

## POISSON POLYTOPES

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We prove the central limit theorem for the volume and the  $f$ -vector of the Poisson random polytope  $\Pi_\eta$  in a fixed convex polytope  $P \subset \mathbb{R}^d$ . Here,  $\Pi_\eta$  is the convex hull of the intersection of a Poisson process  $X$  of intensity  $\eta$  with  $P$ .

**1. Introduction and main results.** Let  $K \subset \mathbb{R}^d$  be a convex set of volume 1. Assume that  $X = X(\eta)$  is a Poisson point process in  $\mathbb{R}^d$  of intensity  $\eta$ . The intersection of  $K$  with  $X(\eta)$  consists of uniformly distributed random points  $X_1, \dots, X_N$  (where  $N$  is a random variable). Define the *Poisson polytope*  $\Pi_\eta$ , as the convex hull  $[X_1, \dots, X_N] = [K \cap X(\eta)]$ .

The study of properties of random convex hulls is a classical subject in stochastic geometry and dates back to 1864. Due to the geometric nature of the available methods, for over one hundred years, investigations mainly concentrated on the expectation of functionals of random convex hulls such as volume or number of vertices; see, for example, the survey of Weil and Wieacker [24].

The first distributional results were only proven twenty years ago. In 1988, Groeneboom [14] obtained the central limit theorem (CLT) for the number of vertices of the Poisson polytope when the convex body  $K$  is the planar disc. In 1994, a CLT for the area of a random polygon in the planar disc was proven by Hsing [16]. Recently, this was generalized to arbitrary dimensions by Reitzner [19], who established a CLT for  $V(\Pi_\eta)$ , the volume of the Poisson polytope, and for  $f_\ell(\Pi_\eta)$ , the number of  $\ell$ -dimensional faces of the Poisson polytope, when the body  $K \subset \mathbb{R}^d$  has smooth boundary.

The situation seems to be much more involved when the underlying convex set is a polytope  $P$ . In the planar case, when  $P$  is a convex polygon, a CLT

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for the number of vertices  $f_0(\Pi_\eta)$  was proven by Groeneboom [14] and a CLT for the area of  $\Pi_\eta$  by Cabo and Groeneboom [12], but it seems that the stated variances are incorrect (see the discussion in Buchta [11]).

The main result of the present paper is the central limit theorem for the Poisson polytope  $\Pi_\eta$  for all dimensions  $d \geq 2$ , when the mother body is a polytope in  $\mathbb{R}^d$ .

**THEOREM 1.1.** *There exists a function  $\varepsilon(\eta)$ , tending to zero as  $\eta \rightarrow \infty$ , such that for every polytope  $P \subset \mathbb{R}^d$  of volume 1,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{V(\Pi_\eta) - \mathbb{E}V(\Pi_\eta)}{\sqrt{\text{Var } V(\Pi_\eta)}} \leq x \right) - \Phi(x) \right| \leq c(P)\varepsilon(\eta)$$

and for all  $\ell = 0, \dots, d-1$ ,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{f_\ell(\Pi_\eta) - \mathbb{E}f_\ell(\Pi_\eta)}{\sqrt{\text{Var } f_\ell(\Pi_\eta)}} \leq x \right) - \Phi(x) \right| \leq c(P)\varepsilon(\eta),$$

where  $c(P)$  is a constant depending only on  $P$ .

**REMARK.** It will turn out that the error term in Theorem 1.1 is

$$\varepsilon(\eta) = (\ln \eta)^{-(d-1)/2+o(1)}.$$

The constant  $c(P)$  depends on the dimension and a power of  $F(P)$ , the number of flags of the polytope  $P$ . A *flag* is a sequence of faces  $F_0, F_1, \dots, F_{d-1}$  of  $P$  such that for all  $i$ ,  $\dim F_i = i$  and  $F_i \subset F_{i+1}$ .

The Poisson polytope  $\Pi_\eta$  is closely related to the random polytope  $P_n$  defined in the following way: fix  $n \in \mathbb{N}$  and choose  $n$  random points  $X_1, \dots, X_n$  independently and uniformly from  $K$ . The *random polytope*  $P_n$  is just the convex hull of these points:  $P_n = [X_1, \dots, X_n]$ . Clearly,  $P_n$  equals in distribution the Poisson polytope  $\Pi_\eta$ , given that the (Poisson-distributed) number of points of  $X \cap K$  is precisely  $n$ .

Starting with Rényi and Sulanke [17] in 1963, there have been many results concerning various properties of  $P_n$  as  $n \rightarrow \infty$ . For instance, the asymptotic behavior of the expectation of the volume  $V(P_n)$ , and of the number,  $f_\ell(P_n)$ , of  $\ell$ -dimensional faces of  $P_n$  ( $\ell = 0, \dots, d-1$ ), have been determined as  $n \rightarrow \infty$ ; see [24] for an extensive survey and also [4, 5] and [18] for more recent results. These results on  $P_n$  imply immediately analogous results for the Poisson polytope  $\Pi_\eta$ . For the sake of completeness, we state here the results concerning the expected volume and number of faces.

THEOREM 1.2. *Assume that  $P$  is a polytope of volume 1. Then*

$$1 - \mathbb{E}V(\Pi_\eta) = \frac{F(P)}{(d+1)^{d-1}(d-1)!} \eta^{-1} \ln^{d-1} \eta (1 + o(1)),$$

$$\mathbb{E}f_\ell(\Pi_\eta) = c(d, \ell) F(P) \ln^{d-1} \eta (1 + o(1)),$$

where  $c(d, \ell) > 0$  is a constant depending on  $d$  and  $\ell$ .

Somewhat surprisingly, the values of these expectations are not needed for the proofs of our main theorems.

The proof of Theorem 1.1 is not simple. It uses a combination of ideas from probability theory and convex geometry. Section 3 contains a short sketch of this proof. First, we have to introduce some notation and background, and, more importantly, the economic cap covering theorem that will be used repeatedly. This is the content of the next section.

**2. Notation and background.** The unit sphere is denoted  $S^{d-1}$ . As usual,  $h_K(u)$  denotes the support function of  $K$  in direction  $u \in S^{d-1}$ :

$$h_K(u) = \max\{u \cdot x : x \in K\}.$$

A *cap*  $C$  of  $K$  is the intersection of  $K$  with a closed half-space. This half-space can be written as  $\{x \in \mathbb{R}^d \mid u \cdot x \geq h_K(u) - t\}$  with  $u \in S^{d-1}$ . Thus,

$$C = K \cap \{x \in \mathbb{R}^d \mid u \cdot x \geq h_K(u) - t\}.$$

The *bounding hyperplane* of  $C$  is the one with equation  $u \cdot x = h_K(u) - t$ . We define, for  $\lambda > 0$ ,  $C^\lambda$  by

$$C^\lambda = K \cap \{x \in \mathbb{R}^d \mid u \cdot x \geq h_K(u) - \lambda t\}.$$

An important role throughout is played by the function  $v : K \rightarrow \mathbb{R}$ , defined as

$$v(z) = \min\{V(K \cap H) : H \text{ is a half-space and } z \in H\}.$$

The *floating body* with parameter  $t$  is just the level set  $K(v \geq t) = \{z \in K : v(z) \geq t\}$ , which is clearly convex. The *wet part* is  $K(v \leq t)$ , that is, where  $v$  is at most  $t$ . The name comes from the three-dimensional picture where  $K$  is a container containing  $t$  units of water.

The *minimal cap* of  $z \in K$  is a cap  $C(z) = C_K(z)$  containing  $z$  such that  $v(z) = V(C(z))$ . It need not be unique. The *center* of the cap  $C = K \cap \{x \in \mathbb{R}^d : u \cdot x \geq h_K(u) - t\}$  is a point  $x \in \partial K$  with  $u \cdot x = h_K(u)$ . The center, again, need not be unique, but this will cause no harm. Assuming that  $x$  is the center of  $C$ , observe that, for  $\lambda \geq 1$ ,

$$C^\lambda \subset x + \lambda(C - x)$$

and thus  $V(C^\lambda) \leq \lambda^d V(C)$  always holds. Also,  $\frac{\lambda}{d} V(C) \leq V(C^\lambda)$  holds as long as  $\lambda t$  is smaller than the width of  $K$  in the direction  $u$ . The proof is simple: let  $L$  be the section that has maximal  $(d-1)$ -dimensional volume among all sections of the form

$$K \cap \{x \in \mathbb{R}^d : u \cdot x \geq h_K(u) - \tau\} \quad \text{when } \tau \in [0, t].$$

Then  $V(C) \leq t V_{d-1}(L)$ . Here,  $V_{d-1}$  stands for  $(d-1)$ -dimensional volume. On the other hand, the double cone with base  $L$ , apexes  $x$  and a point in  $K \cap \{x \in \mathbb{R}^d : u \cdot x = h_K(u) - \lambda t\}$  is contained in  $C^\lambda$  and its volume is at least  $\frac{\lambda t}{d} V_{d-1}(L)$ . So  $\frac{\lambda}{d} V(C) \leq \frac{\lambda t}{d} V_{d-1}(L) \leq V(C^\lambda)$ , which is the inequality we wanted to prove.

Analogously for  $0 < \mu < 1$ , we have  $\mu^d V(C) \leq V(C^\mu) \leq d\mu V(C)$ . For the proof, define  $D = C^\mu$  and  $\lambda = 1/\mu > 1$ . Then  $D$  is a cap of  $K$  and  $D^\lambda = C$ . The inequalities  $\frac{1}{d} \lambda V(D) \leq V(D^\lambda) \leq \lambda^d V(D)$  translate directly to  $\mu^d V(C) \leq V(C^\mu) \leq d\mu V(C)$ . These inequalities will be used often. We call them the *trivial volume estimates*:

$$\begin{aligned} \frac{\lambda}{d} V(C) &\leq V(C^\lambda) \leq \lambda^d V(C) & \text{for } \lambda \geq 1, \\ \mu^d V(C) &\leq V(C^\mu) \leq d\mu V(C) & \text{for } 0 \leq \mu \leq 1, \end{aligned}$$

where the left-hand side of the first inequality only holds for  $C^\lambda \neq K$  and the right-hand side of the second inequality only for  $C \neq K$ . The *Macbeath region*, or *M-region* for short, with center  $z$  and factor  $\lambda > 0$  is

$$M(z, \lambda) = M_K(z, \lambda) = z + \lambda[(K - z) \cap (z - K)].$$

The *M-region* with  $\lambda = 1$  is just the intersection of  $K$  and  $K$  reflected with respect to  $z$ . Thus,  $M(z, 1)$  is convex and centrally symmetric with center  $z$  and  $M(z, \lambda)$  is a homothetic copy of  $M(z, 1)$  with center  $z$  and factor of homothety  $\lambda$ . We define the function  $u : K \rightarrow \mathbb{R}$  by

$$u(z) = V(M(z, 1)).$$

These definitions are from [8, 13] and [3]. The following results come from the same sources. We will use them extensively. We assume that  $K \subset \mathbb{R}^d$  is a convex body of volume 1. Set

$$(2.1) \quad s_0 = (2d)^{-2d}.$$

LEMMA 2.1. *If  $M(x, \frac{1}{2}) \cap M(y, \frac{1}{2}) \neq \emptyset$ , then  $M(x, 1) \subset M(y, 5)$ .*

LEMMA 2.2. *If  $C$  is a cap,  $z \in C$  and  $\lambda > 0$ , then  $K \cap M(z, \lambda) \subset C^{\lambda+1}$ .*

LEMMA 2.3. *If the cap  $C$  is contained in the  $M$ -region  $M(z, \mu)$  and  $\lambda > 0$ , then  $C^\lambda \subset M(z, \lambda\mu)$ .*

LEMMA 2.4. *If the bounding hyperplane of a cap  $C$  is tangent to  $K(v \geq s)$ , then  $s \leq V(C) \leq ds$ .*

Let  $K(v = s) = \partial K(v \geq s)$ . Assume that  $s \leq s_0$  and choose a maximal system of points  $Z = \{z_1, \dots, z_m\}$  on  $K(v = s)$  having pairwise disjoint Macbeath regions  $M(z_i, \frac{1}{2})$ . Such a system will be called *saturated*. Note that  $Z$  (and even  $m$ ) is not uniquely defined. However, for each  $K$  and  $s$ , we fix a saturated system  $Z$ . We write  $Z(s)$  and  $m(s) = |Z(s)|$  when we want to emphasize that our fixed saturated system comes from the level set  $K(v = s)$ . Clearly,  $V(C(z_i)) = s$ . Set

$$K'_i(s) = M(z_i, \frac{1}{2}) \cap C(z_i) \quad \text{and} \quad K_i(s) = C^{6\lambda}(z_i).$$

Note that  $K_i(s)$  is a cap of  $K$  and so, for  $\lambda > 0$ , the set  $K_i^\lambda(s) = C^{6\lambda}(z_i)$  is another cap of  $K$ .

The sets  $K'_i(s)$  and  $K_i(s)$  for  $i = 1, \dots, m(s)$  form what is called an *economic cap covering* in the paper of Bárány and Larman [8]. The following result, the *economic cap covering theorem*, comes from Theorem 6 in [8] and Theorem 7 in [3].

THEOREM 2.5. *For all  $s \in (0, s_0]$  and all convex bodies  $K \subset \mathbb{R}^d$  with  $V(K) = 1$ , we have:*

- (i)  $\bigcup_1^{m(s)} K'_i(s) \subset K(v \leq s) \subset \bigcup_1^{m(s)} K_i(s)$ ;
- (ii)  $s \leq V(K_i(s)) \leq 6^d s$ ,  $i = 1, \dots, m(s)$ ;
- (iii)  $(6d)^{-d} s \leq V(K'_i(s)) \leq 2^{-d} s$ ,  $i = 1, \dots, m(s)$ ;
- (iv) *every  $C$  with  $V(C) \leq s$  is contained in  $M(z_i, 15d) \subset K_i^{3d}(s)$  for some  $i$ .*

The sets  $K'_i(s)$  are pairwise disjoint, all of them have volume  $\geq (6d)^{-d} s$  and are all contained in  $K(v \leq s)$ . This gives an upper bound for  $m(s)$ . Similarly, the sets  $K_i(s)$  cover  $K(v \leq s)$  and all of them have volume  $\leq 6^d s$ . This gives a lower bound for  $m(s)$ . These simple arguments will be used repeatedly and we call them the *usual volume arguments*. Summarizing, we have

$$(2.2) \quad \frac{1}{6^d s} V(K(v \leq s)) \leq m(s) \leq \frac{1}{(6d)^{-d} s} V(K(v \leq s))$$

for  $s \leq s_0$ .

The economic cap covering theorem has the following direct consequence.

CLAIM 2.6. For  $s \leq s_0$  and  $\lambda > 1$ ,

$$K(v \leq \lambda s) \subset \bigcup K_i^{3d^2\lambda}(s).$$

PROOF. It is clear that  $K(v \leq \lambda s)$  is contained in the union of all caps  $C$  with  $V(C) = \lambda s$ . Let  $C$  be a cap with  $V(C) = \lambda s$ . The trivial volume estimates show that the cap  $C^{1/(d\lambda)}$  has volume at most  $s$  and is thus contained in some set  $M(z_i, 15d)$ . Then, by Lemma 2.3,  $C$  is contained in  $M(z_i, 15d^2\lambda)$  which is, by Lemma 2.2, a subset of  $C^{15d^2\lambda+1}(z_i) \subset (C^6(z_i))^{3d^2\lambda} = K_i^{3d^2\lambda}(s)$ .  $\square$

When  $P$  is a polytope of volume  $V(P)$ , the volume of the wet part  $P(v \leq s)$  was determined by Schütt [21], as well as by Bárány and Buchta [6]. As  $s \rightarrow 0$ ,

$$\frac{V(P(v \leq sV(P)))}{V(P)} = \frac{F(P)}{d!d^{d-1}} s \ln^{d-1}\left(\frac{1}{s}\right) (1 + o(1)).$$

Later, we need an estimate for  $m(s)$  and  $V(P(v \leq s))$ , depending on  $P$  only via  $F(P)$ . Such an estimate follows from results in Bárány [3]; see also [4], formula (4).

THEOREM 2.7. If  $P \subset \mathbb{R}^d$  is a polytope with  $V(P) > 0$ , then

$$\underline{c}(d) s \ln^{d-1}\left(\frac{1}{s}\right) \leq \frac{V(P(v \leq sV(P)))}{V(P)} \leq \bar{c}(d) F(P) s \ln^{d-1}\left(\frac{1}{s}\right)$$

and

$$\underline{c}(d) \ln^{d-1}\left(\frac{1}{s}\right) \leq m(sV(P)) \leq \bar{c}(d) F(P) \ln^{d-1}\left(\frac{1}{s}\right)$$

for  $s \leq s_0$ , where  $\underline{c}(d), \bar{c}(d) > 0$  are constants depending on  $d$ .

The second estimate concerning the number of caps,  $m(s)$ , follows from (2.2).

**3. Plan of proof.** This section explains the basic steps of the proof of Theorem 1.1.

*Step 1.* Our proof relies on a precise description of the boundary of a convex polytope. The essential ingredients are good bounds on how many sets  $K'_i(s)$  meet a given cap  $C$  of  $P$  and on the size of the set visible from  $z$  within  $P(v \leq T)$ . These are obtained in Section 4.

*Step 2.* In what follows,  $\alpha, \beta$  are positive constants, to be specified later, that depend only on dimension. Also, we use “ $\ln x$ ” as a shorthand for  $\ln(\ln x)$ . Define

$$(3.1) \quad T = T_\eta = \frac{\alpha \ln \eta}{\eta} \quad \text{and} \quad s = s_\eta = \frac{1}{\eta \ln^\beta \eta}.$$

We wish to show that, with high probability,  $\Pi_\eta$  is sandwiched between  $P(v \geq T)$  and  $P(v \geq s)$ , that is,

$$P(v \geq T) \subset \Pi_\eta \subset P(v \geq s).$$

(For technical reasons, we will have to replace  $T$  by the slightly larger  $T^* = d6^dT$ .) A convenient way to do so is to define a certain event  $A$ , which implies sandwiching, and whose complement,  $\bar{A}$ , has very small probability, namely,  $\mathbb{P}(\bar{A}) \ll F(P) \ln^{-4d^2} \eta$ . This will be achieved in Section 5.

The basic tool for proving our main result is a central limit theorem with weakly dependent random variables. Such an approach has already been used in geometric probability by Avram and Bertsimas [1] who also suggested its use in the study of random convex hulls. For the CLT we are going to use, the weak dependence of random variables is given by the so-called *dependency graph* which is defined as follows. Let  $\zeta_i, i \in \mathcal{V}$ , be a finite collection of random variables. The graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is said to be a dependency graph for  $\zeta_i$  if, for any pair of disjoint sets  $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V}$  such that no edge in  $\mathcal{E}$  goes between  $\mathcal{W}_1$  and  $\mathcal{W}_2$ , the sets of random variables  $\{\zeta_i : i \in \mathcal{W}_1\}$  and  $\{\zeta_i : i \in \mathcal{W}_2\}$  are independent. The following central limit theorem with weak dependence is due to Rinott [20]. A slightly weaker version (that would also work here) was earlier proven by Baldi and Rinott [2].

**THEOREM 3.1 (Rinott).** *Let  $\zeta_i, i \in \mathcal{V}$ , be random variables having a dependency graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . Set  $\zeta = \sum_{i \in \mathcal{V}} \zeta_i$  and  $\sigma^2(\zeta) = \text{Var } \zeta$ . Denote the maximal degree of  $\mathcal{G}$  by  $D$  and suppose that  $|\zeta_i - \mathbb{E}\zeta_i| \leq M$  almost surely. Then, for every  $x$ ,*

$$\left| \mathbb{P}\left(\frac{\zeta - \mathbb{E}\zeta}{\sqrt{\text{Var } \zeta}} \leq x\right) - \Phi(x) \right| \leq \frac{1}{\sqrt{2\pi}} \frac{DM}{\sigma(\zeta)} + 16 \frac{|\mathcal{V}|^{1/2} D^{3/2} M^2}{\sigma^2(\zeta)} + 10 \frac{|\mathcal{V}| D^2 M^3}{\sigma^3(\zeta)}.$$

When using this theorem, one has to define the dependency graph and prove the necessary properties. Also, we need a lower bound on  $\text{Var } \zeta$  (see Theorem 3.3 below) which comes from the companion paper [9].

*Step 3.* Define a graph  $\mathcal{G}$  whose vertex set  $\mathcal{V}$  is  $\{1, 2, \dots, m(T)\}$ , where  $m(T)$  is the size of the fixed saturated system of points on  $P(v = T)$ , as explained just before the cap covering theorem. The corresponding cap covering  $K_1(T), \dots, K_{m(T)}(T)$  is indexed by the vertices of  $\mathcal{G}$ . Two vertices  $i, j \in \mathcal{V}$  form an edge of  $\mathcal{G}$  if the caps  $K_i(T)$  and  $K_j(T)$  are “close to each other,” in a well-defined sense. This definition is crucial and will be explained in Sections 6 and 7. Also, it will be shown that the maximal degree of  $\mathcal{G}$  is  $\ll F(P)^6 \ln^{6(d-1)} \eta$ .

*Step 4.* Assume that the event  $A$  holds which, as mentioned above, implies “sandwiching.” Define the random variables  $\zeta_i, i \in \mathcal{V}$ , and check that  $\mathcal{G}$  is

indeed a dependency graph. The cases of  $\zeta = V(\Pi_\eta)$  and  $\zeta = f_\ell(\Pi_\eta)$  have to be handled somewhat differently. Next, we check that the conditions of Rinott's theorem hold. This will be done in Section 7. This proves the CLT for  $\zeta$  given  $A$ .

*Step 5.* Remove the conditioning on  $A$ . This is simpler for  $\zeta = V(\Pi_\eta)$ , as it is bounded, while  $\zeta = f_\ell(\Pi_\eta)$  is not. Section 8 is devoted to this task. The CLT for  $\zeta$  follows from the CLT for  $\zeta|A$  via the following transference lemma from [10], which has been used in an implicit form in [19] and [23], and possibly elsewhere.

LEMMA 3.2. *Let  $\xi_\eta$  and  $\xi'_\eta$  be two series of random variables with means  $\mu_\eta$  and  $\mu'_\eta$ , variances  $\sigma_\eta^2$  and  $\sigma'^2_\eta$ , respectively. Assume that there are functions  $\varepsilon_1(\eta), \varepsilon_2(\eta), \varepsilon_3(\eta), \varepsilon_4(\eta)$ , all tending to zero as  $\eta$  tends to infinity, such that:*

- (i)  $|\mu'_\eta - \mu_\eta| \leq \varepsilon_1(\eta)\sigma_\eta$ ;
- (ii)  $|\sigma'^2_\eta - \sigma_\eta^2| \leq \varepsilon_2(\eta)\sigma_\eta^2$ ;
- (iii) for every  $x$ ,  $|\mathbb{P}(\xi'_\eta \leq x) - \mathbb{P}(\xi_\eta \leq x)| \leq \varepsilon_3(\eta)$ ;
- (iv) for every  $x$ ,

$$\left| \mathbb{P}\left(\frac{\xi'_\eta - \mu'_\eta}{\sigma'_\eta} \leq x\right) - \Phi(x) \right| \leq \varepsilon_4(\eta).$$

There is then a positive constant  $c$  such that for every  $x$ ,

$$\left| \mathbb{P}\left(\frac{\xi_\eta - \mu_\eta}{\sigma_\eta} \leq x\right) - \Phi(x) \right| \leq c \sum_{i=1}^4 \varepsilon_i(\eta).$$

The transference lemma asserts that if  $\xi'_\eta$  satisfies the CLT (the fourth condition) and  $\xi_\eta$  is sufficiently close to  $\xi'_\eta$  in distribution (the first three conditions), then  $\xi_\eta$  also satisfies the CLT.

REMARK. In [10], the transference lemma is stated with  $\sigma'_\eta$  and  $\sigma'^2_\eta$  on the right-hand side of conditions (i) and (ii). It is easy to see that the present conditions imply those involving  $\sigma'_\eta$ : (ii) shows that  $\sigma'^2_\eta/\sigma_\eta^2$  tends to 1 as  $n \rightarrow \infty$ . Thus,  $\sigma_\eta^2 < 2\sigma'^2_\eta$  for large enough  $n$ . Then (ii) implies  $|\sigma'^2_\eta - \sigma_\eta^2| \leq 2\varepsilon_2(\eta)\sigma'^2_\eta$  and, similarly, (i) implies  $|\mu'_\eta - \mu_\eta| \leq \sqrt{2}\varepsilon_1(\eta)\sigma'_\eta$ .

To apply the central limit theorem and the transference lemma, we need a lower bound on  $\text{Var } \zeta$ . In the companion paper [9], we prove a lower bound for general convex bodies in terms of the volume of the floating body: Theorem 3.1 in [9] says that the variance of  $V(\Pi_\eta)$  is bounded from below by  $\eta^{-1}V(K(v \leq \eta^{-1}))$  and  $\text{Var } f_\ell(\Pi_\eta)$  is bounded by  $\eta V(K(v \leq \eta^{-1}))$ . Using Theorem 2.7, this gives the following result.



THEOREM 3.3. *Assume that  $P$  is a polytope of volume 1. Then*

$$F(P)\eta^{-2}\ln^{d-1}\eta \ll \text{Var } V(\Pi_\eta),$$

$$F(P)\ln^{d-1}\eta \ll \text{Var } f_\ell(\Pi_\eta).$$

Here, we use Vinogradov's " $\gg$ " notation, that is, we write  $f(\eta) \gg g(\eta)$  if there is a constant  $c > 0$ , independent of  $\eta$ , such that  $cf(\eta) > |g(\eta)|$  for all  $\eta \geq \eta_0$ . The constants  $c$  and  $\eta_0$  may, and usually do, depend on the dimension, but not on  $K$ .

The main achievements of this paper, besides the central limit theorems, are the precise sandwiching of  $\Pi_\eta$ , the novel definition of the dependency graph and the proof that its maximal degree is bounded by a power of  $\ln \eta$ . The latter is based on structural properties of the wet part  $P(v \leq t)$  for polytopes.

**4. On the boundary structure of convex polytopes.** In this section, we state some facts about the boundary structure of the polytope  $P$  and its floating body. All proofs in this section, except for those of Claim 4.6 and Lemma 4.7, which are given here, are postponed to Section 9.

So, the polytope  $P$  is fixed and its volume is 1. We need to consider two parameters,  $T$  and  $s$ , which have already been defined in (3.1). However, this is not important for the time being; we only assume that  $2s \leq T$ , say.

Let  $z \in P$  be a point with  $v(z) \leq T$  and write  $[x, z]$  for the closed segment joining  $z$  and a point  $x$ . The following definition is crucial, also having been used by Vu [22]. Set

$$S(z, T) = \{x \in P : [x, z] \cap P(v \geq T) = \emptyset\}.$$

This is the set of points that are *visible from  $z$  within  $P(v < T)$* . We are interested in the size of  $S(z, T)$ .

We again use the notation  $g(s) \ll f(s)$  if  $|g(s)| < cf(s)$  for all  $0 < s \leq t_0$  with constants  $c$  and  $t_0$  depending on the dimension, but not on the underlying convex set.

LEMMA 4.1. *If  $0 < v(z) \leq \frac{1}{2}$ ,  $2v(z) \leq T$ , then*

$$V(S(z, T)) \ll F(P)T \ln^{d-1}\left(\frac{T}{v(z)}\right).$$

Note that since  $S(z, T) \subset P(v \leq T)$ , Theorem 2.7 immediately implies the inequality  $V(S(z, T)) \ll F(P)T \ln^{d-1}(1/T)$ . The improvement from  $1/T$  to  $T/v(z)$  is significant in the range we are interested in.

Consider the economic cap covering from Theorem 2.5 for  $P(v \leq s)$ ,  $s \leq s_0$ , where  $s_0$  is defined in (2.1). The caps  $K_i(s)$  come from a saturated system

$Z(s) = \{z_1, \dots, z_{m(s)}\} \subset P(v = s)$  which is fixed together with  $P$  and  $s$ , as agreed just before the cap covering theorem was presented. We want to know how many  $z_i \in Z(s)$  can be contained in a fixed cap  $C$  of volume  $T$ .

LEMMA 4.2. *Assume that  $C$  is a cap of  $P$  of volume  $T$ . Then, for  $0 < s \leq s_0$ ,  $2s \leq T$ , we have*

$$|Z(s) \cap C| \ll F(P) \ln^{d-1} \left( \frac{T}{s} \right).$$

Next, consider the economic cap covering theorem for  $P(v \leq T)$ . The saturated system  $Y(T) = \{y_1, \dots, y_{m(T)}\}$  on  $P(v = T)$  is again fixed and so are the corresponding covering caps  $K_j(T)$ . [We use the notation  $Y(T)$ ,  $y_j(T)$  and  $m(T)$  in order to avoid confusion with  $Z(s)$ ,  $z_i(s)$  and  $m(s)$ .] We will need a bound on the number of those  $y_j \in Y(T)$  for which  $K_j^\lambda(T)$  contains a fixed  $z \in P(v = s)$ . Here,  $\lambda$  is a constant that depends only on  $d$ .

LEMMA 4.3. *Let  $\lambda \geq 1$  be a constant depending only on  $d$ . Assume that  $0 \leq 2s \leq T \leq (6\lambda)^{-d}s_0$ . If  $z \in P(v = s)$ , then*

$$|\{y_j \in Y(T) : z \in K_j^\lambda(T)\}| \ll F(P) \ln^{d-1} \left( \frac{T}{s} \right).$$

The constant in  $\ll$  depends on  $\lambda$  and, thus, again, only on the dimension.

We will also need a bound on the number of points  $z_j \in Z(s)$  that are contained in  $S(z, T)$  when  $z \in P(v = s)$ .

LEMMA 4.4. *Assume that  $z \in P(v = s)$  and  $0 < s \leq s_0$ ,  $2s \leq T$ . Then*

$$|Z(s) \cap S(z, T)| \ll F(P) \ln^{d-1} \left( \frac{T}{s} \right).$$

The following fact will be needed in the sandwiching step and concerns convex hulls of random points in  $K'_i(T)$ , the small sets in the cap covering theorem. Set  $T^* = d6^d T$ ,  $T \leq s_0$ . In each  $K'_i(T)$ , choose a point  $x_i$  arbitrarily.

CLAIM 4.5. *Under the above conditions,*

$$P(v \geq T^*) \subset [x_1, \dots, x_{m(T)}].$$

We mention in passing that the caps  $K_i^\gamma(T)$  cover  $P(v \leq T^*)$ , where  $\gamma = 3d^3 6^d$ :

$$(4.1) \quad P(v \leq T^*) \subset \bigcup_{i=1}^{m(T)} K_i^\gamma(T).$$

This follows directly from Claim 2.6.

The system  $Z(s) = \{z_1, \dots, z_{m(s)}\}$  on  $P(v = s)$  is saturated, so, for each  $a \in P(v = s)$ , there is a  $z_i$  with  $M(z_i, \frac{1}{2}) \cap M(a, \frac{1}{2}) \neq \emptyset$ . For each  $a$ , we fix such a  $z_i$  and denote it by  $z(a)$ .

CLAIM 4.6. *If a cap  $C$  contains the point  $a \in P(v = s)$ , then  $M(z(a), 1) \subset C^6$ .*

PROOF. This is very simple. As  $z(a)$  satisfies  $M(z(a), \frac{1}{2}) \cap M(a, \frac{1}{2}) \neq \emptyset$  by definition, it follows that Lemmas 2.1 and 2.2 imply that

$$M(z(a), 1) = P \cap M(z(a), 1) \subset P \cap M(a, 5) \subset C^6. \quad \square$$

The following lemma helps to bound the maximal degree of the dependency graph.

LEMMA 4.7. *Assume that  $a, b \in P(v = s)$  and the segment  $[a, b]$  is disjoint from  $P(v \geq T)$ . The segment  $[z(a), z(b)]$  is then disjoint from  $P(v \geq T^*)$ .*

PROOF. Both  $[a, b]$  and  $P(v \geq T)$  are convex, so they can be separated by a hyperplane since they are disjoint. This hyperplane cuts off a cap, say  $C$ , from  $K$  containing  $[a, b]$  and disjoint from  $P(v \geq T)$ . So,  $V(C) \leq dT$ , by Lemma 2.4. Further, Claim 4.6 implies that  $z(a), z(b) \in C^6$ . Consequently,  $[z(a), z(b)] \subset C^6$  and  $V(C^6) \leq d6^d T = T^*$  follows from the trivial volume estimate.  $\square$

**5. Sandwiching  $\Pi_\eta$ .** Recall that the Poisson polytope,  $\Pi_\eta$ , is the convex hull of  $X \cap P$ , where  $X = X(\eta)$  is a Poisson point process of intensity  $\eta$ . We are going to use the well-known fact that, with high probability, the boundary of  $\Pi_\eta$  is contained in a small strip close to the boundary of  $P$ . Results of this type have been proven in [7] and [22]. Here, we need a slightly different, perhaps more refined, estimate.

We make (3.1) more precise and set

$$T = T_\eta = \alpha \frac{\ln \eta}{\eta} \quad \text{with } \alpha = (6d)^d (4d^2 + d - 1).$$

In the following, we assume that  $\eta \geq \eta_0$ , where  $\eta_0$  is chosen such that  $T \leq s_0$ , with  $s_0$  defined as in (2.1). Let  $Y(T)$  be the fixed saturated point set  $\{y_1, \dots, y_{m(T)}\}$  on  $P(v = T)$  according to Theorem 2.5. We get an economic cap covering with caps  $K_j(T)$  and half Macbeath regions  $K'_j(T)$ ,  $j = 1, \dots, m(T)$ . To simplify notation, set  $K_j = K_j(T)$ ,  $K'_j = K'_j(T)$  and  $m_\eta = m(T)$ .

Let  $A'$  be the event that each  $K'_j$  contains at least one point of  $X$ , the Poisson point process with intensity  $\eta$ . Since the number of points in  $K'_j$  is Poisson distributed with parameter  $\eta V(K'_j)$ , from the fact that  $(6d)^{-d}T \leq V(K'_j) \leq 2^{-d}T$ , we have

$$\mathbb{P}(K'_j \cap X = \emptyset) = e^{-\eta V(K'_j)} \leq e^{-(6d)^{-d}\eta T}.$$

Let  $\overline{A}'$  denote the complement of the event  $A'$ . By Theorem 2.7,  $m_\eta \ll F(P) \times \ln^{d-1} \eta$ , so, by Boole's inequality,

$$(5.1) \quad \mathbb{P}(\overline{A}') \leq m_\eta e^{-(6d)^{-d}\eta T} \ll F(P)(\ln \eta)^{-(6d)^{-d}\alpha+d-1} = F(P) \ln^{-4d^2} \eta$$

follows from the choice of  $\alpha$ .

For later reference, we note that

$$(5.2) \quad \mathbb{P}(K'_j \cap X = \emptyset) \geq e^{-2^{-d}\eta T} = \ln^{-2^{-d}\alpha} \eta \geq \ln^{-(3d)^{d+2}} \eta.$$

Now, Claim 4.5 and (5.1) show that, with high probability,  $\Pi_\eta$  contains the floating body  $P(v \geq T^*)$ . (Recall that  $T^* = d6^d T$ .)

$$\mathbb{P}(\Pi_\eta \text{ does not contain } P(v \geq T^*)) \leq \mathbb{P}(\overline{A}') \ll F(P) \ln^{-4d^2} \eta.$$

This is the first half of the sandwiching. For the second half, we make the definition of  $s_n$  in (3.1) more precise and set

$$s = s_\eta = \frac{1}{\eta \ln^\beta \eta} \quad \text{where } \beta = 4d^2 + d - 1.$$

We claim that, with high probability,  $P(v \leq s)$  contains no point of  $X$ . Indeed,  $\eta V(P(v \leq s)) \ll F(P)(\ln \eta)^{-\beta+d-1}$ , by Theorem 2.7, and we get

$$(5.3) \quad \mathbb{P}(X \cap P(v \leq s) \neq \emptyset) = 1 - e^{-\eta V(P(v \leq s))} \ll F(P) \ln^{-4d^2} \eta.$$

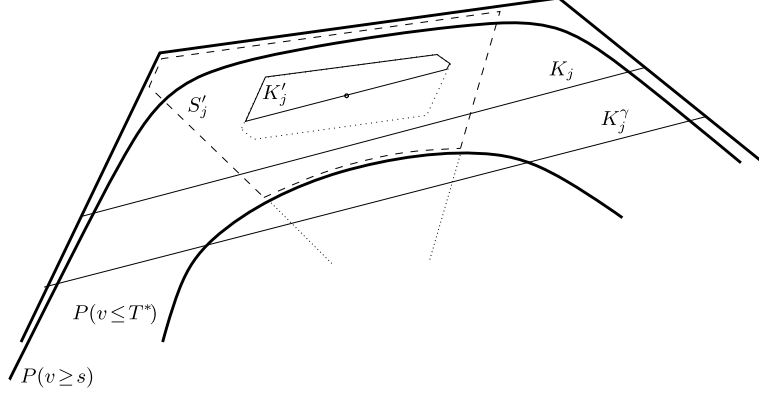
We have just proven that  $\Pi_\eta$  is sandwiched between  $P(v \geq s)$  and  $P(v \geq T^*)$  with high probability:

$$1 - \mathbb{P}(P(v \geq T^*) \subset \Pi_\eta \subset P(v \geq s)) \ll F(P) \ln^{-4d^2} \eta.$$

The proof of the CLT for  $V(\Pi_\eta)$  could be achieved via conditioning on  $A'$ . For  $f_\ell(\Pi_\eta)$ , we need a stronger condition, to be called  $A$ , which will also work for  $V(\Pi_\eta)$ . Set

$$\gamma = 3d^3 6^d.$$

For  $j = 1, \dots, m_\eta$ , let  $S_j = S_j(T)$  be pairwise internally disjoint closed sets with  $\bigcup S_j = P$ ,  $K'_j \subset S_j$  and  $S_j \cap P(v \leq T^*) \subset K_j^\gamma$ . [Recall, from Claim 2.6, that the sets  $K_j^\gamma$  cover  $P(v \leq T^*)$ .] Set  $S'_j = S'_j(T) = S_j \cap P(v \leq T^*)$ , see Figure 1.

FIG. 1. Definition of  $S'_j$ .

Before defining  $A$ , observe that the expected number of points of  $X$  lying in  $S'_j$  is  $\eta V(S'_j)$ . Trivial volume estimates show that

$$(5.4) \quad (6d)^{-d} \alpha \ln \eta \leq \eta V(K'_j) \leq \eta V(S'_j) \leq \eta V(K_j^\gamma) \leq (6\gamma)^d \alpha \ln \eta.$$

Define  $A$  to be the event that each  $K'_j$  contains at least one point,  $P(v \leq s)$  contains no point and each  $S'_j$  contains at most  $3(6\gamma)^d \alpha \ln \eta$  points of  $X$  ( $j = 1, \dots, m_\eta$ ). The following two claims are essential for our proof. We collect the properties of  $\Pi_\eta$  given the event  $A$  and estimate the probability of  $A$ .

CLAIM 5.1. *Given  $A$ , we have  $P(v \geq T^*) \subset \Pi_\eta \subset P(v \geq s)$  and  $|P(v \leq T^*) \cap X| \ll F(P) \ln^{d-1} \eta \ln \eta$ .*

PROOF. This follows immediately from the definition of  $A$  and from the estimate on the volume of  $P(v \leq T^*)$ .  $\square$

$$\text{CLAIM 5.2.} \quad \ln^{-(3d)^{d+2}} \eta \ll \mathbb{P}(\overline{A}) \ll F(P) \ln^{-4d^2} \eta.$$

PROOF. The lower bound follows from  $\mathbb{P}(\overline{A}) \geq \mathbb{P}(K'_1(T) \cap X = \emptyset)$  and from (5.2). For the upper bound, recall (5.1) and (5.3): in (5.1) we showed that  $K'_j \cap X = \emptyset$  for some  $j$  has probability  $\ll F(P) \ln^{-4d^2} \eta$ ; inequality (5.3) shows that  $X \cap P(v \leq s) \neq \emptyset$  with probability  $\ll F(P) \ln^{-4d^2} \eta$ .

So, we only have to estimate the probability that for some  $j$ , the set  $S'_j$  contains more than  $3(6\gamma)^d \alpha \ln \eta \geq \eta V(S'_j)$  points. Let  $N$  denote a Poisson random variable with parameter  $p$ . Then (see, e.g., [19])

$$\mathbb{P}(N \geq 3p) \leq \frac{3}{3-e} e^{-p}.$$

This inequality implies, by setting  $p = \eta V(S'_j)$ , that the probability that  $S'_j$  contains more than  $3(6\gamma)^d \alpha \ln \eta \geq 3p$  points from  $X$  is bounded from above by

$$\frac{3}{3-e} e^{-p} \leq \frac{3}{3-e} \exp(-(6d)^{-d} \alpha \ln \eta) \ll \ln^{-(4d^2+d-1)} \eta.$$

Combining this with the bound  $m_\eta \ll F(P) \ln^{d-1} \eta$  from Theorem 2.7 completes the proof.  $\square$

Set

$$U = U_\eta = \frac{\ln \eta}{\eta} \quad \text{and} \quad U^* = d6^d U.$$

Since we are assuming that  $V(P) = 1$ , Theorem 2.7 tells us that

$$b_1 \frac{\ln^d \eta}{\eta} \leq V(P(v \leq U^*)) \leq b_2 F(P) \frac{\ln^d \eta}{\eta}$$

with positive constants  $b_1, b_2$  depending only on  $d$ . Let  $B$  be the event that  $P(v \geq U^*) \subset \Pi_\eta$  and that  $P(v \leq U^*)$  contains at most  $3b_2 F(P) \ln^d \eta$  points from  $X$ . The following estimate will be useful in Section 8. Its proof is similar to those above, actually even simpler (as there is no need to worry about  $\alpha$ ), and is therefore left to the reader.

LEMMA 5.3.  $\mathbb{P}(\overline{B}) \ll F(P) \eta^{-3d}$ .

**6. The dependency graph.** It is high time to define the dependency graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ . The values of  $s, T$  and  $T^*$  have been given in the previous section. The sets  $K_i = K_i(T)$  and  $K'_i = K'_i(T)$  come from the cap covering theorem. The vertex set,  $\mathcal{V}$ , of the dependency graph is just  $\{1, \dots, m_\eta\}$ .

Define the set  $L_i$  as the union of all  $S'_k$  such that there are points

$$(6.1) \quad a \in S'_i \cap P(v \geq s), \quad b \in S'_k \cap P(v \geq s) \\ \text{with } [a, b] \text{ disjoint from } P(v \geq T^*).$$

Note that  $S'_i \subset L_i$  for all  $i$ . Also,  $S'_k \subset L_i$  holds if and only if  $S'_i \subset L_k$ . Now, distinct vertices  $i, j \in \mathcal{V}$  form an edge in  $\mathcal{G}$  if  $L_i$  and  $L_j$  contain at least one set  $S'_k$  in common,

$$(6.2) \quad ij \in \mathcal{E} \iff \exists k \in \{1, \dots, m_\eta\} \quad \text{such that } S'_k \subset L_i \cap L_j.$$

That this defines a dependency graph for the suitably chosen random variables is proved later, in Lemma 7.1. The main result of this section is an upper bound on the maximal degree  $D$  in  $\mathcal{G}$ .

Some preparation is needed. We need a bound on the number of sets  $S'_k \subset L_i$ .

LEMMA 6.1.  $|\{k : S'_k \subset L_i\}| \ll F(P)^3 (\ln \eta)^{3(d-1)}.$

PROOF. We first show that if  $S'_k \subset L_i$ , then there are also points

$$(6.3) \quad \begin{aligned} a' \in K_i^\gamma \cap P(v=s), \quad b' \in K_k^\gamma \cap P(v=s) \\ \text{with } [a', b'] \text{ disjoint from } P(v \geq T^*). \end{aligned}$$

To simplify notation, we write  $C' = C \cap P(v \geq s)$  when  $C$  is a cap of  $P$ . Clearly,  $C'$  is a cap of  $P(v \geq s)$ . We are going to use the fact that if two caps of a convex body have a point in common, then they also have a point in common from the boundary of the convex set.

Since the segment  $[a, b]$  is disjoint from  $P(v \geq T^*)$ , there is a cap  $C$ , also disjoint from  $P(v \geq T^*)$ , such that  $[a, b] \subset C'$ . Now,

$$a \in C' \cap K_i^{\gamma'}, \quad b \in C' \cap K_k^{\gamma'}.$$

Every one of the two sets above is a nonempty intersection of two caps of  $P(v \geq s)$ . So, each has a point,  $a'$  and  $b'$ , respectively, on the boundary of  $P(v \geq s)$ , which is  $P(v = s)$ . As the segment  $[a', b'] \subset C$ , it is disjoint from  $P(v \geq T^*)$ , which proves (6.3).

Recall that a saturated system  $Z(s) = \{z_1, \dots, z_{m(s)}\}$  has been chosen in  $P(v = s)$ . Also, for each  $x \in P(v = s)$ , we fixed a point  $z(x) \in Z(s)$  so that  $M(x, \frac{1}{2}) \cap M(z(x), \frac{1}{2}) \neq \emptyset$ . We have points  $a', b' \in P(v = s)$  satisfying (6.3). Claim 4.6 shows the existence of points  $z(a'), z(b') \in Z(s)$  such that  $z(a') \in K_i^{6\gamma}$ ,  $z(b') \in K_j^{6\gamma}$  and, by Lemma 4.7, the segment  $[z(a'), z(b')]$  is disjoint from  $P(v \geq T^o)$ , where  $T^o = d6^d T^*$ .

We bound the number of sets  $S'_k$  in  $L_i$  in three steps. In view of Lemma 4.2, with  $C = K_i^{6\gamma}$ , we have

$$|Z(s) \cap K_i^{6\gamma}| \ll F(P) \ln^{d-1} \left( \frac{V(K_i^{6\gamma})}{s} \right) \ll F(P) \ln^{d-1} \eta,$$

where the upper bound for  $V(K_i^{6\gamma})$  comes from (5.4). This is an upper bound on how many  $z(a') \in K_i^{6\gamma}$  there can be, given that the segment  $[z(a'), z(b')]$  starts at  $K_i^{6\gamma}$ .

In the second step, we estimate, for a fixed  $z(a')$ , the number of  $z(b') \in Z(s)$  such that  $[z(a'), z(b')]$  is disjoint from  $P(v \geq T^o)$ . All such  $z(b')$  lie in  $S(z(a'), T^o)$ . So, by Lemma 4.4, the number of such  $z(b')$  is

$$\ll F(P) \ln^{d-1} \left( \frac{T^o}{s} \right) \ll F(P) \ln^{d-1} \eta.$$

In the third step, we estimate the number of  $K_j^{6\gamma}$  that contain a fixed  $z(b') \in Z(s)$ . Lemma 4.3 implies, with  $\lambda = 6\gamma$ , that this number is

$$\ll F(P) \ln^{d-1} \left( \frac{T}{s} \right) \ll F(P) \ln^{d-1} \eta.$$

This argument shows that for a set  $S'_i$ , there are at most  $\ll F(P)^3 \ln^{3(d-1)} \eta$  sets  $S'_k$  which can be connected by some segment  $[a', b']$ . Since every set  $S'_k \subset L_i$  is connected to  $S'_i$  by some segment, the number of sets  $S'_k$  in  $L_i$  is  $\ll F(P)^3 \ln^{3(d-1)} \eta$ .  $\square$

The following result gives the upper bound on the maximal degree  $D$ .

**THEOREM 6.2.**  $D \ll F(P)^6 (\ln \eta)^{6(d-1)}.$

**PROOF.** By (6.2), we have  $ij \in \mathcal{E}$  if  $L_i \cap L_j$  contains some set  $S'_k$ . Clearly, if  $S'_k \subset L_j$ , then, by the definition (6.1), we also have  $S'_j \subset L_k$ . Thus,  $ij \in \mathcal{E}$  if there is some  $k$  such that  $S'_k \subset L_i, S'_j \subset L_k$ , which gives

$$D \leq \max_i \sum_{k: S'_k \subset L_i} |\{j: S'_j \subset L_k\}|.$$

Combined with Lemma 6.1, this gives the bound on the degree of  $\mathcal{G}$ .  $\square$

Thus, the graph  $\mathcal{G}$  has been defined and its maximal degree has been bounded. In the next section, we define the random variables  $\zeta_i$  and show that  $\mathcal{G}$  is a dependency graph.

## 7. The central limit theorem under condition A.

**PROOF OF THE CLT FOR  $V(\Pi_\eta)|A$ .** We introduce  $m_\eta$  random variables  $\zeta_j$  in the following way. For simpler notation, we keep writing  $K'_j$  for  $K'_j(T)$ ,  $K_j$  for  $K_j(T)$ ,  $S_j$  for  $S_j(T)$  and  $S'_j$  for  $S'_j(T)$ . We define  $\zeta_j$  as the missed volume in the set  $S_j$ ,

$$\zeta_j = V(S_j) - V(S_j \cap \Pi_\eta),$$

and  $\zeta$  as the missed volume in the polytope  $P$ ,

$$\zeta = \sum_{j=1}^{m_\eta} \zeta_j = V(P) - V(\Pi_\eta) = 1 - V(\Pi_\eta).$$

In order to prove the CLT for  $V(\Pi_\eta)|A$ , we simply check the conditions of Rinott's theorem. We start with the weak independence condition.

**LEMMA 7.1.** *Given disjoint subsets  $\mathcal{W}_1, \mathcal{W}_2$  of  $\mathcal{V}$  with no edge between them, the random variables  $\{\zeta_i: i \in \mathcal{W}_1\}$  are independent of the random variables  $\{\zeta_j: j \in \mathcal{W}_2\}$  under the conditional distribution of  $X$  given that  $A$  holds.*



PROOF. Under condition  $A$ , the boundary of  $\Pi_\eta$  lies in  $P(s < v \leq T^*)$  and, thus,  $\zeta_j = V(S'_j) - V(S'_j \cap \Pi_\eta)$ . The intersection  $S'_i \cap \Pi_\eta$  is determined by the facets  $[(d-1)\text{-dimensional faces}]$  of  $\Pi_\eta$  intersecting  $S'_i$ . These facets are determined by their vertices. Thus, all vertices that may determine a facet that intersects  $S'_i$  are contained in  $L_i$ . In other words,  $S'_i \cap \Pi_\eta$  is the same as the intersection of  $S'_i$  with the convex hull of  $X \cap L_i$ .

Now, set  $L^k = \bigcup_{i \in \mathcal{W}_k} L_i$  for  $k = 1, 2$ . By definition,  $L^1$  and  $L^2$  are unions of sets  $S'_k$  and have disjoint interiors. Given  $A$ , the  $\zeta_i, i \in \mathcal{W}_1$ , are determined by  $L^1 \cap X$  and the  $\zeta_i, i \in \mathcal{W}_2$ , are determined by  $L^2 \cap X$ . Since  $L^1 \cap X$  is independent of  $L^2 \cap X$ , conditional on  $A$  and otherwise, the claim follows.  $\square$

We have to check two more conditions of Rinott's theorem.

CLAIM 7.2. *Under condition  $A$ ,  $M = \max \|\zeta_j\|_\infty \ll (\ln \eta)/\eta$ .*

PROOF. This is very simple:  $\zeta_j \leq V(S'_j) \ll T \ll (\ln \eta)/\eta$ .  $\square$

CLAIM 7.3. *For  $\ln \eta \gg F(P)^{1/d^2}$ , we have  $\text{Var}(V(\Pi_\eta)|A) \gg F(P)\eta^{-2} \times \ln^{d-1} \eta$ .*

This claim is an easy corollary of Theorem 3.3 and (8.6) from the next section.

Bounds on  $|\mathcal{V}|, D, \zeta_j$  and  $\text{Var} \zeta = \text{Var}(V(\Pi_\eta)|A)$  have been established. Rinott's theorem can be applied. For  $\ln \eta \gg F(P)^{1/d^2}$ , the dominating error term is

$$\frac{|\mathcal{V}|D^2M^3}{\text{Var}(V(\Pi_\eta)|A)^{3/2}} \ll F(P)^{11.5} \frac{(\ln \eta)^{12d-9}}{(\ln \eta)^{(d-1)/2}}$$

as a simple computation shows. If  $\ln \eta$  equals  $F(P)^{1/d^2}$ , then the right-hand side is already  $\gg 1$ , which proves that this error term is valid for all  $\eta$ .  $\square$

PROOF OF THE CLT FOR  $f_\ell(\Pi_\eta)|A$ . The dependency graph remains the same. The random variables  $\zeta_i$  are to be defined, just as in [19], in the following way. Let  $F$  be an  $\ell$ -dimensional face of  $\Pi_\eta$  having  $f_0(S_i, F)$  vertices in  $S_i$  and set

$$\zeta_i = \frac{1}{\ell + 1} \sum_{\text{all } F} f_0(S_i, F).$$

Since, with probability one, no point from  $X$  lies in two  $S_j$ , and each face  $F$  is a simplex with probability one, the sum of the  $\zeta_i$  is equal to  $f_\ell(\Pi_\eta)$

almost surely. The analog of Lemma 7.1 for the new variables  $\zeta_i$  is proved in the same way.

We need to bound  $\max \|\zeta_i\|_\infty$  from above and, also,  $\text{Var } \zeta = \text{Var } f_\ell(\Pi_\eta)$  from below.

CLAIM 7.4. *For  $\ln \eta \gg F(P)^{1/d}$ , we have  $\text{Var}(f_\ell(\Pi_\eta)|A) \gg F(P) \ln^{d-1} \eta$ .*

Again, this follows from Theorem 3.3 and (8.8) in the next section.

CLAIM 7.5.  *$M = \max \|\zeta_i\|_\infty \ll F(P)^{3d} (\ln \eta)^{3d^2}$ .*

PROOF. (Similar to the one in Reitzner [19].) Condition  $A$  ensures that all vertices of  $\Pi_\eta$  lie in  $P(s < v \leq T^*)$ . As we have seen in the proof of Lemma 7.1, each face  $F$  intersecting  $S'_i$  has all of its vertices in  $L_i$ : if  $x \in S'_i$  and  $y \in S'_j$  are vertices of  $F$ , then  $y \in L_i$ . Under condition  $A$ ,  $S'_j$  contains  $\ll \ln \eta$  points from  $X$ . Thus, the number of vertices contributing to  $\zeta_i$  is  $\ll F(P)^3 (\ln \eta)^{(3d-2)}$ , by Lemma 6.1.

The number of  $\ell$ -faces (actually, all subsets of size  $\ell + 1$ ) on this many vertices is  $\ll (F(P)^3 (\ln \eta)^{(3d-2)})^{\ell+1}$ . Each such  $\ell$ -face contributes at most 1 to the value of  $\zeta_i$ . Consequently,

$$\zeta_i \ll (F(P)^3 (\ln \eta)^{(3d-2)})^{\ell+1} \ll F(P)^{3d} (\ln \eta)^{d(3d-2)}$$

since  $\ell + 1 \leq d$ .  $\square$

All conditions of Rinott's theorem have been established. The dominating error term is again the third one and we get the CLT for  $f_\ell(\Pi_\eta)|A$  with error term

$$\frac{|\mathcal{V}| D^2 M^3}{\text{Var}(f_\ell(\Pi_\eta)|A)^{3/2}} \ll F(P)^{15d} \frac{(\ln \eta)^{15d^2}}{(\ln \eta)^{(d-1)/2}}$$

as a simple computation shows.  $\square$

**8. Removing the conditioning.** We are going to use the transference Lemma 3.2.

LEMMA 8.1. *The random variables  $\xi_\eta = V(\Pi_\eta)$  and  $\xi'_\eta = V(\Pi_\eta)|A$  satisfy the conditions of Lemma 3.2 with*

$$\sum \varepsilon_i(\eta) \ll F(P)^{11.5} \ln^{-(d-1)/2+o(1)} \eta.$$

LEMMA 8.2. *The random variables  $\xi_\eta = f_\ell(\Pi_\eta)$  and  $\xi'_\eta = f_\ell(\Pi_\eta)|A$  satisfy the conditions of Lemma 3.2 with*

$$\sum \varepsilon_i(\eta) \ll F(P)^{15d} \ln^{-(d-1)/2+o(1)} \eta.$$

In both cases, the fourth condition of the transference lemma has been proven in the previous section with  $\varepsilon_4 \ll F(P)^{11.5} \ln^{-(d-1)/2+o(1)} \eta$  for the case of volume and with  $\varepsilon_4 \ll F(P)^{15d} \ln^{-(d-1)/2+o(1)} \eta$  for the number of faces. So, our main theorem for  $\Pi_\eta$  follows once the first three conditions of the transference lemma have been checked for the volume and for the number of faces. We will make use of the following simple claim.

CLAIM 8.3. *If  $\zeta$  is a nonnegative random variable and  $A$  is an event, then*

$$|\mathbb{E}(\zeta) - \mathbb{E}(\zeta|A)| \leq (\mathbb{E}(\zeta|A) + \mathbb{E}(\zeta|\bar{A}))\mathbb{P}(\bar{A}).$$

PROOF. It is clear that  $\mathbb{E}(\zeta) = \mathbb{E}(\zeta|A)\mathbb{P}(A) + \mathbb{E}(\zeta|\bar{A})\mathbb{P}(\bar{A})$ . Replacing  $\mathbb{P}(A)$  by  $1 - \mathbb{P}(\bar{A})$  here gives

$$\mathbb{E}(\zeta) - \mathbb{E}(\zeta|A) = (-\mathbb{E}(\zeta|A) + \mathbb{E}(\zeta|\bar{A}))\mathbb{P}(\bar{A})$$

and the claim follows.  $\square$

PROOF OF LEMMA 8.1. We need some preparations. We use Claim 8.3 with  $\zeta = 1 - V(\Pi_\eta)$ . We first estimate  $\mathbb{E}(\zeta^k|\bar{A})$  for  $k = 1, 2$ , the first two moments of  $\zeta|\bar{A}$ . (We will have to do a lot of similar estimations later.) Note that  $0 \leq \zeta^k \leq 1$ .

This is where we use the last paragraph of Section 5. Recall that  $B$  denotes the event that  $P(v \geq U^*) \subset \Pi_\eta$  and  $P(v \leq U^*)$  contains at most  $3b_2 F(P) \ln^d \eta$  points from  $X$ . Here,  $U = (\ln \eta)/\eta$  and  $U^* = d6^d U$ . Lemma 5.3 says that  $\mathbb{P}(\bar{B}) \ll F(P)\eta^{-3d}$ . Let  $I(B)$  denote the indicator function of the event  $B$ . Observe that  $\zeta^k I(B) \leq V(P(v \leq U^*))^k$ . Moreover,  $V(P(v \leq U^*)) \ll F(P)(\ln \eta)^d/\eta$ , by Theorem 2.7. So, we have

$$\begin{aligned} \mathbb{E}(\zeta^k|\bar{A}) &= \mathbb{E}(\zeta^k(1 - I(B))|\bar{A}) + \mathbb{E}(\zeta^k I(B)|\bar{A}) \\ (8.1) \quad &\leq \mathbb{E}((1 - I(B))|\bar{A}) + V(P(v \leq U^*))^k \\ &\ll \mathbb{P}(\bar{B}|\bar{A}) + \left(F(P) \frac{\ln^d \eta}{\eta}\right)^k \ll \left(F(P) \frac{\ln^d \eta}{\eta}\right)^k. \end{aligned}$$

Here, we have used the estimate

$$(8.2) \quad \mathbb{P}(\bar{B}|\bar{A}) \leq \frac{\mathbb{P}(\bar{B})}{\mathbb{P}(\bar{A})} \ll \frac{F(P)\eta^{-3d}}{(\ln \eta)^{-(3d)^{d+2}}} \ll F(P)\eta^{-3d+1},$$

where the lower bound for  $\mathbb{P}(\bar{A})$  comes from Claim 5.2.

As for  $\mathbb{E}(\zeta^k|A)$ , Claim 5.1 tells us that

$$\mathbb{E}(\zeta^k|A) \leq V(P(v \leq T^*))^k \ll \left(F(P) \frac{\ln^d \eta}{\eta}\right)^k.$$

Thus, we get, using Claim 8.3, that

$$(8.3) \quad |\mathbb{E}(\zeta^k|A) - \mathbb{E}(\zeta^k)| \ll \left( F(P) \frac{\ln^d \eta}{\eta} \right)^k \mathbb{P}(\overline{A}).$$

We check condition (ii) first. Since  $\text{Var}(V(\Pi_\eta)) = \text{Var}(1 - \zeta) = \text{Var}(\zeta) = \mathbb{E}(\zeta^2) - (\mathbb{E}(\zeta))^2$  and similarly for  $\text{Var}(V(\Pi_\eta)|A)$ , the aim is to estimate

$$(8.4) \quad \begin{aligned} & |(\mathbb{E}(\zeta^2|A) - (\mathbb{E}(\zeta|A))^2) - (\mathbb{E}(\zeta^2) - (\mathbb{E}(\zeta))^2)| \\ & \leq |\mathbb{E}(\zeta^2|A) - \mathbb{E}(\zeta^2)| + |(\mathbb{E}(\zeta|A))^2 - (\mathbb{E}(\zeta))^2|. \end{aligned}$$

The first term in the last line is bounded in (8.3) with  $k = 2$ . For the second, we have

$$(8.5) \quad \begin{aligned} & |(\mathbb{E}(\zeta|A))^2 - (\mathbb{E}(\zeta))^2| = |\mathbb{E}(\zeta|A) + \mathbb{E}(\zeta)| \cdot |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \\ & \ll F(P) \frac{\ln^d \eta}{\eta} |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \\ & \ll \left( F(P) \frac{\ln^d \eta}{\eta} \right)^2 \mathbb{P}(\overline{A}), \end{aligned}$$

where (8.3) and (8.1) have been applied with  $k = 1$ . We now need the lower bound  $\text{Var}(V(\Pi_\eta)) \gg F(P)\eta^{-2} \ln^{d-1} \eta$  from Theorem 3.3. Combining this lower bound, formulae (8.3), (8.4), (8.5) and Claim 5.2 yields

$$(8.6) \quad \begin{aligned} & |\text{Var}(V(\Pi_\eta)|A) - \text{Var}(V(\Pi_\eta))| \ll \left( F(P) \frac{\ln^d \eta}{\eta} \right)^2 \mathbb{P}(\overline{A}) \\ & \ll F(P)^2 \frac{\ln^{d+1} \eta}{\ln^{4d^2} \eta} \text{Var}(V(\Pi_\eta)). \end{aligned}$$

This shows that condition (ii) of Lemma 8.1 is satisfied with  $\varepsilon_2(\eta) \ll F(P)^2 \times \ln^{-4d^2+d+1} \eta$ . This, together with Theorem 3.3, also immediately proves Claim 7.3, that is,  $\text{Var}(V(\Pi_\eta)|A) \gg F(P)\eta^{-2} \ln^{d-1} \eta$  when  $\ln^{4d^2-d-1} \eta \gg F(P)^2$ .

Finally, (8.3) with  $k = 1$  gives

$$\begin{aligned} & |\mathbb{E}(V(\Pi_\eta)|A) - \mathbb{E}(V(\Pi_\eta))| \\ & = |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \ll F(P) \frac{\ln^d \eta}{\eta} \mathbb{P}(\overline{A}) \\ & \ll F(P)^2 \frac{\ln^d \eta}{\eta \ln^{4d^2} \eta} \ll F(P)^{3/2} \frac{\ln^{(d+1)/2} \eta}{\ln^{4d^2} \eta} \sqrt{\text{Var}(V(\Pi_\eta))}. \end{aligned}$$

Thus, condition (i) is also satisfied with  $\varepsilon_1(\eta) \ll F(P)^{3/2} \ln^{-4d^2+(d+1)/2} \eta$ .

Condition (iii) is the simplest to check: set  $\zeta = I(V(\Pi_\eta)) \leq x$  and apply Claim 8.3. Then

$$\begin{aligned} |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| &= |\mathbb{P}(V(\Pi_\eta) \leq x|A) - \mathbb{P}(V(\Pi_\eta) \leq x)| \\ &\leq 2\mathbb{P}(\overline{A}) \ll F(P) \ln^{-4d^2} \eta \end{aligned}$$

and thus (iii) holds with  $\varepsilon_3(\eta) \ll F(P) \ln^{-4d^2} \eta$ .  $\square$

PROOF OF LEMMA 8.2. This proof is similar to the previous one and so we only point out the main differences. Set  $\zeta = f_\ell(\Pi_\eta)$ . We want to estimate, for  $k = 1, 2$ ,

$$(8.7) \quad \mathbb{E}(\zeta^k|\overline{A}) = \mathbb{E}(\zeta^k(1 - I(B))|\overline{A}) + \mathbb{E}(\zeta^k I(B)|\overline{A}).$$

Note that, given  $B$ ,  $\Pi_\eta$  can have at most  $F(P) \ln^d \eta$  vertices, implying that  $\zeta^k I(B) \ll (F(P) \ln^d \eta)^{k(\ell+1)}$ , which is an upper bound for the second term in (8.7). The first term needs extra care since the random variable  $\zeta$  is not bounded. Let  $N$  be a random variable which is Poisson distributed with mean  $\eta$  and write  $E_m$  for the event  $N = m$ . Of course,  $\zeta \leq m^{\ell+1} \leq m^d$  under condition  $E_m$ . Thus,

$$\begin{aligned} &\mathbb{E}(\zeta^k(1 - I(B))|\overline{A}) \\ &= \sum_{m=0}^{\infty} \mathbb{E}(\zeta^k(1 - I(B))|\overline{A}E_m)\mathbb{P}(E_m) \\ &\leq \sum_{0 \leq m < 3\eta} \mathbb{E}(\zeta^k(1 - I(B))|\overline{A}E_m)\mathbb{P}(E_m) \\ &\quad + \sum_{3\eta \leq m} \mathbb{E}(\zeta^k(1 - I(B))|\overline{A}E_m)\mathbb{P}(E_m) \\ &\leq \sum_{0 \leq m < 3\eta} (3\eta)^{kd} \mathbb{E}((1 - I(B))|\overline{A}E_m)\mathbb{P}(E_m) \\ &\quad + \sum_{3\eta \leq m} m^{kd} \mathbb{E}((1 - I(B))|\overline{A}E_m)\mathbb{P}(E_m) \\ &\ll (3\eta)^{kd} \sum_{0 \leq m < 3\eta} \mathbb{P}(\overline{B}|\overline{A}E_m)\mathbb{P}(E_m) + \sum_{3\eta \leq m} m^{kd} \mathbb{P}(E_m) \\ &\ll (3\eta)^{kd} \mathbb{P}(\overline{B}|\overline{A}) + \sum_{3\eta \leq m} m^{kd} \mathbb{P}(E_m) \ll F(P) \eta^{-d+1}, \end{aligned}$$

where we have used (8.2) and the routine estimation of  $\sum_{3\eta}^{\infty} m^{kd} \mathbb{P}(E_m)$  is omitted. Using this and Claim 5.1 for  $\mathbb{E}(\zeta^k|A)$  yields, for  $k = 1, 2$ ,

$$\mathbb{E}(\zeta^k|A), \mathbb{E}(\zeta^k|\overline{A}) \ll (F(P) \ln^d \eta)^{k(\ell+1)} \leq F(P)^{kd} \ln^{kd^2} \eta.$$

We again need Claim 8.3 and the lower bound from Theorem 3.3 to show that

$$(8.8) \quad |\mathrm{Var}(\zeta|A) - \mathrm{Var}(\zeta)| \ll F(P)^{2d} \ln^{-2d^2-d+1} \eta \mathrm{Var}(\zeta).$$

So, condition (ii) is satisfied. It also follows that  $\mathrm{Var}(\zeta|A) \gg F(P) \ln^{d-1} \eta$  for  $\ln \eta \gg F(P)^{1/d}$ , which is Claim 7.4 from the previous section.

Checking condition (i) follows along the same lines and condition (iii) is straightforward.  $\square$

**9. Proofs of the auxiliary lemmas.** In this section, we assume that  $P$  is a fixed polytope in  $\mathbb{R}^d$  whose volume is 1. We first prove the following claim, where  $\beta = 2ed^3 + 1$  (a  $\beta$  which is different from the one in Section 5).

CLAIM 9.1. *For all  $T$  and all  $z \in P$  satisfying  $0 < v(z) < \frac{1}{2}, v(z) \leq T$ , we have*

$$S(z, T) \subset C^{\beta T/v(z)}(z).$$

PROOF. Set  $s = v(z)$ . Let  $C(z) = \{x \in \mathbb{R}^d | u \cdot x \geq h_P(u) - h_z\}$  be the minimal cap of  $z \in P$  and denote by  $H_h$  the hyperplane  $\{x \in \mathbb{R}^d | u \cdot x = h_P(u) - h\}$ . Then  $H_0$  touches the boundary of  $P$  in a center of the cap,  $H_{h_z}$  is the bounding hyperplane of  $C(z)$  and, thus,  $z \in H_{h_z}$ . Write  $Q_z = P \cap H_{h_z}$ . The following simple geometric arguments show that for every  $h \in [0, h_z]$ , we have

$$(9.1) \quad V_{d-1}(P \cap H_h) \leq 2dV_{d-1}(Q_z)$$

if  $s \leq \frac{1}{2}$ , where  $V_{d-1}(\cdot)$  stands for  $(d-1)$ -dimensional volume. Indeed, the Brunn–Minkowski inequality shows that for some  $h_{\max}$ , the volume of the sections  $P \cap H_h$  is first increasing for  $h \in [0, h_{\max}]$  and then decreasing for  $h \in [h_{\max}, w]$ . Here,  $w$  denotes the width of  $P$  in the direction  $u$ . Thus, if  $h_z \in [0, h_{\max}]$ , equation (9.1) is immediate (with  $2d$  replaced by 1). And if  $h_z > h_{\max}$ , then we have to show that  $V_{d-1}(P \cap H_{h_{\max}}) \leq 2dV_{d-1}(Q_z)$ . Clearly,

$$\frac{1}{d} w V_{d-1}(P \cap H_{h_{\max}}) \leq 1.$$

Since we assume that  $h_z > h_{\max}$  here,  $V_{d-1}(P \cap H_h)$  is decreasing for  $h \in [h_z, w]$  and we also have

$$(w - h_z) V_{d-1}(Q_z) \geq 1 - s \geq \frac{1}{2}.$$

Combining this gives

$$V_{d-1}(P \cap H_{h_{\max}}) \leq \frac{d}{w} \leq \frac{d}{w - h_z} \leq 2dV_{d-1}(Q_z),$$

which is (9.1). It follows that

$$s \leq 2dh_z V_{d-1}(Q_z).$$

Clearly, the set  $S(z, T)$  is the union of caps  $C \subset P(v \leq T)$  such that  $z \in C$ . Let  $C$  be such a cap. Then  $V(C) \leq dT$ , by Lemma 2.4. If  $C$  contains a point of  $H_h$ , then

$$V(C) \geq \frac{1}{d}(h - h_z)V_{d-1}(C \cap Q_z).$$

As is well known (see, e.g., [13]),  $z \in C$  is the center of gravity of  $Q_z$ . A result of Grünbaum [15] then tells us that  $V_{d-1}(C \cap Q_z) \geq \frac{1}{e}V_{d-1}(Q_z)$ . Thus,

$$\frac{1}{2ed^2} \frac{s(h - h_z)}{h_z} \leq \frac{1}{ed}(h - h_z)V_{d-1}(Q_z) \leq V(C) \leq dT.$$

Hence, the distance between an arbitrary point of  $S(z, T)$  and  $H_0$  is at most  $h = \frac{T}{s}2ed^3h_z + h_z \leq (2ed^3 + 1)\frac{T}{s}h_z$ , which shows that, indeed,  $S(z, T) \subset C^{\beta T/s}(z)$ .  $\square$

PROOF OF LEMMA 4.1. Again, setting  $v(z) = s$ , the condition is  $0 < s \leq \frac{1}{2}, 2s \leq T$ . Choose  $\beta$  as in Claim 9.1. Let  $C(z)$  be the minimal cap of  $z$  and set  $C^* = C^{\beta T/s}(z)$  and  $V^* = V(C^*)$ , noting that  $C^*$  is a polytope. By trivial volume estimates,  $V^* \leq (\beta T/s)^d V(C(z)) = (\beta T)^d/s^{d-1}$ . First, assume that  $C^* = P$  and, thus,  $1/T \leq \beta^d(T/s)^{d-1}$ . Then, since  $S(z, T) \subset P(v \leq T)$ , we have

$$V(S(z, T)) \ll F(P)T \ln^{d-1}\left(\frac{1}{T}\right)$$

for  $T \leq s_0$ , by Theorem 2.7, which gives

$$V(S(z, T)) \ll F(P)T \ln^{d-1}\left(\frac{T}{s}\right)$$

for any  $T$  with  $2s \leq T$ . For  $C^* \neq P$ , trivial volume estimates show that  $V(C^*) \geq (\beta T/ds)V(C(z)) = (\beta/d)T$ . Claim 9.1 shows that

$$S(z, T) \subset P(v_P \leq T) \cap C^* \subset C^*(v_{C^*} \leq T),$$

where we have written  $v_{C^*}$  to emphasize that the underlying convex set is now  $C^*$ . By Theorem 2.7, there is a constant  $s_0$  such that for  $T \leq s_0 V^*$ ,

$$\begin{aligned} V(S(z, T)) &\leq V(C^*(v_{C^*} \leq T)) \ll F(C^*)T \ln^{d-1}\left(\frac{V^*}{T}\right) \\ &\leq F(P)T \ln^{d-1}\left(\frac{V^*}{T}\right). \end{aligned}$$

Here, we used the fact that  $F(C^*) \leq F(P)$ , which can be proven quite easily (we omit the proof). In the remaining case,  $s_0 V^* \leq T \leq (d/\beta) V^*$ , we have

$$V(C^*(v_{C^*} \leq T)) \leq V^* \leq s_0^{-1} T \ll F(P) T \ln^{d-1} \left( \frac{V^*}{T} \right).$$

The lemma follows since  $V^*/T \leq \beta^d (T/s)^{d-1}$ .  $\square$

PROOF OF LEMMA 4.2. Assume that  $z_i \in Z(s) \cap C$ . Then, by Lemma 2.2,  $M(z_i, 1) \subset C^2$ . Thus, for  $s \leq s_0$ , the set  $K'_i(s) = M(z_i, \frac{1}{2}) \cap C(z_i)$  lies in  $P(v \leq s) \cap C^2$ . The sets  $K'_i(s)$ ,  $i = 1, \dots, m(s)$ , are pairwise disjoint, so the usual volume argument applies:

$$|Z(s) \cap C| \ll \frac{V(P(v \leq s) \cap C^2)}{s}$$

as  $V(K'_i(s)) \gg s$ . Further,  $P(v \leq s) \cap C^2 \subset C^2(v_{C^2} \leq s)$ , whose volume can be estimated in the same way as in the previous proof. Theorem 2.7 gives

$$V(C^2(v_{C^2} \leq s)) \ll F(C^2) s \ln^{d-1} \left( \frac{V(C^2)}{s} \right) \ll F(P) s \ln^{d-1} \left( \frac{T}{s} \right)$$

for  $s \leq s_0 V(C^2)$  since  $F(C^2) \ll F(P)$ . And, for  $s_0 V(C^2) \leq s \leq s_0$ , the lemma follows from the fact that  $V(C^2(v_{C^2} \leq s)) \leq V(C^2)$  and  $s \leq 2T$ .  $\square$

PROOF OF LEMMA 4.3. Since  $V(K_j^\lambda(T)) \leq \lambda^d V(K_j(T)) \leq (6\lambda)^d T$ , each  $y_j \in Y(T)$  with  $z \in K_j^\lambda(T)$  is contained in  $S(z, (6\lambda)^d T)$ . It is also clear that  $M(y_j, \frac{1}{2}) \cap C(y_j)$  lies in  $S(z, (6\lambda)^d T)$ , once  $y_j \in Y(T)$ . Thus, the usual volume argument applies, with the upper bound on  $V(S(z, (6\lambda)^d T))$  coming from Lemma 4.1.  $\square$

PROOF OF LEMMA 4.4. Let  $C(z)$  be the minimal cap of  $z$ . Claim 9.1 shows that  $S(z, T)$  is contained in the cap  $C := C^{\beta T/s}(z)$  with volume  $V(C) \leq (\beta T)^d / s^{d-1}$ . Lemma 4.2 then applies and gives

$$|Z(s) \cap S(z, T)| \leq |Z(s) \cap C| \ll F(P) \ln^{d-1} \frac{V(C)}{s} \ll F(P) \ln^{d-1} \left( \frac{T}{s} \right)$$

for  $s \leq s_0$  and  $2s \leq V(C)$  since  $V(C)/s \ll (T/s)^d$ . The inequality  $2s \leq V(C)$  follows from the trivial volume estimate if  $C \neq P$ , and from  $s \leq s_0$  if  $C = P$ .  $\square$

PROOF OF CLAIM 4.5. Clearly, it suffices to show that each cap  $C$  whose bounding hyperplane touches  $P(v \geq T^*)$  contains at least one point  $x_i$ . If this is not the case, then there is a cap  $C$  whose bounding hyperplane touches  $P(v \geq T^*)$  with no  $x_i \in C$  and thus no  $M(y_i, \frac{1}{2}) \subset C$  either.



We now claim that  $C^{1/3}$  is disjoint from all Macbeath regions  $M(y_i, \frac{1}{2})$ . Assume, for simpler notation, that  $u \cdot x = h$  with  $h > 0$  is the equation of the bounding hyperplane of  $C$ , and  $u \cdot x = 0$  is the equation of the supporting hyperplane of  $P$  and  $C$ . If  $u \cdot y_i = g$ , then  $M(y_i, 1)$  lies between hyperplanes  $u \cdot x = 2g$  and  $u \cdot x = 0$ . Thus,  $M(y_i, \frac{1}{2})$  lies between hyperplanes  $u \cdot x = \frac{3}{2}g$  and  $u \cdot x = \frac{1}{2}g$ . Here,  $\frac{3}{2}g > h$  holds since, otherwise,  $M(y_i, \frac{1}{2}) \subset C$ . Then  $g > \frac{2}{3}h$ , implying that  $u \cdot x = \frac{1}{3}h$  is a separating hyperplane between  $M(y_i, \frac{1}{2})$  and  $C^{1/3}$ . This proves the claim.

By trivial volume estimates,  $V(C^{1/6})$  is at least  $dT$ . Let  $x_0$  be the point in  $C^{1/6}$  where  $v(x)$  takes its maximal value on  $C^{1/6}$ . By Lemma 2.4,  $V(C^{1/6}) \leq dv(x_0)$  and so  $v(x_0) \geq T$ . This shows the existence of a point  $z \in P(v = T) \cap C^{1/6}$ . However, we then have that  $M(z, \frac{1}{2}) \subset C^{1/3}$  is disjoint from all  $M(y_i, \frac{1}{2})$ , which is impossible since  $Y(T) = \{y_1, \dots, y_{m(T)}\}$  is a saturated system.  $\square$

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