

ALGEBRAIC VERTICES OF NON-CONVEX POLYHEDRA

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ABSTRACT. In this article we define an **algebraic vertex** of a generalized polyhedron and show that the set of algebraic vertices is the smallest set of points needed to define the polyhedron. We prove that the indicator function of a generalized polytope P is a linear combination of indicator functions of simplices whose vertices are algebraic vertices of P . We also show that the indicator function of any generalized polyhedron is a linear combination, with integer coefficients, of indicator functions of cones with apices at algebraic vertices and line-cones.

The concept of an algebraic vertex is closely related to the Fourier–Laplace transform. We show that a point \mathbf{v} is an algebraic vertex of a generalized polyhedron P if and only if the tangent cone of P , at \mathbf{v} , has non-zero Fourier–Laplace transform.

1. INTRODUCTION

We study the vertices of **non-convex polyhedra**, which we also call **generalized polyhedra**, and which we define as the finite union of convex polyhedra in \mathbf{R}^d .

There are many different ways to define a vertex of a generalized polyhedron P , most of them based on properties of the tangent cone to P at a point $\mathbf{v} \in P$. The tangent cone at \mathbf{v} , which we write as $\text{tcone}(P, \mathbf{v})$, is intuitively the collection of all directions that we can ‘see’ if we stand at \mathbf{v} and look into P (see Section 2 for a rigorous definition). We furthermore define a **line-cone** to be a cone that is the union of parallel lines.

One approach is to say that a point \mathbf{v} is a vertex of a generalized polyhedron P if its tangent cone is not a **line-cone**. We call such a point \mathbf{v} a **geometric vertex** of a generalized polyhedron, and it is clear that for a *convex* polyhedron this definition coincides with the usual definition of vertices (see the last Section). In this article we focus on another definition of vertices.

Definition 1. For a generalized polyhedron P , a point $\mathbf{v} \in P$ is called an **algebraic vertex** of P if the indicator function of its tangent cone $\text{tcone}(P, \mathbf{v})$ cannot be represented (up to a set of measure zero) as a linear combination of indicator functions of line-cones.

The theorem of D. Frettlöh and A. Glazyrin [5] states that the indicator function of a convex cone which is not a line-cone cannot be represented as a sum of indicator functions of line-cones, implying that the vertices of an ordinary convex polytope are indeed algebraic vertices.

Our main result is the following description of algebraic vertices, showing that in some sense these generalized vertices form a minimal set of points needed to describe a **generalized polytope**, which is by definition a bounded generalized polyhedron.

Throughout, we denote the **indicator function** of any set $S \subset \mathbf{R}^d$ by $[S]$. In other words $[S](x) = 1$ if $x \in S$, and $[S](x) = 0$ if $x \notin S$.

Theorem 1. Let \mathcal{V}_P be the set of algebraic vertices of a generalized polytope $P \subset \mathbf{R}^d$, and let \mathcal{T}_P be the set of simplices whose vertices lie in \mathcal{V}_P . Then

$$(1) \quad [P] = \sum_{T_i \in \mathcal{T}_P} \alpha_i [T_i],$$

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where the α_i are integers and the equality holds throughout \mathbf{R}^d , except perhaps for a set of measure zero.

Moreover, if $[P]$ is represented (up to measure zero) as a linear combination of indicator functions of some finite number of simplices, then the set of vertices of these simplices must contain \mathcal{V}_P .

It seems that weaker versions of this theorem were known for a long time, and in particular A. Gaifullin [6] showed that the indicator function of a polytope P is a linear combination of indicator functions of simplices whose vertices belong to the set of geometric vertices of the polytope P .

Algebraic vertices of P are closely related to the structure of integral transforms of $[P]$, in particular the Fourier–Laplace transform of $[P]$ (see below), and the Fantappiè transform $\mathcal{F}_P(z)$ [7]. In [7] the authors prove an analogue of Theorem 1, where the role of algebraic vertices is played by linear functions $\langle z, v \rangle$, appearing in the denominator $\prod_{v \in V_F} (1 - \langle z, v \rangle)$ of $\mathcal{F}_P(z)$. They needed, however, an extra genericity assumption on V_F , and were only able to establish a decomposition like the one in (1) with $\alpha_i \in \mathbf{R}$. They stated a conjecture [7, Conjecture 7], that claims the genericity is not needed; this conjecture is essentially settled in the affirmative by our Theorem 1, see Remark 10 for further discussion.

We were informed by Dmitrii Pasechnik [private communication, 2014] that [7, Conjecture 7] follows from results in [4].

Generalized (non-bounded) polyhedra also can be described through their algebraic vertices with the following theorem.

Theorem 2. *Let \mathcal{V}_P be the set of algebraic vertices of a generalized polyhedron $P \subset \mathbf{R}^d$. Then*

$$[P] = \sum_{i=1}^k \alpha_i [D_i] + \sum_{\mathbf{v} \in \mathcal{V}_P} [\text{tcone}(P, \mathbf{v})],$$

for some integers α_i and line-cones D_i , $i = 1, \dots, k$. The equality holds almost everywhere, except perhaps on a set of measure zero.

Moreover, if $[P]$ is represented (up to a set of measure zero) as a linear combination of indicator functions of line-cones and simplicial cones, then the set of apices of these simplicial cones should contain \mathcal{V}_P .

It is well known that the indicator function of a line-cone has vanishing Fourier–Laplace transform (see the definition in section 4). Therefore if \mathbf{v} is not an algebraic vertex, then the Fourier–Laplace transform of the indicator function of its tangent cone also vanishes, because it is a finite linear combination of indicator functions of line-cones.

We show that the opposite also holds. Lemma 6 implies that the indicator function of the tangent cone at an algebraic vertex has non-vanishing Fourier–Laplace transform. We formulate this fact in a more general form.

Theorem 3. *If P is a generalized polyhedron with zero Fourier–Laplace transform, then it does not have algebraic vertices.*

We prove these theorems in a more general form, for elements of the algebra of polyhedra instead of non-convex polyhedra. The article is organized in the following way. In section 2 we define the algebra of polyhedra and tangent cones of generalized polyhedra. In section 3 we prove Theorems 1 and 2. In section 4 we recall the definition of the Fourier–Laplace transform. In section 5 we give the proof of Theorem 3. In the ensuing section we discuss one corollary of the proof, which states that polytopes with zero Fourier–Laplace transform also have a zero Fourier–Laplace transform for each of their *signed sections*. Finally, in the last section, we

discuss various different ways of defining vertices of generalized polyhedra and find various relations between these definitions.

2. THE ALGEBRA OF POLYHEDRAL INDICATOR FUNCTIONS

We define a **polyhedron** as the intersection of a finite number of half-spaces, which is a convex set. Thus, a polyhedron may be unbounded, and a bounded polyhedron is by definition a convex polytope. We define a **generalized polyhedron** as a finite union of polyhedra. Now we can speak about the union, intersection, and subtraction of polyhedra.

For each generalized polyhedron P we can associate the class of sets in \mathbf{R}^d which differ from P on a set of measure zero. Thus, we work with equivalence classes of generalized polyhedra. The above operations extend naturally to these equivalence classes.

Lemma 1. *Each generalized polyhedron $P \subset \mathbf{R}^d$ can be represented as a finite disjoint union of convex polyhedra.*

Proof. Draw a hyperplane through each facet of each convex polyhedron in the union generating P . These hyperplanes cut \mathbf{R}^d into convex cells (some of which are unbounded) that are by definition convex polyhedra. Some of them are contained in P , and their union is exactly P . \square

We call such a representation of P into a disjoint union of convex polyhedra a **convex partition of P** .

Lemma 2. *If P and Q are two generalized polyhedra then the subtraction $T = P \setminus Q$ is also a generalized polyhedron.*

Proof. This argument is almost identical with the previous proof. Draw a hyperplane through each facet of each convex polyhedron of the convex partitions of P and Q . These hyperplanes cut \mathbf{R}^d into convex cells (some of which are unbounded). Some of these cells are contained in T and their union is T . \square

There is a natural algebra structure on the equivalence classes of indicator functions of polyhedra, as follows.

Let \mathcal{P}^d be the vector space, under addition of functions, of real linear combinations of indicator functions of all convex polyhedra in \mathbf{R}^d . We may also define multiplication as the pointwise multiplication of functions. These two operations allow us to consider \mathcal{P}^d as an algebra, which we call the **algebra of polyhedra**.

Again, we are working with equivalent classes of functions. So two functions are equivalent if they differ on a set of measure zero. In fact, \mathcal{P}^d is the algebra of equivalence classes of polyhedra. Furthermore, if P and Q are two generalized polyhedra then:

$$\begin{aligned}[P \cap Q] &= [P] \cdot [Q], \\ [P \cup Q] &= [P] + [Q] - [P] \cdot [Q].\end{aligned}$$

From the previous two lemmas it follows that for any function f belonging to the algebra \mathcal{P}^d , the level set $\{\mathbf{x} \in \mathbf{R}^d \mid f(\mathbf{x}) = \text{constant}\}$ is a generalized polyhedron, because it can be represented as union and subtraction of convex polyhedra in \mathbf{R}^d . So we can represent f as a sum $\sum \alpha_i [P_i]$, where the P_i are disjoint generalized polyhedra, and $\alpha_i \in \mathbf{R}$. We call this sum a **basic decomposition** of f .

If P is a convex polyhedron and \mathbf{v} is a point then we define the tangent cone of P at \mathbf{v} by

$$\begin{aligned}\text{tcone}(P, \mathbf{v}) &= \{\mathbf{v} + \mathbf{x} : \mathbf{v} + \epsilon \mathbf{x} \in P \text{ for some } \epsilon > 0\}, \text{ if } \mathbf{v} \in P, \\ \text{tcone}(P, \mathbf{v}) &= \emptyset, \text{ if } \mathbf{v} \notin P.\end{aligned}$$

If the generalized polyhedron P is a union of polyhedra P_1, \dots, P_n then we define $\text{tcone}(P, \mathbf{v})$ by their union $\cup_{i=1}^n \text{tcone}(P_i, \mathbf{v})$.

Let $\sum \alpha_i[P_i]$ be a basic decomposition of f , and let C_i be the tangent cone at a point \mathbf{v} belonging to the polyhedron P_i . We define the **tangent cone** of the function f , at the point \mathbf{v} , by $\text{tcone}(f, \mathbf{v}) := \sum \alpha_i[C_i]$.

As an aside, an alternate way to define the tangent cone to f is to consider a sufficiently small neighborhood $U_\mathbf{v}$ of \mathbf{v} and for each point $\mathbf{x} \in U_\mathbf{v}$ define $\text{tcone}(f, \mathbf{v})(\mathbf{y}) = f(\mathbf{x})$ for all points \mathbf{y} on the ray $[\mathbf{v}, \mathbf{x}]$.

We are now ready to define the algebraic vertices of a function $f \in \mathcal{P}^d$. We say that \mathbf{v} is an **algebraic vertex** of f if $\text{tcone}(f, \mathbf{v})$ cannot be represented as $\sum \alpha_i[D_i]$, where the D_i are line-cones (cones that can be represented as a union of parallel lines) and $\alpha_i \in \mathbf{R}$.

We say that f is **polyconical function** if it can be represented as a linear combination of indicator functions of simplicial cones C_i with apex at the origin:

$$f = \sum_{i=1}^n \alpha_i[C_i].$$

The collection of **polyconical functions** forms a very natural subalgebra of \mathcal{P}^d . We will also need the following definition.

Definition 2. Let f be a function on \mathbf{R}^d , and let h be an oriented hyperplane, which means that there is a normal vector \mathbf{n}_h , whose direction depends on the orientation of h . Suppose that for almost all (up to measure zero) $\mathbf{x} \in h$ the limits $f_h^+(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} f(\mathbf{x} + \varepsilon \mathbf{n}_h)$ and $f_h^-(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} f(\mathbf{x} - \varepsilon \mathbf{n}_h)$ exist. Then the function $f_h(\mathbf{x}) := f_h^+(\mathbf{x}) - f_h^-(\mathbf{x})$ is called the **signed section of the function f by the signed hyperplane h** .

It is clear that for $f \in \mathcal{P}^d$, both $f_h^+(\mathbf{x})$ and $f_h^-(\mathbf{x})$ belong to \mathcal{P}^{d-1} in the $(d-1)$ -dimensional hyperplane h . Therefore the signed section of $f \in \mathcal{P}^d$ is also an element of the algebra of polyhedra \mathcal{P}^{d-1} defined on h .

In what follows we prove Theorems 1, 2, 3 not only for generalized polyhedra and polytopes, but also for elements of \mathcal{P}^d .

3. PROOFS OF THEOREMS 1 AND 2

Here we prove the statement of the Theorems for functions with bounded support from the algebra \mathcal{P}^d .

Theorem 1'. Assume $f \in \mathcal{P}^d$ has bounded support and let \mathcal{V}_f be the set of algebraic vertices of f . Let \mathcal{T}_f be the set of simplices whose vertices lie in \mathcal{V}_f . Then for suitable real α_i ,

$$f = \sum_{T_i \in \mathcal{T}_f} \alpha_i[T_i].$$

Moreover, if f is represented as a linear combination of indicator functions of simplices, then the set of vertices of these simplices must contain \mathcal{V}_f .

If the coefficients of f in its basic decomposition are all integers, then the α_i are also integers.

Consider any oriented hyperplane h and the signed section f_h of f by h . Usually f_h is zero, but sometimes it is not. If P is convex polytope then $[P]_h$ is non-zero if and only if h contains a facet of P . Using this observation we can define a facet of f in the following way.

Definition 3. If $f \in \mathcal{P}^d$, and the signed section f_h is non-zero for an oriented hyperplane h , then the function f_h is called a **facet** of f .

Note that if f_h is a facet of f with the same orientation as h , then it is also a facet with the opposite orientation of h . It is clear that f_h is a function belonging to \mathcal{P}^{d-1} , on the hyperplane h . We will use the following Lemma.

Lemma 3. *An algebraic vertex of any facet of the function f is an algebraic vertex of f .*

This result follows directly from **Lemma 9** in Section 6. We offer a more direct proof here, though, of Lemma 3 using Definition 1 only.

Proof. We show that if a point \mathbf{v} is not an algebraic vertex of f then it is not an algebraic vertex of any signed plane h containing it. Hence we suppose that $\text{tcone}(f, \mathbf{v}) = \sum \alpha_i [D_i]$, where D_i are line-cones. Note that $[D_i]_h$ are also indicator functions of some line-cone in h . Since taking signed section is a linear map from \mathcal{P}^d to \mathcal{P}^{d-1} , it follows that $\text{tcone}(f_h, \mathbf{v}) = \sum \alpha_i [D_i]_h$. Therefore \mathbf{v} is not an algebraic vertex of f_h . \square

Lemma 3 implies, by an easy induction, the following corollary.

Corollary 4. *Any non-zero $f \in \mathcal{P}^d$ with bounded support has an algebraic vertex.*

Proof of Theorem 1'. The proof is by induction on the dimension. For the 1-dimensional space the statement is trivial.

Let's fix a point \mathbf{p} in \mathbf{R}^d . For each facet f_h of f choose the orientation of h so that p lies in the positive halfspace of h . We can associate a signed “pyramid function” Pyr_h as follows: if $\mathbf{x} \in h$ and \mathbf{y} is a point on the interval (\mathbf{p}, \mathbf{x}) define $\text{Pyr}_h(\mathbf{y}) = f_h(\mathbf{x})$ and zero otherwise. (So, if f_h is an indicator function of a two-dimensional polygon in \mathbf{R}^3 then Pyr_h will be the indicator function of the pyramid whose apex is \mathbf{p} and whose base is the polygon.)

It is clear that

$$(2) \quad f = \sum_h \text{Pyr}_h,$$

where the sum is taken over all facets of f . Indeed for each point \mathbf{y} consider the tail $[\mathbf{y}, \infty)$ of the ray $[\mathbf{p}, \mathbf{y})$, intersecting hyperplanes the h_1, h_2, \dots, h_k that are supporting facets of f . As their orientation was chosen properly, y lies on the positive side of each h_i , and f equals 0 at points that are far away on the ray, we see that $f(\mathbf{y})$ equals the sum of f_{h_i} .

By the induction hypothesis each facet f_h is a linear combination of indicator functions of simplices:

$$f_h = \sum \beta_{h,i} [T_{h,i}],$$

where $T_{h,i}$ are simplices whose vertices are algebraic vertices of f_h (which are algebraic vertices of f by Lemma 3). Note that

$$(3) \quad \text{Pyr}_h = \sum \beta_{h,i} [\text{simplex}(\mathbf{p}, T_{h,i})],$$

where the $\text{simplex}(\mathbf{p}, T_{h,i})$ is a simplex with base $T_{h,i}$ and apex \mathbf{p} , and \mathbf{p} is an arbitrary point of \mathbf{R}^d .

Now we let \mathbf{p} be any algebraic vertex of f , which exists by Corollary 4. We conclude that the vertices of all simplices $\text{Pyr}(\mathbf{p}, T_{h,i})$ are algebraic vertices of f .

Therefore combining equalities (2) and (3) we obtain the required representation:

$$f = \sum_h \sum \beta_{h,i} [\text{simplex}(\mathbf{p}, T_{h,i})].$$

It is clear that if the coefficients in the basic decomposition of f are integers, then the coefficients of the basic decomposition of its signed sections are also integers. Therefore in the representation of f which we obtain by this induction the coefficients are integers.

Finally we show that if $f = \sum_1^m \alpha_i [T_i]$, where each T_i is a simplex, then the set of vertices of T_i contains \mathcal{V}_f . Suppose $\mathbf{v} \in \mathcal{V}_f$ is not a vertex of any of the simplices T_i . Then the tangent cone $\text{tcone}([T_i], \mathbf{v})$ is a line-cone for every i . Since $\text{tcone}(f, \mathbf{v}) = \sum_i \alpha_i \text{tcone}([T_i], \mathbf{v})$, the tangent cone of f at \mathbf{v} can be represented as a sum of line-cones. This contradicts the fact that \mathbf{v} is an algebraic vertex of f . \square

Theorem 2'. *Let \mathcal{V}_f be the set of algebraic vertices of $f \in \mathcal{P}^d$. Then*

$$f = \sum_{i=1}^k \alpha_i [D_i] + \sum_{\mathbf{v} \in \mathcal{V}_f} \text{tcone}(f, \mathbf{v}),$$

for some real numbers α_i and line-cones D_i , $i = 1, \dots, k$.

Moreover, if f is represented as a linear combination of indicator functions of line-cones and simplicial cones, then the set of apices of these simplicial cones should contain \mathcal{V}_f .

If the coefficients of f in its basic decomposition are all integers, then α_i and β_i are also integers.

Proof. Draw a hyperplane through each facet of each polyhedron in the basic decomposition of f . These hyperplanes separate the space into convex polyhedra P_i , which tile the space face-to-face. Denote the set of vertices of this convex partition by \mathcal{V} . On each of these polyhedra P_i the function f takes the same constant value, so:

$$f = \sum_i \gamma_i [P_i].$$

We note that each convex polyhedron can be represented as a linear combination of tangent cones of its faces of all dimensions, also known as the Brianchon–Gram relation [2, Theorem 9.5]. Since the tangent cone of a face of dimension ≥ 1 is a line-cone, we may write:

$$[P_i] = \sum_{\mathbf{v} \in \mathcal{V}} \text{tcone}(P_i, \mathbf{v}) + \sum_j \alpha_{i,j} [D_j],$$

where D_j are line-cones which are tangent cones corresponding to faces of P_i of dimension ≥ 1 .

Observe now that if \mathbf{v} is not an algebraic vertex of f then $\sum_i \gamma_i \text{tcone}(P_i, \mathbf{v}) = \text{tcone}(f, \mathbf{v})$ can be represented as a linear combination of indicator functions of line-cones. We see that f is a linear combination of indicator functions of line-cones, plus a linear combination of tangent cones at algebraic vertices of f . This proves the first part of the Theorem.

It is clear that if the coefficients of f in its basic decomposition are all integers, then in each step we obtain a representation of f with integer coefficients.

Finally, we show that if f is represented as a linear combination of indicator functions of line-cones and simplicial cones, then the set of apices of these simplicial cones should contain \mathcal{V}_f . This is similar to the proof of the corresponding statement in Theorem 1'. Suppose that

$$f = \sum_i \alpha_i [D_i] + \sum_i \beta_i [C_i],$$

for some real α_i and β_i , line-cones D_i , $i = 1, \dots, k$, and simplicial cones C_i , $i = 1, \dots, m$. Let $\mathbf{v} \in \mathcal{V}_f$. Note that

$$\text{tcone}(f, \mathbf{v}) = \sum_i \alpha_i \text{tcone}([D_i], \mathbf{v}) + \sum_i \beta_i \text{tcone}([C_i], \mathbf{v}).$$

If \mathbf{v} is not a vertex of some cone C_i , then all $\text{tcone}([C_i], \mathbf{v})$ are line-cones, and $\text{tcone}([D_i], \mathbf{v})$ are line-cones as well. Therefore we obtain a representation of $\text{tcone}(f, \mathbf{v})$ as a linear combination of indicator functions of line-cones, contradicting the assumption that \mathbf{v} is an algebraic vertex. \square

4. THE FOURIER–LAPLACE TRANSFORM

In this section we recall the definition of the Fourier–Laplace transform and list some of its properties. For more details see [1, Chapter 8] and [2, Chapter 10], or the original works [8] and [9].

The Fourier–Laplace transform of a generalized d -polytope P is defined as follows:

$$\widehat{[P]}(\mathbf{z}) := \int_P e^{\langle \mathbf{z}, \mathbf{x} \rangle} d\mathbf{x},$$

valid for all $\mathbf{z} \in \mathbf{C}^d$.

It may be tempting to also define the Fourier–Laplace transform of generalized polyhedra P in the same manner, but we cannot do so because we run into the problem that there may not be any value of $\mathbf{z} \in \mathbf{C}^d$ for which the latter integral converges, as happens for example when P consists of positive and negative orthants.

We first define the Fourier–Laplace transform of a **convex polyhedral cone** K whose apex is the origin by

$$(4) \quad \widehat{[K]}(\mathbf{z}) := \int_K e^{\langle \mathbf{z}, \mathbf{x} \rangle} d\mathbf{x},$$

which converges for all $\mathbf{z} \in \mathbf{C}^d$, for which the real part $\Re(-\mathbf{z})$ lies in the interior of the **dual cone** K° , that is $K^\circ = \{\mathbf{y} \in \mathbf{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0, \forall \mathbf{x} \in K\}$. We recall that a **simplicial cone** K is a convex cone in \mathbf{R}^d with exactly d edges (also called generators), say $\mathbf{w}_1, \dots, \mathbf{w}_d$, and for such a simplicial cone with apex at the origin, we have the elementary fact that (see [1, 2]):

$$(5) \quad \widehat{[K]}(\mathbf{z}) := \frac{|\det K|}{\langle \mathbf{w}_1, \mathbf{z} \rangle \cdots \langle \mathbf{w}_d, \mathbf{z} \rangle} = \frac{|\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_d|}{\langle \mathbf{w}_1, \mathbf{z} \rangle \cdots \langle \mathbf{w}_d, \mathbf{z} \rangle},$$

where we define $|\det K|$ to be the absolute value of the determinant of the matrix whose columns are the vectors $\mathbf{w}_j \in \mathbf{R}^d$.

Let $K_{\mathbf{v}}$ be a cone K shifted by a vector \mathbf{v} . From Equation (4) it follows that $\widehat{[K_{\mathbf{v}}]}(\mathbf{z})$ must be defined as $\widehat{[K]}(\mathbf{z}) \cdot e^{\langle \mathbf{z}, \mathbf{v} \rangle}$.

Note that if the edges \mathbf{w}_1 and \mathbf{w}_2 of a simplicial cone K are parallel to each other then it degenerates to a line-cone and the value in the numerator in (5) equals zero. Since each line-cone can be represented as disjoint union of these degenerated line-cones it is natural to define the Fourier–Laplace transform of line-cones as zero. Finally, using the Brianchon–Gram identity it is possible to extend the definition of the Fourier–Laplace transform to all elements of the algebra of polyhedra \mathcal{P}^d (for full details see Theorem 8.4 in [1]).

It is clear from the above discussion that the Fourier–Laplace transform of $f \in \mathcal{P}^d$ can be written in the form

$$(6) \quad \sum_{\mathbf{v} \in \mathcal{V}} e^{\langle \mathbf{v}, \mathbf{z} \rangle} F_{\mathbf{v}}(\mathbf{z}),$$

where $F_{\mathbf{v}}$ is a finite linear combination of functions of the form $\prod_1^d \langle \mathbf{w}_j, \mathbf{z} \rangle)^{-1}$, and $\mathcal{V} \subset \mathbf{R}^d$ is a finite set. For instance, for \mathcal{V} we can choose the set of geometric vertices of a basic decomposition of f . Theorem 2' shows that $\mathcal{V} = \mathcal{V}_f$ is an appropriate choice. We will show later that, in fact, \mathcal{V} essentially coincides with \mathcal{V}_f .

We also should note that $\widehat{f}(\mathbf{z})$ is an analytic function of \mathbf{z} , and so it follows that if it vanishes on an open d -dimensional set, then it vanishes on all of \mathbf{R}^d .

5. POLYHEDRA WHOSE FOURIER–LAPLACE TRANSFORM VANISHES

Here we prove Theorem 3 for any function from the algebra of polyhedra.

Theorem 3'. *If $f \in \mathcal{P}^d$ has zero Fourier–Laplace transform, then it has no algebraic vertices, or equivalently by Theorem 2', we have*

$$f = \sum_{i=1}^k \alpha_i [D_i]$$

for some real numbers α_i and line-cones D_i , $i = 1, \dots, k$.

Let \mathcal{V}_f be the set of algebraic vertices of f , and assume that $\mathcal{V}_f \neq \emptyset$. Rearrange the members in the representation of f as the sum from Theorem 2':

$$f = \sum_{i=1}^k \alpha_i [D_i] + \sum_{\mathbf{v} \in \mathcal{V}_f} \text{tcone}(f, \mathbf{v}).$$

The proof consist of two parts. Lemma 6 below implies that the Fourier–Laplace transform of $\text{tcone}(f, \mathbf{v})$ is non-zero because it cannot be represented as a linear combination of line-cones.

All the algebraic vertices $\mathbf{v} \in \mathcal{V}_f$ are distinct. Then Lemma 7 shows that the Fourier–Laplace transform of their sum should be non-zero as well.

The following lemma is crucial to a lot of the analysis that ensues, showing in particular that if we consider the union of pointed cones which lie in the interior of a half space, then the transform of this union cannot vanish.

Lemma 5. *Suppose we have n internally disjoint pointed cones C_i lying in the upper halfspace of \mathbf{R}^d . Let $g = \sum_{i=1}^n \alpha_i [C_i]$. Then*

$$\widehat{g}(\cdot) = 0 \text{ if and only if all } \alpha_i = 0.$$

Proof. Let C be the cone generated by all generators of the cones C_i . Thus C is a pointed cone. Suppose $\widehat{g}(\mathbf{z}) = 0$ for all \mathbf{z} belonging to the interior of the dual cone C° . We assume by contradiction that all α_i are different from zero. Let \mathbf{w} be one of its extreme ray directions of C , and let C_1, \dots, C_k be the cones that have \mathbf{w} as a generator. We assume without loss of generality that $\|\mathbf{w}\| = 1$. We are going to show that $\alpha_i = 0$ for $i = 1, \dots, k$. This contradiction will finish the proof.

From here we proceed by induction on the dimension d . For $d = 1$ there is a unique cone, which is the ray generated by \mathbf{w} . So $\widehat{g}(\mathbf{z}) = \frac{\alpha_1}{\langle \mathbf{w}, \mathbf{z} \rangle}$, and it is clear that $\alpha_1 = 0$. The case $d = 2$ is also somewhat special since then there is only one cone C_1 containing \mathbf{w} . Let \mathbf{z} be the unique unit vector orthogonal to \mathbf{w} and having $\langle \mathbf{z}, \mathbf{w}_j \rangle > 0$ for all other generators \mathbf{w}_j . Define for $\varepsilon > 0$

$$\mathbf{u} = \mathbf{u}_\varepsilon = \frac{\mathbf{z} + \varepsilon \mathbf{w}}{\|\mathbf{z} + \varepsilon \mathbf{w}\|}.$$

If ε is small enough then \mathbf{u} is a unit vector with $\langle \mathbf{u}, \mathbf{w}_j \rangle > 0$ for all generators including \mathbf{w} as well. It is easy to check using (5) that $\widehat{[C_i]}(\mathbf{u})$ are bounded for all $i > 1$. Denoting the second generator of C_1 by \mathbf{w}_1 ,

$$\widehat{[C_1]}(\mathbf{u}) = \frac{|\mathbf{w} \wedge \mathbf{w}_1|}{\langle \mathbf{w}, \mathbf{u} \rangle \langle \mathbf{w}_1, \mathbf{u} \rangle} = \frac{|\mathbf{w} \wedge \mathbf{w}_1|}{\varepsilon \langle \mathbf{w}_1, \mathbf{z} \rangle (1 + o(1))},$$

So $\widehat{g}(\mathbf{u}) = 0$ for all $\varepsilon > 0$ is only possible if $\alpha_1 = 0$.

Now for the step $d - 1 \rightarrow d$ with $d > 2$, we write $G = \{\mathbf{x} \in \mathbf{R}^d : \langle \mathbf{w}, \mathbf{x} \rangle = 0\}$; this is a copy of \mathbf{R}^{d-1} of course. Let C'_i denote the orthogonal projection of C_i to G for $i = 1, \dots, k$. Let $\mathbf{z} \in G$ be a unit vector with $\langle \mathbf{z}, \mathbf{w}_j \rangle > 0$ for all generators different from \mathbf{w} . The set of such \mathbf{z} 's

is exactly the set of unit normal vectors to those support hyperplanes of C that contain only \mathbf{w} from the generators of C . Observe now that all C'_i , $i = 1, \dots, k$ lie in $\{\mathbf{x} \in G : \langle \mathbf{x}, \mathbf{z} \rangle > 0\}$ which is an open halfspace in G . Define $\mathbf{u} = \mathbf{u}_\varepsilon$ the same way as above; so \mathbf{u} is a unit vector with $\langle \mathbf{w}_j, \mathbf{u} \rangle > 0$ for all generators including \mathbf{w} as well.

Note that for $i = 1, \dots, k$

$$\widehat{[C_i]}(\mathbf{u}) = \frac{\widehat{[C'_i]}(\mathbf{z}) + O(\varepsilon)}{\varepsilon}.$$

Indeed, suppose C_i is generated by vectors $\mathbf{w}_1, \dots, \mathbf{w}_{d-1}$, $\mathbf{w}_d = \mathbf{w}$. Denote by \mathbf{w}'_j the projection of \mathbf{w}_j on the hyperplane G . Then

$$\widehat{[C_i]}(\mathbf{u}) = \frac{|\mathbf{w}_1 \wedge \dots \wedge \mathbf{w}_d|}{\prod_{j=1}^d \langle \mathbf{w}_j, \mathbf{u} \rangle} = \frac{|\mathbf{w}'_1 \wedge \dots \wedge \mathbf{w}'_{d-1}|}{\prod_{j=1}^{d-1} (\langle \mathbf{w}'_j, \mathbf{z} \rangle + \varepsilon \langle \mathbf{w}, \mathbf{w}_j \rangle)} \cdot \frac{\|\mathbf{w}\|}{\varepsilon \|\mathbf{w}\|^2} = (\widehat{[C'_i]}(\mathbf{z}) + O(\varepsilon)) \cdot \frac{1}{\varepsilon}.$$

Note that $\widehat{[C_i]}(\mathbf{u})$ for $i > k$ is bounded as ε goes to zero. Therefore

$$0 = \widehat{g}(\mathbf{u}) = \frac{1}{\varepsilon} \sum_1^k \alpha_i \widehat{[C'_i]}(\mathbf{z}) + O(1).$$

Multiplying by ε and taking the limit $\varepsilon \rightarrow 0$ gives the equation $\sum_1^k \alpha_i \widehat{[C'_i]}(\mathbf{z}) = 0$. This holds for all unit vectors $\mathbf{z} \in G$ for which $\langle \mathbf{z}, \mathbf{w}_j \rangle > 0$ for every generator of the cones C_i , $i = 1, \dots, k$, which is the same as $\langle \mathbf{z}, \mathbf{w}'_j \rangle > 0$ for every generator of the cones C'_i , $i = 1, \dots, k$. Observe that $\sum_1^k \alpha_i \widehat{[C'_i]}(\mathbf{z})$ is the Fourier–Laplace transform (taken in $G = \mathbf{R}^{d-1}$) of the function $\sum_1^k \alpha_i [C'_i]$ and so by the induction hypothesis all $\alpha_i = 0$ for $i = 1, \dots, k$. \square

Lemma 6. *A polyconical function g has zero Fourier–Laplace transform if and only if it is a linear combination of line-cones.*

Proof. One direction is easy: if g is a linear combination of line-cones, then it has zero Fourier–Laplace transform because all line-cones have zero Fourier–Laplace transform. For the other direction we consider the decomposition of g into simplicial cones:

$$g = \sum_{i=1}^k \alpha_i [C_i].$$

We will show how to reduce the proof to the context of Lemma 5, namely, it suffices to prove the theorem in a simpler case, when

- (i) all the C_i are internally disjoint pointed cones, and
- (ii) there is a unit vector \mathbf{v} such that $\langle \mathbf{v}, \mathbf{w} \rangle > 0$ for every generator \mathbf{w} of every C_i .

To see this we start with a simplicial cone C whose generators are $\mathbf{w}_1, \dots, \mathbf{w}_d$, that is, these vectors are linearly independent and every point in C is a linear combination with non-negative coefficients of the \mathbf{w}_j , that is, $C = \text{pos}\{\mathbf{w}_1, \dots, \mathbf{w}_d\}$. Let, further, \mathbf{v} be a unit vector with $\langle \mathbf{v}, \mathbf{w}_j \rangle \neq 0$ for any j . Define \mathbf{w}_j^* as \mathbf{w}_j if $\langle \mathbf{v}, \mathbf{w}_j \rangle > 0$ and $-\mathbf{w}_j$ if $\langle \mathbf{v}, \mathbf{w}_j \rangle < 0$. Set $C^* = \text{pos}\{\mathbf{w}_1^*, \dots, \mathbf{w}_d^*\}$. We start with a simple

Claim 1. *There are coefficients $\varepsilon \in \{-1, 1\}$ and $\varepsilon_i \in \{-1, 0, 1\}$ and line-cones D_i such that $[C] = \varepsilon [C^*] + \sum_i^d \varepsilon_i [D_i]$.*

Proof of the claim. There is nothing to prove if $\langle \mathbf{v}, \mathbf{w}_j \rangle > 0$ for all j . Assume $\langle \mathbf{v}, \mathbf{w}_1 \rangle < 0$, say, and observe that

$$\begin{aligned} [C] &= [\text{pos}\{\mathbf{w}_1, \dots, \mathbf{w}_d\}] \\ &= [\text{pos}\{\mathbf{w}_1, -\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}] \setminus [\text{pos}\{-\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}]. \end{aligned}$$

The cone $\text{pos}\{\mathbf{w}_1, -\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\}$ is a line-cone and the simplicial cone $\text{pos}\{-\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_d\} = \text{pos}\{\mathbf{w}_1^*, \mathbf{w}_2, \dots, \mathbf{w}_d\}$ has one fewer generators than C with $\langle \mathbf{v}, \mathbf{w} \rangle < 0$. Continuing this way we end up with the required formula, finishing the proof of the claim.

We return now to the reduction of our Lemma to Lemma 5. Applying the Claim to each cone C_i we have

$$g = \sum_{i=1}^n \alpha_i [C_i] = \sum_{i=1}^n \alpha_i \varepsilon_i [C_i^*] + \sum_{j=1}^k \gamma_j [D_j]$$

where $\varepsilon_i = \pm 1$ and the D_j are line-cones. It follows that the Fourier–Laplace transform of $g^* = \sum_{i=1}^n \alpha_i \varepsilon_i [C_i^*]$ is zero.

The advantage of g^* is that it is a linear combination (of the indicator functions) of simplicial cones C_i^* and each C_i^* lies in the open halfspace $H = \{\mathbf{x} \in \mathbf{R}^d : \langle \mathbf{x}, \mathbf{v} \rangle > 0\}$.

Consider now all the hyperplanes containing a facet of some C_i^* . The complement of their union in \mathbf{R}^d consists of finitely many cones, and on each such cone, F say, g^* is constant. This constant is zero if the cone F does not lie completely in H . Thus $g^* = \sum_{j=1}^m \beta_j [F_j]$ where each $\beta_j \neq 0$ and the F_j are pairwise internally disjoint cones, each contained in H . This finishes the proof of the reduction, and of the Lemma. \square

Lemma 7. Suppose we are given a collection of n distinct points $\mathbf{v}_i \in \mathbf{R}^d$ and n polyconical functions f_i , whose Fourier–Laplace transform \widehat{f}_i is not the zero function.

Let $g(\mathbf{z}) := \sum_{i=1}^n \alpha_i f_i(\mathbf{z} - \mathbf{v}_i)$, with all $\alpha_i \in \mathbf{R}$. Then

$$\widehat{g}(\cdot) = 0 \text{ if and only if all } \alpha_i = 0.$$

Proof. Let P be the convex hull of the points \mathbf{v}_i . Without loss of generality we can assume that \mathbf{v}_1 is an extreme point of P . We translate everything so that \mathbf{v}_1 becomes the origin. Let K be the dual cone to the tangent cone of P at \mathbf{v}_1 , and fix \mathbf{u} to be any point in the interior of K .

Denote by $g_i(\mathbf{z}) := f_i(\mathbf{z} - (\mathbf{v}_i - \mathbf{v}_1))$. It is enough to prove that $\sum \alpha_i g_i$ has zero Fourier–Laplace transform if and only if all $\alpha_i = 0$.

As the apex of every cone defining g_i is $\mathbf{v}_i - \mathbf{v}_1$, and its Fourier–Laplace transform is of the form

$$e^{\langle \mathbf{v}_i - \mathbf{v}_1, \mathbf{z} \rangle} F_i(\mathbf{z}),$$

where $F_i(\mathbf{z})$ is a linear combination of functions of the form $(\prod_1^d \langle \mathbf{w}_j, \mathbf{z} \rangle)^{-1}$. Then for every $i > 1$ the value $\widehat{g}_i(t\mathbf{u})$ tends to 0 like $o(e^{tc_i})$, with $c_i < 0$, as $t \rightarrow \infty$. But the function $\widehat{g}_1(\mathbf{z})$ does not have an exponential in the numerator, since its associated vertex is the origin, and hence $\widehat{g}_1(t\mathbf{u}) = t^{-d} \cdot \widehat{g}_1(\mathbf{u})$ tends to 0 like $O(t^{-d})$. Therefore $\widehat{g}_1(\mathbf{u}) = 0$ for all $\mathbf{u} \in K$. Since K is d -dimensional, we see that $\widehat{g}_1(\mathbf{z}) = 0$ for all \mathbf{z} and hence \widehat{f}_1 vanishes identically. This last conclusion contradicts our assumption about the f_i 's. \square

Remark 8. Using the same argument in the Lemma above, we return to formula (6) and claim $F_\mathbf{v}$ is zero if $\mathbf{v} \notin \mathcal{V}_f$. This will show that the representation of the Fourier–Laplace transform in the form of (6) is essentially unique. The proof is simple. Suppose there are two representations. We then subtract the two representations from each other to obtain

$$0 = \sum_{\mathbf{v} \in \mathcal{V}'} e^{\langle \mathbf{v}, \mathbf{z} \rangle} F_\mathbf{v}(\mathbf{z}),$$

for some finite set \mathcal{V}' . Now repeating the arguments from the proof of Lemma 7 leads to a contradiction.

6. SIGNED SECTIONS OF POLYHEDRA

The following theorem shows the strong correlation between $f \in \mathcal{P}^d$ and its signed sections f_h .

Theorem 4. *Let $f \in \mathcal{P}^d$. Then $\widehat{f}(\cdot) = 0$ if and only if $\widehat{f}_h = 0$ for any oriented hyperplane h .*

We first prove it for polyconical functions.

Lemma 9. *The Fourier–Laplace transform of a polyconical function f is zero if and only if for all hyperplanes h through the origin, the $(d - 1)$ -dimensional Fourier–Laplace transform of the signed section f_h is zero.*

Proof. Note that the $(d - 1)$ -dimensional Fourier–Laplace transform of every signed section of a line-cone is zero, as the signed section itself is a line cone.

Now applying Claim 1 to the cones generating f we get the function f^* whose support lies in a halfspace H . From Lemma 5 it follows that $\widehat{f}(\cdot)$ is zero if and only if f^* is zero.

But f^* vanishes if and only if all of its signed sections vanish. Supports of signed sections of f^* lie in H . Therefore, by Lemma 5 again, the condition that the Fourier–Laplace transform of all signed sections of f^* are zero is equivalent to the condition that all their $(d - 1)$ -dimensional Fourier–Laplace transforms are zero. But the Fourier–Laplace transform of a signed section f_h^* coincides with the Fourier–Laplace transform of a signed section f_h , for any h , since they differ from each other on linear combinations of line-cones. \square

Proof of Theorem 4. By Theorem 3, if $\widehat{f}(\cdot) = 0$, then at each point \mathbf{v} , the Fourier–Laplace transform of the tangent cone to f equals zero, and vice versa.

Suppose $\widehat{f}(\cdot) = 0$ and h is some hyperplane. Consider the tangent cone $C = \text{tcone}(f, \mathbf{v})$ of a point \mathbf{v} from h . By Lemma 9 its Fourier–Laplace transform is zero, therefore the Fourier–Laplace transform of its signed section C_h is also zero. This holds for any point \mathbf{v} from h and therefore $\widehat{f}_h = 0$.

In the other direction the proof is analogous. We let \mathbf{v} be any point, and $C := \text{tcone}(f, \mathbf{v})$. For any hyperplane h through \mathbf{v} we have $\widehat{f}_h = 0$, and therefore by Lemma 7, $\widehat{C}_h = 0$. Lemma 9 now implies that $\widehat{C} = 0$. Since this holds for any point \mathbf{v} , we conclude that $\widehat{f}(\cdot) = 0$. \square

7. WHAT IS A VERTEX OF A GENERALIZED POLYTOPE?

There are several ways to define a vertex of generalized polytope P (or of $f \in \mathcal{P}^d$). In this section we discuss these possibilities.

In the introduction \mathbf{v} is defined as a geometric vertex of P if $\text{tcone}(P, \mathbf{v})$ is not a line-cone. Figure 1 shows a generalized polytope in which two edges touch each other at the point \mathbf{v} . According to the above definition, \mathbf{v} is a geometric vertex. But it is not very natural to consider this point as a vertex, because it appears occasionally and can disappear after a slight mutation of polytope.

One may try to define a geometric vertex when not all connected components of $\text{tcone}(P, \mathbf{v})$ are line-cones. Such a vertex is shown on Figure 2. Is it natural to take this point for a vertex? Perhaps not.

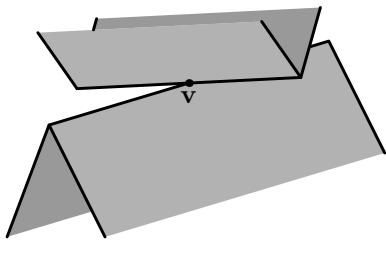


Fig. 1.

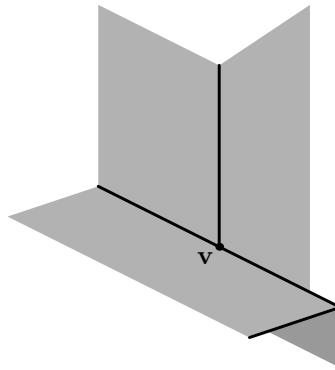


Fig. 2.

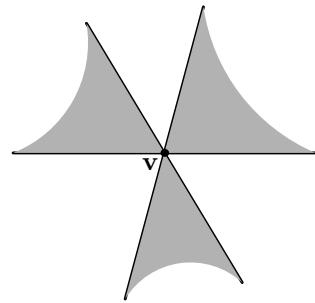


Fig. 3.

Here is another way to define a vertex (see [7]).

Definition 4. A point v is a **combinatorial vertex** of a generalized polytope P if its tangent cone cannot be partitioned into line-cones.

Lemma 5 in [7] shows that this definition is equivalent to the following one.

Definition 5. A point v is a **combinatorial vertex** of a generalized polytope P if, in every dissection P into disjoint simplices, the point v is a vertex of some simplex of the dissection.

It is well-known that some non-convex polytopes P cannot be triangulated into simplices using only vertices of P . The most famous example is the Schönhardt polytope [10], which is a modification of the triangular prism.

So if we try to define a triangulation of non-convex polytopes we are either forced to use simplices with vertices not only from the set of vertices of P (which are clearly discernible in the case of the Schönhardt polytope), or we are forced to use overlapping simplices by subtracting simplices from each other (see [7, Remark 5] for a representation of the Schönhardt polytope). In view of this phenomena, it is natural to modify the latter two definitions by allowing signed versions instead of unions. We thus arrive at our definition of algebraic vertices, which we repeat from the introduction.

Definition 1'. For a generalized polyhedron P , a point $v \in P$ is called an **algebraic vertex** of P if the indicator function of its tangent cone $\text{tcone}(P, v)$ cannot be represented as a linear combination of indicator functions of line-cones.

Definition 6. A point v is an **algebraic vertex** of P if in every representation of the indicator function of P as a linear combination of indicator functions of simplices, the point v is a vertex of some simplex.

Theorem 1 shows that the set of vertices that satisfy these two definitions coincide.

Remark 10. In [7], instead of the Fourier–Laplace transform the authors consider the Fantappiè transform \mathcal{F}_P (which is closely related to the Fourier–Laplace transform) of a generalized polytope P , see the discussion following Theorem 1. The authors of [7] show that if the set of combinatorial vertices of P is in general position (i.e. the affine span of each $(d+1)$ -tuple equals \mathbf{R}^d) then it is a set of vertices $V_{\mathcal{F}}$ appearing in \mathcal{F}_P . Using our approach it is possible to show that $V_{\mathcal{F}}$ is exactly the set of algebraic vertices of P . The proof is similar to the proof of Lemma 7 and Remark 8.

Finally, Figure 3 shows the tangent cone of the generalized polytope P which is defined as a union of three cones having a common apex v , such that their sides lie on three lines. The Fourier–Laplace transform of $\text{tcone}(P, v)$ is zero, so v is not a vertex (see [1], Problem 9.3 for a more general statement). But is it not, really? The interested reader should decide.

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REFERENCES

- [1] A. Barvinok. *Integer Points in Polyhedra*, volume 452. European Mathematical Society, 2008.
- [2] M. Beck and S. Robins. *Computing the continuous discretely: Integer-point enumeration in polyhedra*. Springer, 2007.
- [3] M. Brion. Points entiers dans les polyèdres convexes. *Ann. Sci. Ecole Norm. Sup.*, 21(4):653–663, 1988.
- [4] M. Brion and M. Vergne. Arrangement of hyperplanes. I. Rational functions and Jeffrey–Kirwan residue. *Annales scientifiques de l'Ecole normale supérieure*, 32(5):715–741, 1999.
- [5] D. Frettlöh and A. Glazyrin. The Lonely Vertex Problem. *Contributions to Algebra and Geometry*, 50(1):71–79, 2009.
- [6] A. A. Gaifullin. Generalization of Sabitov’s Theorem to Polyhedra of Arbitrary Dimensions. *Discrete & Computational Geometry*, 52(2):195–220, 2014.
- [7] N. Gravin, D. V. Pasechnik, B. Shapiro, and M. Shapiro. On moments of a polytope. *arXiv preprint arXiv:1210.3193*, 2012.
- [8] J. Lawrence. Polytope volume computation. *Mathematics of Computation*, 57(195):259–271, 1991.
- [9] A. V. Pukhlikov and A. G. Khovanskii. The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes. *Algebra i analiz*, 4(4):188–216, 1992.
- [10] E. Schönhardt. Über die Zerlegung von Dreieckspolyedern in Tetraeder. *Mathematische Annalen*, 98(1):309–312, 1928.

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