, z_{k+1+n}]$. Similarly translate T_k a little so that the translated copy intersects C in x_2, y_2 and $[x_2, y_2]$ lies in $U(C)$, and is much shorter than $[z_k, z_{k+1}]$ and, most importantly, it is much shorter than $[x_1, y_1]$, namely, $m|x_2 - y_2| < |x_1 - y_1|$. This is clearly possible.

Now we choose points w_1 resp. w_2 from the caps cut off from C^o by the segment $[x_1, y_1]$ and $[x_2, y_2]$ so that, for $i = 1, 2$, the triangles $\triangle_i = \text{conv}\{x_i, y_i, w_i\}$ are homothetic. This is possible again. Note that $[x_1, w_1]$ and $[x_2, w_2]$ are parallel, and so are $[y_1, w_1]$ and $[y_2, w_2]$.

The next target is construct a convex curve Γ_k from z_k to z_{k+1} going through x_2 and y_2 that lies in $U(C)$, has a unique tangent at every point, and this tangent coincides with the line through x_2, w_2 at x_2 and with the line through y_2, w_2 at y_2. Also, an analogous curve Γ_{k+n} is needed from z_{k+n} to z_{k+1+n}.

This is quite easy. The unique parabola arc connecting x_2 to y_2 within $\triangle_2$ that touches the sides $[x_2, w_2]$ at x_2 and $[w_2, y_2]$ at y_2 is the middle piece of Γ_k. To connect this arc by a convex curve to z_k (say) within $U(C)$ choose a point $w \in C$ on the arc between z_k and y_2 so close to y_2 that the triangle $\triangle$ delimited by $T(z)$, the line through y_2, w_2, and the segment $[y_2, z]$ lies in $U(C)$. The analogous parabola arc in $\triangle$ gives the next piece of Γ_k, and then add to this piece the subarc of C between w and z_k. The middle piece of Γ_k is continued to z_{k+1} the same way.

The convex curve Γ_{k+n} connecting z_{k+n} to z_{k+1+n} is constructed the same way. Note that the tangents to Γ_k at x_2 (resp. y_2) are parallel with the tangents to Γ_{k+n} at x_1 (and y_1).

The curves Γ_k for $k = 0, \ldots, 2n-1$ together form a convex curve $\Gamma \in C_1$. It has parallel tangents at $x_1 \in \Gamma_{k+n}$ and $x_2 \in \Gamma_k$, and also at y_1 and y_2. Thus $[x_1, x_2]$ and $[y_1, y_2]$ are affine diameters of Γ and $m|x_1 - y_1| < |x_2 - y_2|$. As this holds for every k, $\Gamma \notin \mathcal{F}_{n,m}$. Thus $\mathcal{F}_{n,m}$ is indeed nowhere dense.

It follows that $\mathcal{C}_2 = \mathcal{C}_1 \setminus \bigcup_{n,m} \mathcal{F}_{n,m}$ is comeagre in $\mathcal{C}_1$. We show next that every $C \in \mathcal{C}_2$ satisfies the requirement of the lemma. So we are given $\varepsilon > 0$ and a short subarc C_0 of C. Take a positive integer m with $1/m < \varepsilon$. For a suitably large n, C_0 contains an arc of the form $A_{k,n}$. As $C \notin \mathcal{F}_{n,m}$, there are distinct points $x, y \in A_{k,n}$ with

$$\frac{|x - y|}{|x^* - y^*|} \leq \frac{1}{m} < \varepsilon.$$

This finishes the proof. □

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