CONE-VOLUME MEASURE AND STABILITY

KÁROLY J. BÖRÖCZKY AND MARTIN HENK

Abstract. We show that the cone-volume measure of a convex body with centroid at the origin satisfies the subspace concentration condition. This implies, among others, a conjectured best possible inequality for the U-functional of a convex body. For both results we provide stronger versions in the sense of stability inequalities.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ having non-empty interiors, i.e., $K \in \mathcal{K}^n$ is a convex compact subset of the n-dimensional Euclidean space $\mathbb{R}^n$ with $\operatorname{int}(K) \neq \emptyset$. As usual, we denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ with associated Euclidean norm $\| \cdot \|$. $S^{n-1} \subset \mathbb{R}^n$ denotes the $(n-1)$-dimensional unit sphere, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The norm associated to a o-symmetric convex body $K \in \mathcal{K}^n$ is denoted by $\| \cdot \|_K$, i.e., $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda\, K\}$.

For $K \in \mathcal{K}^n$, we write $S_K(\cdot)$ and $h_K(\cdot)$ to denote its surface area measure and support function, respectively, and ν_K to denote the Gauß map assigning the exterior unit normal $\nu_K(x)$ to an $x \in \partial_* K$, where $\partial_* K$ consists of all points in the boundary ∂K of K having an unique outer normal vector. If the origin o lies in the interior of $K \in \mathcal{K}^n$, the *cone-volume measure of* K on S^{n-1} is given by

$$\mathrm{V}_K(\omega) = \int_\omega \frac{h_K(u)}{n}\, dS_K(u) = \int_{\nu_K^{-1}(\omega)} \frac{\langle x, \nu_K(x) \rangle}{n}\, d\mathcal{H}_{n-1}(x), \tag{1.1}$$

where $\omega \subset S^{n-1}$ is a Borel set and, in general, $\mathcal{H}_k(x)$ denotes the k-dimensional Hausdorff-measure. Instead of $\mathcal{H}_n(\cdot)$, we also write $\mathrm{V}(\cdot)$ for the n-dimensional volume.

The name cone-volume measure stems from the fact that if K is a polytope with facets $F_1, \ldots, F_m$ and corresponding exterior unit normals $u_1, \ldots, u_m$, then

$$\mathrm{V}_K(\omega) = \sum_{i=1}^m \mathrm{V}([o, F_i])\delta_{u_i}(\omega). \tag{1.2}$$

Date: July 29, 2014.
2010 *Mathematics Subject Classification.* 52A40, 52B11.
Key words and phrases. cone-volume measure, subspace concentration condition, U-functional, centro-affine inequalities, log-Minkowski Problem, centroid, polytope.

Here δ_u is the Dirac delta measure on S^{n-1} at $u \in S^{n-1}$, and for $x_1, \ldots, x_m \in \mathbb{R}^n$ and subsets $S_1, \ldots, S_L \subseteq \mathbb{R}^n$ we denote the convex hull of the set $\{x_1, \ldots, x_m, S_1, \ldots, S_l\}$ by $[x_1, \ldots, x_m, S_1, \ldots, S_l]$. With this notation $[o, F_i]$ is the cone with apex o and basis F_i.

In recent years, cone-volume measures have appeared and were studied in various contexts, see, e.g., F. Barthe, O. Guedon, S. Mendelson and A. Naor [6], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [8, 9], M. Gromov and V.D. Milman [17], M. Ludwig [28], M. Ludwig and M. Reitzner [29], E. Lutwak, D. Yang and G. Zhang [32], A. Naor [34], A. Naor and D. Romik [35], G. Paouris and E. Werner [36], A. Stancu [42], G. Zhu [45, 46].

In particular, cone-volume measure are the subject of the *logarithmic Minkowski problem*, which is the particular interesting limiting case $p = 0$ of the general L_p-Minkowski problem – one of the central problems in convex geometric analysis. It is the task:

Find necessary and sufficient conditions for a Borel measure μ on S^{n-1} to be the cone-volume measure V_K of $K \in \mathcal{K}^n$ (with o in its interior).

In the recent paper [9], K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang characterize the cone-volume measures of origin-symmetric convex bodies. In order to state their result we have to introduce the subspace concentration condition. We say that a Borel measure μ on S^{n-1} satisfies the *subspace concentration condition* if for any linear subspace $L \subset \mathbb{R}^n$, we have

$$\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n}\, \mu(S^{n-1}), \tag{1.3}$$

and equality in (1.3) for some L implies the existence of a complementary linear subspace $\widetilde{L}$ such that

$$\mu(\widetilde{L} \cap S^{n-1}) = \frac{\dim \widetilde{L}}{n}\, \mu(S^{n-1}), \tag{1.4}$$

and hence $\operatorname{supp} \mu \subset L \cup \widetilde{L}$, i.e., the support of the measure "lives" in $L \cup \widetilde{L}$.

Via the subspace concentration condition, the logarithmic Minkowski problem was settled in [9] in the symmetric case.

Theorem 1.1 ([9])**.** *A non-zero finite even Borel measure on S^{n-1} is the cone-volume measure of an origin-symmetric convex bodies if and only if it satisfies the subspace concentration condition.*

This result was proved earlier for discrete measures on S^1, i.e., for polygons, by A. Stancu [40, 41]. For cone-volume measures of origin-symmetric polytopes (cf. (1.2)) the necessity of (1.3) was independently shown by M. Henk, A. Schürmann and J.M.Wills [23] and B. He, G. Leng and K. Li [22].

We recall that the centroid of a k-dimensional convex compact set $M \subset \mathbb{R}^n$ is defined as

$$c(M) = \mathcal{H}_k(M)^{-1} \int_M x \, d\mathcal{H}_k(x).$$

The centroid seems also be the right and natural position of the origin in order to extend Theorem 1.1 to arbitrary convex bodies. In fact, in [24] it was shown by M. Henk and E. Linke that the necessity part of Theorem 1.1 also holds for polytopes with centroid at the origin, i.e.,

Theorem 1.2 ([24])**.** *Let $K \in \mathcal{K}^n$ be a polytope with centroid at the origin. Then its cone-volume measure V_K satisfies the subspace concentration condition.*

Our first result is an extension of Theorem 1.2 to convex bodies.

Theorem 1.3. *Let $K \in \mathcal{K}^n$ with centroid at the origin. Then its cone-volume measure satisfies the subspace concentration condition.*

While the subspace concentration condition is also the sufficiency property to characterize cone-volume measures among even non-trivial Borel measures, the cone-volume measure of a convex body $K \in \mathcal{K}^n$ whose centroid is the origin should satisfy some extra properties. For example, in Proposition 4.1 we prove that the measure of any open hemisphere is at least $\frac{1}{2n}$.

If the origin is the not the cetroid of the convex body, then the subspace concentration condition may not hold anymore. In fact, it was recently shown by G. Zhu [45] that for unit vectors $u_1, \ldots, u_m \in S^{n-1}$ in general position, $m \geq n+1$, and arbitrary positive numbers $\gamma_1, \ldots, \gamma_m$ there exists a polytope P with outer unit normals u_i with $\mathrm{V}_P(\{u_i\}) = \gamma_i$, $1 \leq i \leq m$. In other words, Zhu settled the logarithmic Minkowski problem for discrete measures whose support is in general position. In general, the centroid of such a polytope P is not the origin, and a full characterization of cone-volume measures of arbitrary polytopes/bodies is still a challenging and important problem.

We note that (1.4) is a kind of condition on the cone-volume measure which is independent of the choice of the origin.

Lemma 1.4. *If $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$, and $\operatorname{supp} \mathrm{V}_K \subset L \cup \widetilde{L}$ for the proper complementary linear subspaces $L, \widetilde{L} \subset \mathbb{R}^n$, then*

$$\mathrm{V}_K(L \cap S^{n-1}) = \frac{\dim L}{n}\, \mu(S^{n-1}).$$

Let us provide the simple argument leading to Lemma 1.4. It follows from Minkowski's uniqueness theorem that $K = M + \widetilde{M}$ where M, $\widetilde{M}$ are contained in affine spaces orthogonal to L, $\widetilde{L}$, respectively. By Fubini's theorem, we conclude (1.4) for V_K and the subspaces $L, \widetilde{L}$.

For a convex body K containing the origin in its interior, E. Lutwak, D. Yang and G. Zhang [30] defined the $\mathrm{SL}(n)$ invariant quantity $\mathrm{U}(K)$ as an integral over subsets $(u_1, \ldots, u_n) \in S^{n-1} \times \cdots \times S^{n-1}$, by

$$\mathrm{U}(K) = \left(\int_{u_1 \wedge \ldots \wedge u_n \neq 0} d\mathrm{V}_K(u_1) \cdots d\mathrm{V}_K(u_n) \right)^{\frac{1}{n}},$$

where $u_1 \wedge \ldots \wedge u_n \neq 0$ means that the vectors $u_1, \ldots, u_n$ are linearly independent. The U-functional has been proved useful in obtaining strong inequalities for the volume of projection bodies [30]. For information on projection bodies we refer to the books by Gardner [15] and Schneider [39], and for more information on the importance of centro-affine functionals we refer to C. Haberl and L. Parapatits [21, 29] and the references within.

We readily have $\mathrm{U}(K) \leq V(K)$, and equality holds if and only if $V_K(L \cap S^{n-1}) = 0$ for any non-trivial subspace of $\mathbb{R}^n$ according to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [10]. As a consequence of Theorem 1.3 we prove here a lower bound on $\mathrm{U}(K)$ in terms of $V(K)$ which was conjectured in [10].

Theorem 1.5. *Let $K \in \mathcal{K}^n$ with centroid at the origin. Then*

$$U(K) \geq \frac{(n!)^{1/n}}{n} V(K),$$

with equality if and only if K is a parallepiped.

In particular, $\mathrm{U}(K) > (1/\mathrm{e})V(K)$. For polytopes, Theorem 1.5 was shown in [24], where the special cases if K is an origin-symmetric polytope, or if $n = 2, 3$ were verified by B. He, G. Leng and K. Li [22], and G. Xiong [44], respectively.

In order to state another consequence of Theorem 1.5 we need the notation of an *isotropic measure*, going back to K.M. Ball's reformulation of the Brascamp-Lieb inequality in [2]. A Borel measure μ on S^{n-1} is called *isotropic* if

$$\mathrm{Id}_n = \int_{S^{n-1}} u \otimes u \, d\mu(u),$$

where Id_n is the $n \times n$-identity matrix and $u \otimes u$ the standard tensor product, i.e., $u \otimes u = u\, u^{\intercal}$.

Equating traces shows that for an isotropic measure $\mu(S^{n-1}) = n$. The subspace concentration condition of a Borel measure μ on S^{n-1} is equivalent to have an *isotropic* normalized linear image of μ, i.e., that is, there exists a $\Phi \in \mathrm{GL}(n)$ such that

$$\mathrm{Id}_n = \frac{n}{\mu(S^{n-1})} \int_{S^{n-1}} \frac{\Phi u}{\|\Phi u\|} \otimes \frac{\Phi u}{\|\Phi u\|} \, d\mu(u). \tag{1.5}$$

The equivalence in this general form is due to K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [10], while the discrete case was earlier handled by E. A. Carlen, and D. Cordero-Erausquin [11], and J. Bennett, A. Carbery, M. Christ and T. Tao [7] in their study of the Brascamp-Lieb inequality. Moreover, the case of a measure μ when strict inequality holds for all subspaces in (1.3) is due to B. Klartag [27]. Isotropic measures on S^{n-1} are discussed also e.g. in F. Barthe [3, 4], E. Lutwak, D. Yang and G. Zhang [31, 33]. We note that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits [25], see, e.g., F. Barthe and

D. Cordero-Erausquin [5], O. Guedon and E. Milman [20] and B. Klartag [26].

Now from Theorem 1.5 and by the equivalence (1.5) we immediately conclude

Corollary 1.6. *Every convex body $K \in \mathcal{K}^n$ has an affine image, whose cone-volume measure is isotropic.*

This, in particular, answers a question posed by E. Lutwak, D. Yang and G. Zhang [32].

In order to present stronger stability versions of Theorem 1.3 and Theorem 1.5 we need two notions of distance between the "shapes" of two convex bodies. Let $K, M \in \mathcal{K}^n$, and let $K' = K - c(K)$, $M' = M - c(M)$ be their translates whose centroids are the origin. Then we define

$$\begin{aligned} \delta_{\rm hom}(K,M) &= \min\{\lambda \geq 0 : \exists t > 0,\ M' \subset t\,K' \subset {\rm e}^{\lambda} M'\}, \\ \delta_{\rm vol}(K,M) &= \frac{{\rm V}\big(M' \Delta t\,K'\big)}{{\rm V}(M)},\ t = \frac{{\rm V}(M)^{1/n}}{{\rm V}(K)^{1/n}}, \end{aligned}$$

where $A\Delta B$ denotes the symmetric difference of two sets, i.e., $A\Delta B = A \setminus B \cup B \setminus A$.

Then both $\delta_{\rm hom}$ and $\delta_{\rm vol}$ are metrics on the space of convex bodies in $\mathbb{R}^n$ whose volumes are 1, and centroids are the origin.

Theorem 1.7. *Let $K \in \mathcal{K}^n$ with centroid at the origin, and let*

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{d-\varepsilon}{n}\,\mathrm{V}(K)$$

for a non-trivial linear subspace L with $\dim L = d$ and $\varepsilon \in (0, \varepsilon_0)$. Then there exist an $(n-d)$-dimensional compact convex set $C \subset L^{\perp}$, and a complementary d-dimensional compact convex set M such that

$$\delta_{\rm hom}(K, C+M) \leq \gamma_h \varepsilon^{1/(5n)} \quad \text{and} \quad \delta_{\rm vol}(K, C+M) \leq \gamma_v \varepsilon^{1/5},$$

where $\varepsilon_0, \gamma_h, \gamma_v > 0$ depend only on n.

Here $L^{\perp}$ denotes the orthogonal complement of L, and M is called a complementary compact convex set of C, if the linear spaces generated by M and C are complementary.

Observe that the range of ε, i.e., ε_0, in Theorem 1.7 has to depend on the dimension. For if, let $K \in \mathcal{K}^n$ be a simplex whose centroid is the origin, and let L be generated by d outer normals of the simplex, $d \in \{1, \ldots, n-1\}$. Then we have $\mathrm{V}_K(L \cap S^{n-1}) = \frac{d}{n+1}\,\mathrm{V}(K)$.

Actually, if L is 1-dimensional, then a more precise version of Theorem 1.7 holds.

Theorem 1.8. *Let $K \in \mathcal{K}^n$ with centroid at the origin, and let*

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{1-\varepsilon}{n}\,\mathrm{V}(K)$$

for a linear subspace L with $\dim L = 1$ *and* $\varepsilon \in (0, \tilde{\varepsilon}_0)$. *Then there exist* $(n-1)$*-dimensional compact convex set* $C \subset L^\perp$ *with* $c(C) = o$, *and* $x, y \in \partial K$ *such that* $y = -e^s x$ *where* $|s| < \tilde{\gamma}_v \varepsilon^{\frac{1}{6}}$, $[x, y] + C \subset K$, *and*

$$K \subset [x, y] + (1 + \tilde{\gamma}_h \varepsilon^{\frac{1}{6n}})C \quad \text{and} \quad \mathrm{V}(K) \le (1 + \tilde{\gamma}_v \varepsilon^{\frac{1}{6}})\mathrm{V}([x, y] + C),$$

where $\tilde{\varepsilon}_0, \tilde{\gamma}_h, \tilde{\gamma}_v > 0$ *depend only on* n.

We use this theorem in order to deduce the following stability version of Theorem 1.5.

Theorem 1.9. *Let* $K \in \mathcal{K}^n$ *with centroid at the origin, and let*

$$\mathrm{U}(K) \le (1 + \varepsilon)\frac{(n!)^{1/n}}{n}\mathrm{V}(K)$$

for $\varepsilon \in (0, \varepsilon_*)$. *Then there exists a* K *containing parallepiped* P, *such that for any facet* F *of* P, *we have*

$$\mathcal{H}_{n-1}(F \cap K) \ge (1 - \gamma_* \varepsilon^{\frac{1}{6}})\mathcal{H}_{n-1}(F),$$

where $\varepsilon_*, \gamma_* > 0$ *depend only on* n. *In particular, we have*

$$(1 - \gamma \varepsilon^{\frac{1}{6n}})P \subset K \quad \text{and} \quad \mathrm{V}(P \backslash K) \le \gamma \varepsilon^{\frac{1}{6}} \mathrm{V}(K).$$

The paper is organized as follows. In the next section we collect some basic facts and notations from convexity which will be used later on. The third section is devoted to the proof of Theorem 1.3. In Section 4 we show another characetristic property of cone-volume measures of convex bodies with centroid at the origin. The proofs of Theorem 1.7, 1.8 are given in Section 8 and are prepared in Sections 5–7. Finally, in Section 9 we prove Theorem 1.5.

Acknowledgements. We are grateful to Rolf Schneider for various ideas shaping this paper. We also acknowledge fruitful discussions with Daniel Hug and David Preiss about the Gauß-Green theorem.

2. Preliminaries

Good general references for the theory of convex bodies are provided by the books of Gardner[15], Gruber[18], Schneider[39] and Thompson[43].

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of convex body $K \in \mathcal{K}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{\langle x, y\rangle : y \in K\}.$$

A boundary point $x \in \partial K$ is said to have a unit outer normal (vector) $u \in S^{n-1}$ provided $\langle x, u\rangle = h_K(u)$. $x \in \partial K$ is called singular if it has more than one unit outer normal, and $\partial_* K$ is the set of all non-singular boundary points. It is well known that the set of singular boundary points of a convex body has $\mathcal{H}_{n-1}$-measure equal to 0. For each Borel set $\omega \subset S^{n-1}$, the inverse spherical image of ω is the set of all points of ∂K which have an outer unit normal belonging to ω. Since the inverse spherical image of ω differs from

$\nu_K^{-1}(\omega) \subseteq \partial_* K$ by a set of $\mathcal{H}_{n-1}$-measure equal to 0, we will often make no distinction between the two sets.

For $K \in \mathcal{K}^n$ the Borel measure S_K on S^{n-1} given by

$$S_K(\omega) = \mathcal{H}_{n-1}(\nu_K^{-1}(\omega))$$

is called the (Aleksandrov-Fenchel-Jessen) surface area measure. Observe that

$$\mathrm{V}(K) = \mathrm{V}_K(S^{n-1}) = \int_{S^{n-1}} \frac{h_K(u)}{n} d\, S_K(u).$$

As usual, for two subsets $C, D \subseteq \mathbb{R}^n$ and reals $\nu, \mu \geq 0$ the Minkowski combination is defined by

$$\nu\, C + \mu\, D = \{\nu\, c + \mu\, d : c \in C,\, d \in D\}.$$

By the celebrated Brunn-Minkowski inequality we know that the n-th root of the volume of the Minkowski combination is a concave function. More precisely, for two convex compact sets $K_0, K_1 \subset \mathbb{R}^n$ and for $\lambda \in [0, 1]$ we have

$$\mathrm{V}((1-\lambda)\, K_0 + \lambda\, K_1)^{1/n} \geq (1-\lambda)\, \mathrm{V}(K_0)^{1/n} + \lambda\, \mathrm{V}(K_1)^{1/n} \tag{2.1}$$

with equality for some $0 < \lambda < 1$ if and only if K_0 and K_1 lie in parallel hyperplanes or are homothetic, i.e., there exist $t \in \mathbb{R}^n$ and $\mu \geq 0$ such that $K_1 = t + \mu\, K_0$ (see also [16]).

Let $f : C \to \mathbb{R}_{>0}$ be a positive function on an open convex subset $C \subset \mathbb{R}^n$ with the property that there exists a $k \in \mathbb{N}$ such that $f^{1/k}$ is concave. Then by the (weighted) arithmetic-geometric mean inequality

$$\begin{aligned} f((1-\lambda)\, x + \lambda\, y) &= \Big(f^{1/k}((1-\lambda)\, x + \lambda\, y)\Big)^k \\ &\geq \Big((1-\lambda) f^{1/k}(x) + \lambda f^{1/k}(y)\Big)^k \\ &\geq f^{1-\lambda}(x) \cdot f^{\lambda}(y). \end{aligned}$$

This means that f belongs to the class of log-concave functions which by the positivity of f is equivalent to

$$\ln f((1-\lambda)\, x + \lambda\, y) \geq (1-\lambda) \ln f(x) + \lambda \ln f(y)$$

for $\lambda \in [0, 1]$. Hence, for all $x, y \in C$ there exists a subgradient $g(y) \in \mathbb{R}^n$ such that (cf., e.g., [38, Sect. 23])

$$\ln f(x) - \ln f(y) \leq \langle g(y), x - y \rangle. \tag{2.2}$$

If f is differentiable at y, the subgradient is the gradient of $\ln f$ at y, i.e., $g(y) = \nabla \ln f = \frac{1}{f(y)} \nabla f(y)$.

For a subspace $L \subseteq \mathbb{R}^n$, let $L^{\perp}$ be its orthogonal complement, and for $X \subseteq \mathbb{R}^n$ we denote by $X|L$ its orthogonal projection onto L, i.e., the image of X under the linear map forgetting the part of X belonging to $L^{\perp}$.

Here, for a convex body $K \in \mathcal{K}^n$ and a d-dimensional subspace L, $1 \leq d \leq n-1$, we are interested in the function measuring the volume of K intersected with planes parallel to $L^\perp$, i.e., in the function

$$f_{K,L} : L \to \mathbb{R}_{\geq 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)), \tag{2.3}$$

where $k = n - d$ is the dimension of $L^\perp$. By the Brunn-Minkowski inequality and the remark above, $f_{K,L}$ is a log-concave on function on $K|L$ which is positive at least in the relative interior of $K|L$ (cf. [1]). $f_{K,L}$ is also called the k-dimensional X-ray of K parallel to $L^\perp$ (cf. [15]). By well-known properties of concave functions we also know

Proposition 2.1.

i) $\ln f_{K,L}$ – *and thus* $f_{K,L}$ – *is continuous on* $\operatorname{int}(K)|L$. *Moreover,* $\ln f_{K,L}$ – *and thus* $f_{K,L}$ – *are Lipschitzian on any compact subset of* $(\operatorname{int} K)|L$.
ii) $\ln f_{K,L}$ – *and thus* $f_{K,L}$ – *is on* $\operatorname{int}(K)|L$ *almost everywhere differentiable, i.e., there exists a dense subset* $D \subseteq \operatorname{int}(K)|L$, *where* $\nabla f_{K,L}$ *exists.*

Proof. For i) see, e.g., [39, Theorem 1.5.3], and for ii) see, e.g., [38, Theorem 25.5]. □

Now for $K \in \mathcal{K}^n$ with centroid at 0, i.e., $c(K) = 0$, we have by Fubini's theorem with respect to the decomposition $L \oplus L^\perp$

$$\begin{aligned} 0 &= \int_K x \,\mathrm{d}\mathcal{H}_n(x) \\ &= \int_{K|L} \left(\int_{(\hat{x}+L^\perp)\cap K} \tilde{x} \,\mathrm{d}\mathcal{H}_k(\tilde{x}) \right) \mathrm{d}\mathcal{H}_d(\hat{x}) \\ &= \int_{K|L} f_{K,L}(\hat{x})\, c((\hat{x} + L^\perp) \cap K) \,\mathrm{d}\mathcal{H}_d(\hat{x}). \end{aligned}$$

Writing $c((\hat{x} + L^\perp) \cap K) = \hat{x} + \tilde{y}$ with $\tilde{y} \in L^\perp$ gives

$$\int_{K|L} f_{K,L}(\hat{x})\, \hat{x} \,\mathrm{d}\mathcal{H}_d(\hat{x}) = 0. \tag{2.4}$$

3. Proof of Theorem 1.3

For the proof of Theorem 1.3 we first establish some more properties of the function $f_{K,L}$, where we always assume that $L \subset \mathbb{R}^n$ is a d-dimensional linear subspace, $1 \leq d \leq n-1$, with k-dimensional orthogonal complement $L^\perp$. We recall that a function f is said to be upper semicontinuous on $K|L$ if whenever $x, y_m \in K|L$ for $m \in \mathbb{N}$ and y_m tends to x, then

$$f(x) \geq \limsup_{m\to\infty} f(y_m).$$

Lemma 3.1. *The function* $f_{K,L}$ *is upper semicontinuous on* $K|L$.

Proof. Let $x, y_m \in K|L$ for $m \in \mathbb{N}$ be such that $\lim_{m\to\infty} y_m = x$. According to the Blaschke selection principle (cf., e.g., [39]), we may assume that the sequence of compact convex sets

$$C_m = [(y_m + L^{\perp}) \cap K] - y_m \subset L^{\perp}$$

tends to a compact convex set $C \subset L^{\perp}$ in the Hausdorff topology. Since the k-volume of a compact convex set in $L^{\perp}$ is a continuous functional, we have $\mathcal{H}_k(C) = \lim_{m\to\infty} f_{K,L}(y_m)$. However, $x + C \subset K$, and therefore $f_{K,L}(x) \geq \mathcal{H}_k(C)$. □

An immediate consequence is that for sequences from the relative interior of $K|L$, $f_{K,L}$ behaves "continiuously", i.e.,

Corollary 3.2. *Let $o \in \operatorname{int} K$ and $x \in K|L$. Then $\lim_{m\to\infty} f_{K,L}(e^{\frac{-1}{m}} x) = f_{K,L}(x)$.*

Proof. Since $o \in \operatorname{int} K$, we get by the concavity of $f_{K,L}^{1/k}$ that

$$f_{K,L}(e^{\frac{-1}{m}} x) \geq e^{\frac{-k}{m}} f_{K,L}(x).$$

Since $f_{K,L}$ is also upper semicontinuous on $K|L$ by Lemma 3.1, we conclude the corollary. □

Although the gradient $\nabla f_{K,L}$ might not be bounded, its norm belongs to the space $L^1(K|L)$ of absolute integrable functions.

Lemma 3.3. *$\|\nabla f_{K,L}\| \in L^1(K|L)$, and thus the function $x \mapsto \langle \nabla f_{K,L}(x), x\rangle$ is in $L^1(K|L)$, as well.*

Proof. Let $f = f_{K,L}$. Since $\nabla f(x) = \nabla (f^{\frac{1}{k}})^k(x) = k f^{\frac{k-1}{k}}(x) \nabla f^{\frac{1}{k}}(x)$ for almost all $x \in K|L$, it is sufficient to prove $\|\nabla h\| \in L^1(K|L)$ for the concave function $h = f^{\frac{1}{k}}$. However, by the Brunn-Minkowski theorem, the graph X of the function h over $K|L$ is part of the boundary of a $(d+1)$-dimensional compact convex set. Thus

$$\int_{K|L} \|\nabla h\| \, d\mathcal{H}_d(x) \leq \int_{K|L} \sqrt{1 + \|\nabla h\|^2} \, d\mathcal{H}_d(x) = \mathcal{H}_d(X) < \infty.$$

□

The next two statements, which are the core ingredients of the proof of Theorem 1.3 have been proved in the special case of polytopes in [24].

Proposition 3.4. *If $o \in \operatorname{int} K$, then*

$$n \operatorname{V}_K(L \cap S^{n-1}) = d \operatorname{V}(K) + \int_{K|L} \langle \nabla f_{K,L}(x), x\rangle \, d\mathcal{H}_d(x).$$

Proof. Let $f = f_{K,L}$, and let $F(x) = f(x)x$ for $x \in K|L$, which is a Lipschitz vector field on any compact subset of $(\operatorname{int} K)|L$ (cf. (2.1) i)). To state the Gauß-Green divergence theorem for Lipschitz vector fields on Lipschitz domains, we follow W.F. Pfeffer [37]. Naturally,

$$E_m = e^{\frac{-1}{m}} K|L \subset (\operatorname{int} K)|L$$

is a compact Lipschitz domain for $m \geq 1$, and hence $\overline{\partial}_\star E_m = \overline{\partial} E_m$ according to Proposition 4.1.2 in [37], where $\overline{\partial}(E_m)$ denotes the (relative) boundary with respect to the linear space L.

Therefore Theorem 6.5.4 in [37] (going back to H. Federer [13]) yields that

$$\int_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \, d\mathcal{H}_{d-1}(x) = \int_{E_m} \operatorname{div} F(x) \, d\mathcal{H}_d(x). \tag{3.1}$$

If $y \in \overline{\partial}(K|L)$ then $\nu_{K|L}(y) = \nu_{E_m}(e^{\frac{-1}{m}} y)$; thus the left hand side of (3.1) is

$$\begin{aligned}\int_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \, d\mathcal{H}_{d-1}(x) =& e^{\frac{-(d-1)}{m}} \int_{\overline{\partial}(K|L)} \langle F(e^{\frac{-1}{m}} y), \nu_{K|L}(y)\rangle \, d\mathcal{H}_{d-1}(y) \\ =& e^{\frac{-d}{m}} \int_{\overline{\partial}(K|L)} f(e^{\frac{-1}{m}} y)\langle y, \nu_{K|L}(y)\rangle \, d\mathcal{H}_{d-1}(y).\end{aligned}$$

Therefore, Corollary 3.2 and the Lebesgue dominated convergence theorem yield

$$\lim_{m\to\infty} \int_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \, d\mathcal{H}_{d-1}(x) = \int_{\overline{\partial}(K|L)} f(y)\langle y, \nu_{K|L}(y)\rangle \, d\mathcal{H}_{d-1}(y). \tag{3.2}$$

Now, in order to evaluate the right hand side let $X = \partial K \cap (L^\perp + \overline{\partial}(K|L))$. Then the set of smooth points of ∂K in X, i.e., $\partial_\star K \cap X$ coincides with the set of points in $\nu_K^{-1}(L \cap S^{n-1})$. In addition, if $z \in X \cap \partial_\star K$, then $\nu_{K|L}(y) = \nu_K(z)$ for $y = z|L$, and thus (1.1) and (3.2) give

$$\begin{aligned}\lim_{m\to\infty} \int_{\overline{\partial} E_m} \langle F(x), \nu_{E_m}(x)\rangle \, d\mathcal{H}_{d-1}(x) &= \int_X \langle z, \nu_K(z)\rangle \, d\mathcal{H}_{n-1}(z) \\ &= n \, \mathrm{V}_K(L \cap S^{n-1}). \end{aligned} \tag{3.3}$$

Next, if $\nabla f(x)$ exists at $x \in \operatorname{int}(K)|L$, then

$$\operatorname{div} F(x) = d\, f(x) + \langle x, \nabla f(x)\rangle.$$

Therefore the right hand side of (3.1) is (cf. Proposition 2.1 ii), Lemma 3.3)

$$\int_{E_m} \operatorname{div} F(x) \, d\mathcal{H}_d(x) = d \int_{E_m} f(x) \, d\mathcal{H}_d(x) + \int_{E_m} \langle x, \nabla f(x)\rangle \, d\mathcal{H}_d(x).$$

Since $\int_{K|L} f(x) \, d\mathcal{H}_d(x) = \mathrm{V}(K)$, we deduce that

$$\lim_{m\to\infty} \int_{E_m} \operatorname{div} F(x) \, d\mathcal{H}_d(x) = dV(K) + \int_{K|L} \langle x, \nabla f(x)\rangle \, d\mathcal{H}_d(x). \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) completes the proof of the proposition. □

If K is an o-symmetric convex body, we know by the Brunn-Minkowski inequality (2.1) that $f_{K,L}(x)$ attains its maximum at the origin 0. Hence, in view of (2.2) we know that $\langle \nabla f_{K,L}(x), x\rangle \leq 0$ for almost every $x \in K|L$. Although, this is no longer true for bodies with centroid at 0, the next proposition shows that it is true in the average.

Proposition 3.5. *If $c(K) = o$, then*

$$\int_{K|L} \langle \nabla f_{K,L}(x), x\rangle \, d\mathcal{H}_d(x) \leq 0,$$

with equality if and only if $f_{K,L}$ is constant on $K|L$.

Proof. Again, let $f = f_{K,L}$ and let $g : K|L \to L$ be a subgradient of f. For $z \in (\text{int}K)|L$, applying (2.2) to $y = o$ and $x = z$ first, and next to $y = z$ and $x = o$, we deduce that

$$\langle g(z), z\rangle \leq \ln f(z) - \ln f(0) \leq \langle g(o), z\rangle, \tag{3.5}$$

where g is a subgradient of f. In particular, if ∇f exists at $z \in (\text{int}K)|L$, then $\langle \nabla f(z), z\rangle \leq \langle g(o), z f(z)\rangle$. Together with the property $c(K) = 0$ we get from (2.4) that

$$\int_{K|L} \langle \nabla f(z), z\rangle \, d\mathcal{H}_d(z) \leq \int_{K|L} \langle g(o), z f(z)\rangle \, d\mathcal{H}_d(z) = 0. \tag{3.6}$$

Let us assume that equality holds in (3.6), and hence for almost all $z \in (\text{int}\, K)|L$ in (3.5). In particular, we have $\ln f(x) - \ln f(0) = \langle g(o), x\rangle$, and in turn $f(x) = f(0)e^{\langle g(o),x\rangle}$ for almost all $x \in (\text{int}\, K)|L$. Since f is continuous on $(\text{int}\, K)|L$, Corollary 3.2 yields that $f(x) = f(0)e^{\langle g(o),x\rangle}$ for all $x \in K|L$. However $f^{\frac{1}{k}}$ is concave, therefore $g(o) = o$, or in other words, f is constant. □

Now, we are ready to give the proof of Theorem 1.3.

Proof of Theorem 1.3. Combining Propositions 3.4 and 3.5 yields that

$$V_K(L \cap S^{n-1}) = \frac{d}{n} V(K) + \frac{1}{n} \int_{K|L} \langle \nabla f_{K,L}(x), x\rangle \, d\mathcal{H}_d(x) \leq \frac{d}{n} V(K).$$

Let us assume that equality holds, and hence $f_{K,L}(x) = f_{K,L}(o)$ for $x \in K|L$ according to Proposition 3.5. Let $C(x) = K \cap (x + L^{\perp})$ for $x \in K|L$. For any $x \in K|L$, there exists $\eta > 0$ such that $-\eta x \in K|L$, and hence

$$\tfrac{\eta}{1+\eta} C(x) + \tfrac{1}{1+\eta} C(-\eta x) \subset C(o).$$

Therefore $f_{K,L}(x) = f_{K,L}(-\eta x) = f_{K,L}(o)$ and the equality characterization of the Brunn-Minkowski inequality (2.1) implies that $C(x)$ is a translate of $C(o)$.

Choose linearly independent $v_1, \ldots, v_d \in K|L$ such that $v_0 = -v_1 - \ldots - v_d \in K|L$, as well. By $\sum_{i=0}^d v_i = o$ we have $\sum_{i=0}^d \frac{1}{d+1} C(v_i) \subset C(o)$, and we deduce that $\sum_{i=0}^{d}\frac{1}{d+1}\,c(C(v_i))=c(C(o))$. In particular,

$$c(C(o))\in\Pi=\mathrm{aff}\{c(C(v_0)),\ldots,c(C(v_d))\},$$

where aff{} denotes the affine hull. Next let $x\in K|L$. There exists $\eta>0$ such that $-\eta x\in[v_0,\ldots,v_d]$, and so $\lambda x+\sum_{i=0}^{d}\lambda_i v_i=o$, where $\lambda+\sum_{i=0}^{d}\lambda_i=1$ and $\lambda,\lambda_i\geq 0$ for $i=0,\ldots,d$. It follows as above that $\lambda\,C(x)+\sum_{i=0}^{d}\lambda_i C(v_i)=C(o)$, and hence $c(C(x))\in\Pi$, as well. Therefore, writing $M=\Pi\cap(K+L^{\perp})$ and $C=C(o)-c(C(o))$, we get $K=C+M$. In particular, $\mathrm{supp}\,V_K\subset L\cup\Pi^{\perp}$ and $L\cap\Pi^{\perp}=\{0\}$. □

4. Another property of the cone-volume measure if the centroid is the origin

Let us recall two basic notions about convex bodies. Firstly, a convex body in $\mathbb{R}^n$ is called a cylinder if it is of the form $[p,q]+C$ for $p,q\in\mathbb{R}^n$ and an $(n-1)$-dimensional convex compact set C; $p+C$ and $q+C$ are called bases of the cylinder.

Secondly, let $v\in S^{n-1}$, and let M be a convex body in $\mathbb{R}^n$. For any t with $-h_M(-v)<t<h_M(v)$, we replace the section $M\cap(tv+v^{\perp})$ with the $(n-1)$-ball of the same $(n-1)$-measure, centered at tv in $tv+v^{\perp}$. Here, $v^{\perp}$ is the abbreviation for the linear space orthogonal to v.

The closure $\widetilde{M}$ of the union of these $(n-1)$-balls is called the Schwarz rounding of M with respect to $\mathbb{R}v$. It is a convex body by the Brunn-Minkowski theorem, and readily satisfies $\mathrm{V}(\widetilde{M})=\mathrm{V}(M)$. If $\widetilde{M}$ is a cylinder, then all sections of the form $M\cap(tv+v^{\perp})$ are of the same $(n-1)$-measure, and hence the equality case of the Brunn-Minkowski theorem yields that M is a cylinder, as well. For more on Schwarz rounding we refer to [18].

Proposition 4.1. *Let $K\in\mathcal{K}^n$ with $c(K)=o$ and $V(K)=1$. Then*

$$\mathrm{V}_K(\Omega)\geq\frac{1}{2n},$$

for any open hemisphere $\Omega\subset S^{n-1}$. Equality holds if and only if K is a cylinder whose generating segment is orthogonal to the linear $(n-1)$-space bounding the hemisphere S.

Proof. Let $\Omega\subset S^{n-1}$ be an open hemisphere, and let $v\in S^{n-1}$ such that

$$\Omega=\left\{u\in S^{n-1}:\,\langle u,v\rangle>0\right\}.$$

For any convex body $M\in\mathcal{K}^n$ with $o\in\mathrm{int}\,M$ and $x\in M|v^{\perp}$, let

$$f_M(x)=\max\{t\in\mathbb{R}\,:x+tv\in M\},$$

and let $\varphi_M(x)=x+f_M(x)\,v$.

In particular the points of ∂M where all exterior normals have acute angle with v are of the form $\varphi_M(x)$ for $x\in\mathrm{int}\,M|v^{\perp}$. Therefore

$$\mathrm{V}_M(\Omega)=\mathrm{V}(\Xi_M)\ \text{ for }\Xi_M=\bigcup_{x\in M|v^{\perp}}[o,\varphi_M(x)].$$

For $x \in (\mathrm{int}\, M|v^\perp)\backslash\{o\}$, let $z = \theta^{-1}x \in \partial M|v^\perp$ for some $\theta \in (0,1)$. Since $[\varphi_M(z), o, \varphi_M(o)] \subset \Xi_M$, we have

(4.1) $x+\mathbb{R}v$ intersects Ξ_M in a segment of length at least $(1-\theta)\|\varphi_M(o)\|$, with equality if and only if $[\varphi_M(z), \varphi_M(o)] \subset \partial M$.

Now, let $\lambda = f_K(o)$, and hence $\lambda v \in \partial K$. After a linear transformation we may assume that the tangent hyperplane H at λv is given by $H = \lambda v + v^\perp$.

We shake K down to H, i.e., for each $x \in K|v^\perp$, we translate the section $(x+\mathbb{R}v)\cap K$ by $(\lambda - f_K(x))v$ and hence one endpoint lands in H. We write K' to denote the resulting convex body, which satisfies

$$K'|v^\perp = K|v^\perp = C - \lambda\, v \quad \text{for } C = K' \cap H.$$

In addition $\mathrm{V}(K') = \mathrm{V}(K) = 1$, and $\Xi_{K'}$ is the cone $[o, C]$.

For $x \in (\mathrm{int}\, K|v^\perp)\backslash\{o\}$, it follows by (4.1) that $x+\mathbb{R}v$ intersects Ξ_K in a segment of length at least the length of $\Xi_{K'} \cap (x+\mathbb{R}v)$. Therefore, Fubini's theorem yields

$$\mathrm{V}(\Xi_K) \geq \mathrm{V}(\Xi_{K'}). \tag{4.2}$$

Furthermore, Fubini's theorem implies that

$$\begin{aligned} \langle c(K'), u\rangle &= \langle c(K), u\rangle = 0 \quad \text{for } u \in v^\perp; \\ \langle c(K'), v\rangle &\geq \langle c(K), v\rangle = 0 \quad \text{with equality if and only if } K' = K. \end{aligned}$$

We deduce

$$c(K') = \eta v \quad \text{for } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = K. \tag{4.3}$$

Next let $\widetilde{K}$ be the Schwarz rounding of K' with respect to $\mathbb{R}v$. It follows from the rotational symmetry of $\widetilde{K}$ that $\langle c(\widetilde{K}), u\rangle = 0$ for $u \in v^\perp$, and by Fubini's theorem that $\langle c(\widetilde{K}), v\rangle = \langle c(K'), v\rangle$, which in turn yield by (4.3) and $\mathrm{V}(\widetilde{K}) = \mathrm{V}(K') = 1$ that

$$c(\widetilde{K}) = c(K') = \eta v \quad \text{for } \eta \geq 0, \text{ with } \eta = 0 \text{ if and only if } K' = K. \tag{4.4}$$

We conclude by (4.2) and (4.4) that

$$\mathrm{V}(\Xi_K) \geq \mathrm{V}(\Xi_{\widetilde{K}-c(\widetilde{K})}) \quad \text{with equality if and only if } K' = K. \tag{4.5}$$

Finally we compare $\widetilde{K}$ to the cylinder Z over the $(n-1)$-ball $H\cap\widetilde{K}$, where $\mathrm{V}(Z) = \mathrm{V}(\widetilde{K}) = 1$ and Z and K lie on the same side of H. We deduce from the rotational symmetry of Z that $\langle c(Z), u\rangle = 0$ for $u \in v^\perp$. On the other hand, the rotational symmetry of $\widetilde{K}$ and $\widetilde{K}|v^\perp = (H \cap \widetilde{K}) - \lambda v$ yield that

$$\langle x, v\rangle > -h_Z(-v) > \langle y, v\rangle \quad \text{for all } x \in \mathrm{int} Z\backslash\widetilde{K} \text{ and } y \in \widetilde{K}\backslash Z.$$

Therefore,

$$c(Z) = \tau v \quad \text{for } \tau \geq \eta, \text{ with } \tau = \eta \text{ if and only if } Z = \widetilde{K}.$$

We conclude by (4.4) and (4.5) that

$$\mathrm{V}(\Xi_K) \geq \mathrm{V}(\Xi_{Z-c(Z)}) = 1/(2n) \quad \text{with equality iff } K' = K \text{ and } Z = \widetilde{K}.$$

In turn, we get Proposition 4.1. □

5. Some properties of the symmetric volume distance

First we show that the distance $\delta_{\rm hom}$ can be estimated in terms of $\delta_{\rm vol}$. These types of estimates have been around, only we were not able to locate them in the form we need.

Lemma 5.1. *Let $K\in\mathcal{K}^n$ with $c(K)=o$.*

(i) *If $Q\subset K$ is a convex body with $\mathrm{V}(K\backslash Q)\leq t\,\mathrm{V}(K)$ for $t\in(0,\frac{1}{e})$, then $(1-(et)^{1/n})K\subset Q$.*

(ii) *If Q is a convex body with $\mathrm{V}(K\Delta Q)\leq t\mathrm{V}(K)$ for $t\in(0,\frac{1}{4^n e})$, then $(1-(et)^{1/n})K\subset Q\subset(1+4(et)^{1/n})K$.*

Proof. The main tool is the following result due to B. Grünbaum [19]. If $M\in\mathcal{K}^n$, and H^+ is a half space containing $c(M)$, then

$$\mathrm{V}(M\cap H^+)\geq \mathrm{V}(M)/e. \tag{5.1}$$

To prove (i), let $\lambda=0$ if $o\notin\operatorname{int}Q$, and let $\lambda>0$ be maximal with the property that $\lambda K\subset Q$ otherwise. In addition, let $x=o$ if $o\notin\operatorname{int}Q$, and let x be a common boundary point of Q and λK otherwise. Therefore, there exists a half space H_1^+ such that x lies on its boundary, and $H_1^+\cap\operatorname{int}Q=\emptyset$. Now there exists a $y\in K$ such that $x=\lambda y$, and hence x is the centroid of $x+(1-\lambda)K=\lambda y+(1-\lambda)K\subset K$. It follows from (5.1) that

$$t\mathrm{V}(K)\geq \mathrm{V}(H_1^+\cap K)\geq \mathrm{V}(H_1^+\cap(x+(1-\lambda)K))\geq \mathrm{V}((1-\lambda)K)/e,$$

and thus $t\geq\frac{(1-\lambda)^n}{e}$.

To prove (ii), we observe that $\lambda K\subset Q$ for $\lambda=1-(et)^{1/n}$ by (i). We may assume that $Q\backslash K\neq\emptyset$, and let $\mu>1$ be minimal with the property that $Q\subset\mu K$. For a common boundary point z of Q and μK, let $w\in K$ such that $z=\mu w$. In particular, w is the centroid of

$$w+\frac{\lambda(\mu-1)}{\mu}\,K\subset\frac{1}{\mu}\,z+\frac{\mu-1}{\mu}\,Q\subset Q.$$

In addition there exists a half space H_2^+ such that w lies on its boundary, and $H_2^+\cap\operatorname{int}K=\emptyset$. We deduce again from (5.1) that

$$t\mathrm{V}(K)\geq\mathrm{V}(H_2^+\cap Q)\geq V\left(H_2^+\cap\left(w+\frac{\lambda(\mu-1)}{\mu}\,K\right)\right)\geq\frac{\lambda^n(\mu-1)^n}{\mu^n e}\,\mathrm{V}(K).$$

Now $t<\frac{1}{4^n e}$ yields that $\lambda>\frac12$ and $2(e\,t)^{1/n}<\frac12$, which in turn implies that $\mu\leq(1-2(et)^{1/n})^{-1}<1+4(et)^{1/n}$. □

Corollary 5.2. *Let $K,Q\in\mathcal{K}^n$. Then*

$$\delta_{\rm hom}(K,Q)\leq 12\,\delta_{\rm vol}(K,Q)^{1/n}\qquad \text{if } \delta_{\rm vol}(K,Q)<\tfrac{1}{4^n e},$$
$$\delta_{\rm vol}(K,Q)\leq 3n\,\delta_{\rm hom}(K,Q)\qquad \text{if } \delta_{\rm hom}(K,Q)<\tfrac{1}{2n}.$$

Proof. We use that $1+s<e^s<1+2s$ and $1-s<e^{-s}<1-\frac{s}{2}$ if $s\in(0,1)$.

We may assume that $c(K)=c(Q)=o$, and $\mathrm{V}(K)=\mathrm{V}(Q)=1$. In particular, $\mathrm{V}(K\Delta Q)=\delta_{\mathrm{vol}}(K,Q)$, and hence the estimates for the exponential function and Lemma 5.1 yield with $s=\delta_{\mathrm{vol}}(K,Q)$ that

$$e^{-2e^{1/n}s^{1/n}}K\subset(1-(se)^{\frac{1}{n}})K\subset Q\cap K\subset Q.$$

Using the analogous formula $e^{-2e^{1/n}s^{1/n}}Q\subset K$, we conclude the first estimate.

For the second estimate, let $t=\delta_{\mathrm{hom}}(K,Q)$. It follows that $e^{-t}K\subset Q\subset e^tK$, thus $\mathrm{V}(K\Delta Q)\leq e^{nt}-e^{-nt}<3nt$. □

Our next goal is Lemma 5.4 stating that one does not need to insist on the common centroid in the definition of δ_{vol}. We prepare the argument by the following observation.

Lemma 5.3. *Let $K\in\mathcal{K}^n$ and $x\in\mathbb{R}^n$. Then*

$$\mathrm{V}(K\Delta(x+K))\leq 2n\|x\|_{K-K}\mathrm{V}(K).$$

Proof. We may assume that $x\neq o$. Let $y,z\in K$ such that $x=\|x\|_{K-K}(y-z)$, and hence

$$\|x\|_{K-K}=\|x\|/\|y-z\|.$$

Applying Steiner symmetrization with respect to the hyperplane $x^\perp$ shows that

$$\mathrm{V}(K)\geq\frac{\|y-z\|}{n}\,\mathcal{H}_{n-1}(K|x^\perp).$$

We deduce by Fubini's theorem that

$$\mathrm{V}(K\Delta(x+K))\leq 2\|x\|\mathcal{H}_{n-1}(K|x^\perp)\leq 2n\|x\|_{K-K}\mathrm{V}(K).$$

□

Lemma 5.4. *Let $K,Q\in\mathcal{K}^n$ with $c(K)=o$ and $\mathrm{V}(K\Delta Q)\leq t\mathrm{V}(K)$ for $t\in(0,\frac{1}{4^ne})$. Then*

$$\|c(Q)\|_{K-K}\leq 4nt\quad\text{ and }\quad\delta_{\mathrm{vol}}(K,Q)\leq 9n^2t.$$

Proof. We may assume that $\mathrm{V}(K)=1$, and the minimal volume so called Löwner ellipsoid E containing $K-K$ is a ball (see, e.g., [18]). In particular, $n^{-1/2}E\subset K-K\subset E$, and the Brunn-Minkowski and Rogers-Shephard theorems yield that $2^n\leq\mathrm{V}(K-K)\leq\binom{2n}{n}$. Since the volume of a centrally convex body over the volume of its Loewner ellipsoid is at least $2^n/(n!\mathrm{V}(B^n))$ according to K. Ball [2], we have

$$2^n\leq\mathrm{V}(E)\leq\binom{2n}{n}\frac{n!}{2^n}\mathrm{V}(B^n)<\sqrt{3}\cdot\frac{2^nn^n}{e^n}\mathrm{V}(B^n).$$

It follows that

$$\frac{2}{\sqrt{e\pi}}B^n\subset K-K\subset nB^n\ \text{ and }\ \frac{1}{n}\|x\|\leq\|x\|_{K-K}\leq 2\|x\|. \tag{5.2}$$

Therefore, to prove Lemma 5.4, it is sufficient to verify the corresponding estimate for $\|c(Q)\|$.

If $c(Q) = o$, then we are done, otherwise let $u = c(Q)/\|c(Q)\|$. We have $Q \subset 2K \subset 2nB^n$ by Lemma 5.1 and (5.2), and $\mathrm{V}(Q) \geq 1 - t$ implies $\mathrm{V}(Q)^{-1} < 2$. By (5.2) we also have

$$\|c(Q)\|_{K-K} \leq 2\|c(Q)\| = 2\mathrm{V}(Q)^{-1}\langle u, c(Q)\rangle = 2\mathrm{V}(Q)^{-1}\left\|\int_Q \langle u, x\rangle\, dx\right\|,$$

and since $c(K) = o$ we get

$$\begin{aligned} \|c(Q)\|_{K-K} &\leq 2\mathrm{V}(Q)^{-1}\left\|\int_Q \langle u, x\rangle\, dx\right\| \\ &= 2\mathrm{V}(Q)^{-1}\left\|\int_{Q\setminus K} \langle u, x\rangle\, dx - \int_{K\setminus Q} \langle u, x\rangle\, dx\right\| \\ & 4\int_{K\Delta Q} |\langle u, x\rangle|\, dx \leq 4nt. \end{aligned} \tag{5.3}$$

Let $K' = K + c(Q)$, thus Lemma 5.3 and (5.3) imply that $\mathrm{V}(K\Delta K') \leq 8n^2t$. We observe that $Q' = c(Q) + \mathrm{V}(Q)^{-1/n}(Q - c(Q))$ satisfies $c(Q') = c(Q)$, $\mathrm{V}(Q') = 1$, and $\mathrm{V}(Q'\Delta Q) \leq t$ by $1 - t \leq \mathrm{V}(Q) \leq 1 + t$ (cf. Lemma 5.1). Therefore

$$\delta_{\mathrm{vol}}(K, Q) = \mathrm{V}(K'\Delta Q') \leq \mathrm{V}(K'\Delta K) + \mathrm{V}(K\Delta Q) + \mathrm{V}(Q\Delta Q') < 9n^2t.$$

□

6. Some consequences of the stability of the Brunn-Minkowski inequality

Concerning the Brunn-Minkowski theory, including the properties of mixed volumes, the main reference is R. Schneider [39]. We use the Brunn-Minkowski theory in $L^\perp$ in the terminology of Theorem 1.7, whose dimension is $k = n - d$. For $k, m \geq 1$, let

$$I_m^k = \{(i_1, \ldots, i_m) : \, i_j \in \mathbb{N}, \; j = 1, \ldots, m \text{ and } i_1 + \ldots + i_m = k\}.$$

For compact convex sets $C_1, \ldots, C_m$ in $\mathbb{R}^k$ and $(i_1, \ldots, i_m) \in I_m^k$, the non-negative mixed volumes $\mathrm{V}(C_1, i_1; \ldots; C_m, i_m)$ were defined by H. Minkowski in a way such that if $\alpha_1, \ldots, \alpha_m \geq 0$, then

$$\mathcal{H}_k\left(\sum_{j=1}^m \alpha_j C_j\right) = \sum_{(i_1,\ldots,i_m)\in I_m^k} \mathrm{V}(C_1, i_1; \ldots; C_m, i_m)\alpha_1^{i_1} \cdot \ldots \cdot \alpha_m^{i_m}. \tag{6.1}$$

The mixed volume $\mathrm{V}(C_1, i_1; \ldots; C_m, i_m)$ actually depends only on the C_j with $i_j > 0$, does not depend on the order how the pairs C_j, i_j are indexed, and we frequently ignore the pairs C_j, i_j with $i_j = 0$. We have $\mathrm{V}(C_1, k) =$

$\mathcal{H}_k(C_1)$, and $\mathrm{V}(C_1, i_1; \ldots; C_m, i_m) > 0$ if each C_j is k-dimensional. It follows by the Alexandrov-Fenchel inequality that

$$\mathrm{V}(C_1, i_1; \ldots; C_m, i_m)^k \geq \prod_{j=1}^{m} \mathcal{H}_k(C_j)^{i_j}. \tag{6.2}$$

An important special case of (6.2) is the classical Minkowski inequality, which says

$$\mathrm{V}(C_1, 1; C_2, k-1)^k \geq \mathcal{H}_k(C_1)\mathcal{H}_k(C_2)^{k-1}. \tag{6.3}$$

Equality holds for k-dimensional C_1 and C_2 in the Minkowski inequality (6.3) if and only if C_1 and C_2 are homothetic. We remark that the equality conditions in the Alexandrov-Fenchel inequality (6.2) are not yet clarified in general.

Now the Alexandrov-Fenchel inequality (6.2), and actually already the Minkowski inequality (6.3) yields the classsical (general) Brunn-Minkowski theorem stating that if $C_1, \ldots, C_m$ are compact convex sets in $\mathbb{R}^k$, and $\alpha_1, \ldots, \alpha_m \geq 0$, then (cf. (2.1))

$$\mathcal{H}_k\left(\sum_{j=1}^{m} \alpha_j C_j\right)^{1/k} \geq \sum_{j=1}^{m} \alpha_i \mathcal{H}_k(C_i)^{1/k}. \tag{6.4}$$

Equality holds for k-dimensional $C_1, \ldots, C_m$ and positive $\alpha_1, \ldots, \alpha_m$ in the Brunn-Minkowski inequality (6.4) if and only if C_1 and C_j are homothetic for $j = 2, \ldots, m$.

We need the following stability version of the Minkowski inequality (6.3) due to A. Figalli, F. Maggi and A. Pratelli [14]. If C_1, C_2 are k-dimensional compact convex sets in $\mathbb{R}^k$, and

$$\mathrm{V}(C_1, 1; C_2, k-1)^k \leq (1+\varepsilon)\mathcal{H}_k(C_1)\mathcal{H}_k(C_2)^{k-1} \tag{6.5}$$

for small $\varepsilon \geq 0$, then [14] proves that

$$\delta_{\mathrm{vol}}(C_1, C_2) \leq \tilde{\gamma}_v \varepsilon^{1/2} \tag{6.6}$$

where the explicit $\tilde{\gamma}_v > 0$ depends only on the dimension k.

We remark that here we only work out the estimate with respect to the symmetric volume distance δ_{vol}, and then just use Corollary 5.2 for δ_{hom}. Actually, V.I. Diskant [12] proved that (6.5) implies

$$\delta_{\mathrm{hom}}(C_1, C_2) \leq \tilde{\gamma}_h \varepsilon^{1/k} \tag{6.7}$$

for an unknown $\tilde{\gamma}_h > 0$ depending only on k. We note that (6.6) and Corollary 5.2 readily yields a version of (6.7) with exponent $\frac{1}{2k}$ instead of $\frac{1}{k}$.

Combining the stability versions (6.6) and (6.7) with Lemma 5.3 and Lemma 5.4 leads to the following stability version of the Brunn-Minkowski inequality.

Lemma 6.1. *For any $k \geq 1$, $m \geq 2$ and $\omega \in (0,1]$, there exist positive $\varepsilon_0(k,m,\omega)$ and $\gamma(k,m,\omega)$ depending on k, m and ω such that if k-dimensional compact convex sets $C_0, C_1, \ldots, C_m$ in $\mathbb{R}^k$, and $\alpha_1, \ldots, \alpha_m > 0$ satisfy that $\alpha_i/\alpha_j \geq \omega$ and $\mathcal{H}_k(C_i) = V$ for $i,j = 1,\ldots,m$, and*

$$\alpha_1 C_1 + \ldots + \alpha_m C_m \subset C_0 \quad \text{and} \quad \mathcal{H}_k(C_0) \leq e^{\varepsilon}(\alpha_1 + \ldots + \alpha_m)^k V$$

for $\varepsilon \in (0, \varepsilon_0(k,m,\omega))$, then for $i = 1,\ldots,m$, we have

$$\begin{aligned} \delta_{\mathrm{vol}}(C_i, C_0) &\leq \gamma(k,m,\omega)\varepsilon^{1/2}, \\ \left\| c(C_0) - \sum_{i=1}^m \alpha_i c(C_i) \right\|_{C_0 - C_0} &\leq (\alpha_1 + \ldots + \alpha_m)\gamma(k,m,\omega)\varepsilon^{1/2}. \end{aligned}$$

Proof. Since $\mathcal{H}_k(\alpha_1 C_1 + \ldots + \alpha_m C_m) \geq (\alpha_1 + \ldots + \alpha_m)^k V$ according to the Brunn-Minkowski inequality, we may assume that $\alpha_1 C_1 + \ldots + \alpha_m C_m = C_0$ by Lemma 5.4. For $1 \leq i < j \leq m$, we apply the Alexandrov-Fenchel inequality (6.2) to each term in (6.1) except for $k\alpha_i\alpha_j^{k-1}\mathrm{V}(C_i, 1; C_j, k-1)$ and deduce that

$$k\alpha_i\alpha_j^{k-1}\mathrm{V}(C_i, 1; C_j, k-1) \leq k\alpha_i\alpha_j^{k-1}V + (e^{\varepsilon} - 1)(\alpha_1 + \ldots + \alpha_m)^k V.$$

Here $(\alpha_1 + \ldots + \alpha_m)^k \leq (\frac{m}{\omega})^k \alpha_i \alpha_j^{k-1}$, and hence

$$\mathrm{V}(C_i, 1; C_j, k-1) \leq \left(1 + \frac{2}{k}\left(\frac{m}{\omega}\right)^k \varepsilon\right) V.$$

Thus (6.6) yield

$$\delta_{\mathrm{vol}}(C_i, C_j) \leq \bar{\gamma}(k,m,\omega)\varepsilon^{1/2} \tag{6.8}$$

for $\bar{\gamma}(k,m,\omega)$ depending only on k, m and ω. To compare to C_0, we may assume that $V = 1$, $\alpha_1 + \ldots + \alpha_m = 1$ and $c(C_i) = o$ for $i = 1,\ldots,m$. Let $M = C_1 \cap \ldots \cap C_m$.

It follows from (6.8) that

$$\mathcal{H}_k(C_i \backslash M) \leq m \cdot \bar{\gamma}(k,m,\omega)\varepsilon^{1/2}, \ i = 1,\ldots,m,$$

and hence $\mathcal{H}_k(M) \geq 1 - m \cdot \bar{\gamma}(k,m,\omega)\varepsilon^{1/2}$. Since $M \subset C_i$ for $i = 1,\ldots,m$ yields $M \subset C_0 = \sum_{i=1}^m \alpha_i C_i$, and $\mathcal{H}_k(C_0) \leq e^{\varepsilon}$, we deduce

$$\mathcal{H}_k(C_0 \Delta C_i) \leq 2\bar{\gamma}(k,m,\omega)\varepsilon^{1/2}, \ i = 1,\ldots,m.$$

Therefore Lemma 5.3 and Lemma 5.4 imply the required estimates for $\delta_{\mathrm{vol}}(C_i, C_0)$ and $c(C_0)$. □

To prove the next Proposition 6.3, we need the following observation.

Lemma 6.2. *If M is a convex body in $\mathbb{R}^d$ such that $-M \subset \eta M$ for $\eta \geq 1$, then there exists an d-simplex $T \subset M$ whose centroid is the origin such that $M \subset \eta d^{3/2} T$.*

Proof. We may assume that the John ellipsoid E of maximal volume contained in $M\cap(-M)$ is Euclidean ball, and let $T\subset M\cap(-M)$ be an inscribed regular simplex. Then $\eta^{-1}M\subset M\cap(-M)\subset\sqrt{d}E\subset d^{3/2}T$. □

For Proposition 6.3 we use the notation of the previous sections, i.e., $K\in\mathcal{K}^n$ is a convex body with $c(K)=o$, $d,k\in\{1,\ldots,n-1\}$ with $d+k=n$, and L is a d-dimensional linear subspace. For $x\in K|L$, we set

$$f(x)=f_{K,L}(x)=\mathcal{H}_k(K\cap(x+L^{\perp})).$$

Proposition 6.3. *There exist $t_0,\gamma>0$ depending on n with the following properties. Let $t\in(0,t_0)$, let $M_*\subset K|L$ be a d-dimensional convex compact set, and let $K_*=K\cap(M_*+L^{\perp})$. If $e^{-t}\le f(x)/f(o)\le e^t$ holds for any $x\in M_*$, then there exist a k-dimensional compact convex set $C\subset L^{\perp}$, and a complementary d-dimensional compact convex set M such that*

$$\delta_{\rm vol}(K,C+M)\le\gamma\max\left\{\frac{\mathrm{V}(K\backslash K_*)}{\mathrm{V}(K)},t^{1/2}\right\}.$$

Proof. Since $c(K)=o$ we have $-K\subset nK$. Hence $-K|L\subset nK|L$ and we may choose, according to Lemma 6.2, $v_0,\ldots,v_d\in e^{-s}K|L$, for some $s>0$, such that $v_0+\ldots+v_d=o$, and

$$e^{-s}K|L\subset n^{5/2}[v_0,\ldots,v_d]. \tag{6.9}$$

For $x\in e^{-s}K|L$, let $K(x)=K\cap(x+L^{\perp})$, and let

$$\widetilde{K}(x)=\frac{f(o)^{1/k}}{f(x)^{1/k}}\,K(x),\quad\text{and hence } \mathcal{H}_k(\widetilde{K}(x))=f(o). \tag{6.10}$$

We define

$$\begin{aligned}A&=\mathrm{aff}\{c(K(v_0)),\ldots,c(K(v_d))\},\\ M&=\{y\in A:(y+L^{\perp})\cap e^{-s}K\neq\emptyset\},\\ C&=K(o)-c(K(o)).\end{aligned}$$

We compare K_* to $M+C$. To this end we consider the affine bijection $\varphi:L\to A$ defined by the correspondance $\{\varphi(x)\}=A\cap(x+L^{\perp})$ for $x\in L$. In particular,

$$\varphi(v_i)=c(K(v_i)),\ i=0,\ldots,d\ \text{ and }\ \varphi(o)=\frac{1}{d+1}\sum_{i=0}^{d}c(K(v_i)). \tag{6.11}$$

Let $x\in e^{-s}K|L$. We have $\frac{-1}{2n^{5/2}}x\in\frac12[v_0,\ldots,v_d]$ according to (6.9), thus

$$\frac{-1}{2n^{5/2}}\,x=\sum_{i=0}^{d}\alpha_iv_i\ \text{ where }\sum_{i=0}^{d}\alpha_i=1\text{ and }\alpha_i\ge\frac{1}{2(d+1)},\ i=0,\ldots,d.$$

We define

$$\begin{aligned}\tilde{\beta} &= \frac{\beta f(x)^{1/k}}{f(o)^{1/k}} \quad \text{where} \quad \beta = \frac{1}{1+2n^{5/2}};\\ \tilde{\beta}_i &= \frac{\beta_i f(v_i)^{1/k}}{f(o)^{1/k}} \quad \text{where} \quad \beta_i = \frac{\alpha_i 2n^{5/2}}{1+2n^{5/2}},\ i=0,\ldots,d,\end{aligned}$$

and hence $\beta + \sum_{i=0}^d \beta_i = 1$ and $\beta x + \sum_{i=0}^d \beta_i v_i = o$. The condition on the function f yields that

$$e^{-t/k} \leq \tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d \leq e^{t/k},$$

and the ratio of any two of $\tilde{\beta}, \tilde{\beta}_0, \ldots, \tilde{\beta}_d$ is at least $1/(4n^{5/2})$. In particular,

$$e^t(\tilde{\beta} + \tilde{\beta}_0 + \ldots + \tilde{\beta}_d)^k f(o) \geq \mathcal{H}_k(K(o)),$$

and the convexity of K implies (cf. (6.10))

$$\tilde{\beta}\widetilde{K}(x) + \sum_{i=0}^d \tilde{\beta}_i \widetilde{K}(v_i) = \beta K(x) + \sum_{i=0}^d \beta_i K(v_i) \subset K(o).$$

We deduce from Lemma 6.1, the stability version of the Brunn-Minkowski inequality, that there exists $\gamma^* > 0$ depending on n such that for $i=0,\ldots,d$, we have

$$\delta_{\rm vol}(K(v_i), K(o)),\ \delta_{\rm vol}(K(x), K(o)) \leq \gamma^* t^{1/2}, \tag{6.12}$$

$$\left\| c(K(o)) - \beta c(K(x)) - \sum_{i=1}^d \beta_i c(K(v_i)) \right\|_{K(o)-K(o)} \leq \gamma^* t^{1/2}. \tag{6.13}$$

Naturally, if $k=1$, then even t can be written instead of $t^{1/2}$ on the right hand side of (6.12) and (6.13), but we ignore this possibility.

First we asssume that $x=o$. In this case (6.11) and (6.13) yield

$$\|c(K(o)) - \varphi(o)\|_{K(o)-K(o)} \leq \gamma^* t^{1/2}. \tag{6.14}$$

Next let $x \in e^{-s}K|L$ be arbitrary. We have $\beta\varphi(x) + \sum_{i=0}^d \beta_i \varphi(v_i) = \varphi(o)$ because φ is affine. We recall that $C = K(o) - c(K(o))$. Let

$$w = c(K(o)) - \beta c(K(x)) - \sum_{i=1}^d \beta_i c(K(v_i)).$$

Since $\beta\varphi(x) = \varphi(o) - \sum_{i=0}^d \beta_i\varphi(v_i)$, we have

$$\begin{aligned}
\big\|c(K(x)) - \varphi(x)\big\|_{C-C} &= \frac{\|\beta c(K(x)) - \beta\varphi(x))\|_{C-C}}{\beta} \\
&\le \frac{\|\beta c(K(x)) + w - \beta\varphi(x))\|_{C-C}}{\beta} + \frac{\|-w\|_{C-C}}{\beta} \\
&= \frac{\|c(K(o)) - \varphi(o) - \sum_{i=0}^d \beta_i(c(K(v_i)) - \varphi(v_i))\|_{C-C}}{\beta} \\
&\quad + \frac{\|c(K(o)) - \beta c(K(x)) - \sum_{i=1}^d \beta_i c(K(v_i))\|_{C-C}}{\beta}.
\end{aligned}$$

As $\varphi(v_i) = c(K(v_i))$ according to (6.11), it follows by (6.13) and (6.14) that

$$\|c(K(x)) - \varphi(x)\|_{C-C} \le \frac{2\gamma^*}{\beta} \cdot t^{1/2} < 6n^{5/2}\gamma^* t^{1/2}. \tag{6.15}$$

For $x \in e^{-s}K|L$, we deduce in order from (6.15), (6.12) and (6.10) that

$$\begin{aligned}
\mathcal{H}_k\big((C+\varphi(x))\Delta K(x)\big) &\le \mathcal{H}_k\big((C + \varphi(x))\Delta(C + c(K(x)))\big) \\
&\quad + \mathcal{H}_k\big((C + c(K(x)))\Delta(\widetilde{K}(x) - c(\widetilde{K}(x)) + c(K(x)))\big) \\
&\quad + \mathcal{H}_k\big((\widetilde{K}(x) - c(\widetilde{K}(x)) + c(K(x)))\Delta K(x)\big) \\
&\quad < 9n^{5/2}\gamma^* t^{1/2}\mathcal{H}_k(C).
\end{aligned}$$

Hence, by Fubini's theorem we get

$$\mathrm{V}(K_*\Delta(M + C)) < 9n^{5/2}\gamma^* t^{1/2}\mathrm{V}(M + C)$$

and Lemma 5.4 yields the required estimate for δ_{vol}. □

7. Some more properties of $f_{K,L}(x)$

Here we establish some more properties of the log-concave function (cf. (2.3))

$$f_{K,L} : L \to \mathbb{R}_{\ge 0} \text{ with } x \mapsto \mathcal{H}_k(K \cap (x + L^\perp)),$$

and use the notation as introduced in Section 2, i.e., $K \in \mathcal{K}^n$ is an n-dimensional convex body with $c(K) = 0$, L is a d-dimensional subspace L, $1 \le d \le n-1$, and we set $k = n-d$. Since we will keep K and L fixed, we just write $f(x)$ instead of $f_{K,L}(x)$. As in Section 2 let $g(x)$ be the subgradient of $f(x)$, and we recall that $g(x) = \nabla f(x)/f(x)$ almost everywhere on $\mathrm{int}\,(K)|L$.

For $\eta \ge 0$, we set

$$\begin{aligned}
M_\eta &= \{x \in K|L : \ln f(x) - \ln f(o) \ge \langle g(o), x\rangle - \eta\}, \\
K_\eta &= K \cap (M_\eta + L^\perp).
\end{aligned}$$

Since $\ln f$ is concave, both M_η and K_η are compact and convex.

Lemma 7.1. *Let $\eta \ge 0$. Then*

$$\int_{K|L} \langle\nabla f(x), x\rangle d\mathcal{H}_d(x) \le -\eta\mathrm{V}(K\backslash K_\eta).$$

Proof. Let $x \in (\mathrm{int}\, K)|L$ and $\eta \geq 0$, and let us assume $\ln f(x) - \ln f(o) \leq \langle g(o), x\rangle - \eta$. Then by (2.2) we have $\langle g(x), x\rangle \leq \langle g(o), x\rangle - \eta$. Hence if ∇f exists at $x \in (\mathrm{int}\, K)|L$, then

$$\begin{aligned} \langle \nabla f(x), x\rangle &\leq 0 \ \text{ provided that } x \in M_\eta, \\ \langle \nabla f(x), x\rangle &\leq \langle g(o), f(x)x\rangle - f(x)\eta \ \text{ provided that } x \notin M_\eta. \end{aligned}$$

We conclude the lemma by (2.4) and $\mathrm{V}(K\backslash K_\eta) = \int_{(K|L)\backslash M_\eta} f(x)\, dx$. □

Lemma 7.2. *Let* $\eta \in [0, 1]$. *If* $\mathrm{V}(K\backslash K_\eta) \leq \mathrm{V}(K)/(2^n e)$, *then*

$$e^{-\tau} \leq \frac{f(x)}{f(o)} \leq e^{\tau} \quad \text{for } \tau = 7n^{3/2}\eta^{1/2} \text{ and } x \in M_\eta.$$

Proof. By Lemma 5.1 we have $\frac{1}{2}K \subset K_\eta$, and $f(x) \geq f(o)e^{\langle g(o),x\rangle - \eta}$ for $x \in K_\eta$. We claim that for $\pm y \in K_\eta$

$$|\langle g(o), y\rangle| \leq 3\sqrt{k\eta}. \tag{7.1}$$

The concavity of $f^{1/k}$ yields that

$$\begin{aligned} f(o)^{1/k} &\geq \frac{f(y)^{1/k} + f(-y)^{1/k}}{2} \geq f(o)^{1/k} e^{-\eta/k} \frac{e^{\langle g(o),y\rangle/k} + e^{\langle g(o),-y\rangle/k}}{2} \\ &\geq f(o)^{1/k} e^{-\eta/k} \left(1 + \left(\frac{\langle g(o), y\rangle}{2k}\right)^2\right). \end{aligned}$$

Since $e^t < 1 + 2t$ for $t \in [0, 1]$, we conclude (7.1).

It follows from $\frac{1}{2}K \subset K_\eta$ and $-K \subset nK$ that $\frac{1}{2}(K|L) \subset M_\eta$ and $-(K|L) \subset n(K|L)$. In particular, if $x \in M_\eta$ is arbitrary, then $\pm y \in M_\eta$ for $y = \frac{1}{2n}x$. We deduce from (7.1) that $|\langle g(o), x\rangle| = 2n|\langle g(o), y\rangle| \leq 6n\sqrt{k\eta}$. Therefore, the lemma follows from $f(o)e^{\langle g(o),x\rangle - \eta} \leq f(x) \leq f(o)e^{\langle g(o),x\rangle}$. □

8. Proofs of Theorem 1.7 and Theorem 1.8

For the proofs of the two stability theorems 1.7 and 1.8, let $K \in \mathcal{K}^n$ with $c(K) = o$, and let

$$\mathrm{V}_K(L \cap S^{n-1}) > \frac{d - \varepsilon}{n} \mathrm{V}(K)$$

for a non-trivial linear subspace L with $\dim L = d$ and $\varepsilon \in (0, (2^n e)^{-5})$. As before, for $x \in K|L$ let

$$f(x) = \mathcal{H}_k(K \cap (x + L^\perp)).$$

According to Proposition 3.4, the condition on $\mathrm{V}_K(L \cap S^{n-1})$ is equivalent with

$$\int_{K|L} \langle \nabla f(x), x\rangle d\mathcal{H}_d(x) > -\varepsilon \mathrm{V}(K). \tag{8.1}$$

Proof of Theorem 1.7. We set $\eta = \varepsilon^{4/5}$, and use the notation of Lemma 7.1. It follows from (8.1) and Lemma 7.1 that

$$\mathrm{V}(K\backslash K_\eta) < \varepsilon^{1/5}\mathrm{V}(K) < \mathrm{V}(K)/(2^n e),$$

and from Lemma 7.2 that

$$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \quad \text{for } t = 7n^{3/2}\varepsilon^{2/5} \text{ and } x \in M_\eta.$$

We assume that ε is small enough in order to apply Proposition 6.3 with $M_* = M_\eta$ and $t = 7n^{3/2}\varepsilon^{2/5}$. We deduce the existence of an $(n-d)$-dimensional compact convex set $C \subset L^\perp$, and complementary d-dimensional compact convex set M such that

$$\delta_{\mathrm{vol}}(K, C+M) \leq \gamma_v \varepsilon^{1/5}.$$

In turn Corollary 5.2 implies that

$$\delta_{\mathrm{hom}}(K, C+M) \leq \gamma_h \varepsilon^{1/(5n)},$$

completing the proof of Theorem 1.7. □

Proof of Theorem 1.8. Here we have $d = 1$. We may assume that $L = \mathbb{R}$, and $K|L = [-a, b]$ where $0 < a \leq b$. Since $c(K) = o$ implies $-K \subset nK$ according to B. Grünbaum [19], we have $b \leq na$.

We set $\eta = \varepsilon^{2/3}$, and use again the notation of Lemma 7.1. We deduce from (8.1) and Lemma 7.1 that

$$\mathrm{V}(K\backslash K_\eta) < \varepsilon^{1/3}\mathrm{V}(K) < \mathrm{V}(K)/(2^n e), \tag{8.2}$$

and from Lemma 7.2 that

$$e^{-t} \leq \frac{f(x)}{f(o)} \leq e^t \quad \text{for } t = 7n^{3/2}\varepsilon^{1/3} \text{ and } x \in M_\eta. \tag{8.3}$$

It follows from Lemma 5.1 and (8.2) that $\frac{1}{2}[-a, b] \subset M_\eta$, therefore the concavity of $\ln f$ and (8.3) yield that

$$f(x) \leq e^{2t} f(o) \quad \text{for } x \in [-a, b]. \tag{8.4}$$

Let $M_\eta = [-a_\eta, b_\eta]$ for $a_\eta, b_\eta > 0$. Since $K\backslash K_\eta$ contains two cones, one with base $K(-a_\eta)$ and height $a - a_\eta$, and one with base $K(b_\eta)$ and height $b - b_\eta$, we get by (8.3), (8.2) and (8.4) that

$$\begin{aligned} \frac{a - a_\eta + b - b_\eta}{n} e^{-t} f(o) &\leq \frac{a - a_\eta + b - b_\eta}{n}(f(-a_\eta) + f(b_\eta)) \\ &\leq \mathrm{V}(K\backslash K_\eta) < \varepsilon^{\frac{1}{3}}\mathrm{V}(K) \leq \varepsilon^{\frac{1}{3}} e^{2t} f(o)(a+b). \end{aligned}$$

In particular,

$$\mathcal{H}_1(M_\eta) = a_\eta + b_\eta > (1 - 2n\varepsilon^{\frac{1}{3}})(a+b).$$

Here and below $\gamma_1, \gamma_2, \dots$ denote positive constants depending on n. We deduce by (8.3) that if ε is small enough, then

$$\begin{aligned} af(-a)+bf(b) &= n\mathrm{V}_K(L\cap S^{n-1}) > (1-\varepsilon)\mathrm{V}(K) > (1-\varepsilon)\mathcal{H}_1(M_\eta)e^{-t}f(o) \\ &> (1-\gamma_1\varepsilon^{\frac{1}{3}})(a+b)f(o). \end{aligned}$$

Since $b\geq a$ and $\frac{a}{a+b}\geq\frac{1}{n+1}$ by $b\leq na$, (8.4) implies that if ε is small enough, then

$$f(-a), f(b) \geq (1-\gamma_2\varepsilon^{\frac{1}{3}})f(o).$$

As $\ln f$ is concave, we have

$$f(x)\geq(1-\gamma_2\varepsilon^{\frac{1}{3}})f(o) \quad \text{for } x\in[-a,b].$$

However $\frac{a}{a+b}\,C(b)+\frac{b}{a+b}\,C(-a)\subset C(o)$, where $C(x)=K\cap(x+L^\perp)$. Thus Lemma 6.1 yields that

$$\delta_{\mathrm{vol}}(C(o),C(-a))\leq\gamma_3\varepsilon^{\frac{1}{6}} \quad\text{and}\quad \delta_{\mathrm{vol}}(C(o),C(b))\leq\gamma_3\varepsilon^{\frac{1}{6}}. \tag{8.5}$$

Hence, with

$$\widetilde{C}=(C(-a)-\tilde{x})\cap(C(b)-\tilde{y}) \quad\text{for } \tilde{x}=c(C(-a)) \text{ and } \tilde{y}=c(C(b)).$$

It follows from (8.4) and (8.5) that

$$[\tilde{x},\tilde{y}]+\widetilde{C}\subset K \quad\text{and}\quad \mathrm{V}(K)\leq(1+\gamma_4\varepsilon^{\frac{1}{6}})\mathrm{V}([\tilde{x},\tilde{y}]+\widetilde{C}).$$

Using Lemma 5.4, we replace $\widetilde{C}$ by a suitably smaller homothetic copy C such that $c(C)=o$, and obtain that there exist $x\in\tilde{x}+\widetilde{C}$ and $y\in\tilde{y}+\widetilde{C}$ satisfying $o\in[x,y]$, $e^{-s}\|x\|\leq\|y\|\leq e^{s}\|x\|$ for $s=\gamma_5\varepsilon^{\frac{1}{6}}$, and

$$[x,y]+C\subset K \quad\text{and}\quad \mathrm{V}(K)\leq(1+\gamma_6\varepsilon^{\frac{1}{6}})\mathrm{V}([x,y]+C).$$

Finally, if $z\in[-a,b]$, then $-z/n\in[-a,b]$ and $\frac{1}{n+1}\,C(z)+\frac{n}{n+1}\,C(-z/n)\subset C(o)$. Therefore Lemma 5.1, Lemma 6.1 and the estimates above imply

$$K\subset[x,y]+(1+\gamma_5\varepsilon^{\frac{1}{6n}})C,$$

completing the proof of Theorem 1.8. □

9. Stability of the U-functional U(K)

Let $m\in\{1,\dots,n\}$. In this section, a finite sequence $u_1,\dots,u_m$ always denote points of S^{n-1}, and by $\mathrm{lin}\,\{X\}$ we denote the linear hull of a set X. As in [24], we define $\sigma_m(K)>0$ by

$$\sigma_m(K)^m=\int_{u_1\wedge\ldots\wedge u_m\neq 0}1\,d\mathrm{V}_K(u_1)\cdots d\mathrm{V}_K(u_m).$$

In particular, $\sigma_1(K)=\mathrm{V}(K)$, $\sigma_n(K)=U(K)$, and for $m<n$, we have

(9.1)

$$\sigma_{m+1}(K)^{m+1}=$$
$$\int_{u_1\wedge\ldots\wedge u_m\neq 0}\big(\mathrm{V}(K)-\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u_1,\dots,u_m\})\big)\,d\mathrm{V}_K(u_1)\cdots d\mathrm{V}_K(u_m).$$

As $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u_1,\ldots,u_m\}) \leq \frac{m}{n}\,\mathrm{V}(K)$ for linearly independent $u_1,\ldots,u_m$ according to Theorem 1.3, we deduce that

$$\sigma_{m+1}(K)^{m+1} \geq \left(1-\frac{m}{n}\right)\mathrm{V}(K)\sigma_m(K)^m. \tag{9.2}$$

Therefore the inequality of Theorem 1.5 follows from

$$\mathrm{U}(K)^n \geq \frac{1}{n}\,\mathrm{V}(K)\sigma_{n-1}(K)^{n-1} \geq \ldots \geq \frac{(n-1)!}{n^{n-1}}\,\mathrm{V}(K)^{n-1}\sigma_1 = \frac{n!}{n^n}\,\mathrm{V}(K)^n.$$

Now we assume that

$$\mathrm{U}(K) \leq (1+\varepsilon)\frac{(n!)^{1/n}}{n}\,\mathrm{V}(K)$$

where $\varepsilon > 0$ is small enough to satify all estimates below. In particular, $\varepsilon < \frac{1}{4n^3}\,\tilde{\varepsilon}_0$, where $\tilde{\varepsilon}_0$ comes from Theorem 1.8. Applying (9.1) for $m=1$, (9.2) for $m\geq 2$, and using $(1+\varepsilon)^n\frac{n-1}{n} < \frac{n-1}{n}+2n\varepsilon$ gives

$$\int_{S^{n-1}}(\mathrm{V}(K)-\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\}))\,d\mathrm{V}_K(u) \leq \left(\frac{n-1}{n}+2n\varepsilon\right)\mathrm{V}(K)^2. \tag{9.3}$$

For any $X\subset S^{n-1}$, there exists $u\in X$ maximizing $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\})$ because different 1-dimensional subspaces have disjoint intersections with S^{n-1}. We consider linearly independent $v_1,\ldots,v_n\in S^{n-1}$ such that v_1 maximizes $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\})$ for $u\in S^{n-1}$, and v_i maximizes $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\})$ for all $u\in S^{n-1}\backslash\mathrm{lin}\{v_1,\ldots,v_{i-1}\}$ if $i=2,\ldots,n$. Let $L=\mathrm{lin}\{v_1,\ldots,v_{n-1}\}$, and let $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{v_n\}) = (\frac{1}{n}-t)\mathrm{V}(K)$, and hence $t\in[0,\frac{1}{n}]$ (cf. (1.3)). Thus we have

$$\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{v_i\}) \;\geq\; (\tfrac{1}{n}-t)\mathrm{V}(K) \quad \text{for } i=1,\ldots,n, \tag{9.4}$$

$$\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\}) \;\leq\; (\tfrac{1}{n}-t)\mathrm{V}(K) \quad \text{for } u\in S^{n-1}\backslash L. \tag{9.5}$$

We deduce from (9.3), (9.5) and $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{u\}) \leq \frac{1}{n}\,\mathrm{V}(K)$ for $u\in S^{n-1}\cap L$ that

$$(\tfrac{n-1}{n}+t)\mathrm{V}(K)\mathrm{V}_K(S^{n-1}\backslash L)+\tfrac{n-1}{n}\,\mathrm{V}(K)\mathrm{V}_K(S^{n-1}\cap L) \leq \left(\tfrac{n-1}{n}+2n\varepsilon\right)\mathrm{V}(K)^2.$$

Since $\mathrm{V}_K(S^{n-1}\backslash L) \geq \frac{1}{n}\,\mathrm{V}(K)$ according to Theorem 1.3, we conclude that $t\leq 2n^2\varepsilon$. In particular, $\mathrm{V}_K(S^{n-1}\cap\mathrm{lin}\{v_i\}) \geq (\frac{1}{n}-2n^2\varepsilon)\mathrm{V}(K)$ for $i=1,\ldots,n$ by (9.4).

From Theorem 1.8 we find for $i=1,\ldots,n$, that there exist an $(n-1)$-dimensional compact convex set $C_i\subset v_i^\perp$ with $c(C_i)=o$, and $x_i,y_i\in\partial K$ such that $y_i=-e^{s_i}x$, where $|s_i|<n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}$, and for $i=1,\ldots,n$, we have

$$[x_i,y_i]+C_i \;\subset\; K \tag{9.6}$$

$$\mathrm{V}(K\backslash([x_i,y_i]+C_i)) \;\leq\; n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}\mathrm{V}(K) \tag{9.7}$$

$$K \;\subset\; [x_i,y_i]+(1+2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})C_i. \tag{9.8}$$

We may assume that v_i is an exterior normal at x_i, $i=1,\ldots,n$. After a linear transformation of K, we may also assume that $v_1,\ldots,v_n$ form and orthonormal system, and $\langle v_i, x_i - y_i\rangle = 2$. In particular,

$$e^{-\tau} < \langle v_i, x_i\rangle, \langle -v_i, y_i\rangle < e^{\tau},\ \tau = n\tilde{\gamma}_v\varepsilon^{\frac{1}{6}}. \tag{9.9}$$

In what follows, we write $\gamma_1, \gamma_2, \ldots$ for positive constants depending on n only. It follows from combining (9.6), (9.7) and (9.9) that

$$1 - \gamma_1\varepsilon^{\frac{1}{6}} < \mathcal{H}_{n-1}(C_i)/\mathcal{H}_{n-1}(C_j) < 1 + \gamma_1\varepsilon^{\frac{1}{6}} \quad \text{for } i, j \in \{1, \ldots, n\}. \tag{9.10}$$

For any $i \neq j \in \{1, \ldots, n\}$, we write

$$\begin{aligned} w_i(v_j) &= h_{C_i}(v_j) + h_{C_i}(-v_j), \\ a_i(v_j) &= \max\left\{\mathcal{H}_{n-2}(C_i \cap (tv_j + v_j^{\perp})) : -h_{C_i}(-v_j) \leq t \leq h_{C_i}(v_j)\right\}, \end{aligned}$$

and recall that $h_{C_i}(x)$ denotes the support function. Hence $w_i(v_j)$ is the width of C_i in the direction of v_j. Calculating $\mathcal{H}_{n-1}(C_i)$ by integrating along $\mathbb{R}v_j$ leads to

$$\tfrac{1}{n-1}\, w_i(v_j)a_i(v_j) \leq \mathcal{H}_{n-1}(C_i) \leq w_i(v_j)a_i(v_j) \quad \text{for } i \neq j \in \{1, \ldots, n\}. \tag{9.11}$$

Let $p \neq q \in \{1, \ldots, n\}$. We choose $t_1 \geq t_* \geq t_0$ such that

$$\begin{aligned} \langle v_p, t_1x_p\rangle &= h_{C_q}(v_p) \\ \langle -v_p, t_0x_p\rangle &= h_{C_q}(-v_p) \\ \mathcal{H}_{n-2}(C_q \cap (t_*x_p + v_p^{\perp})) &= a_q(v_p). \end{aligned}$$

It follows from (9.9) and (9.8) that

$$\begin{aligned} t_1 - t_0 &> w_q(v_p)/2, \\ C_q \cap (t_*x_p + v_p^{\perp}) &\subset t_*x_p + (1 + 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})C_p. \end{aligned} \tag{9.12}$$

Therefore $a_p(v_q) \geq (1 + 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})^{-(n-2)}a_q(v_p)$, and hence interchanging the role of p and q leads to

$$1 - \gamma_2\varepsilon^{\frac{1}{6n}} < a_q(v_p)/a_q(v_p) < 1 + \gamma_2\varepsilon^{\frac{1}{6n}}.$$

We deduce from (9.10) and (9.11) that

$$\frac{1}{2n} < \frac{w_p(v_q)}{w_q(v_p)} < 2n. \tag{9.13}$$

Now combining (9.6) and (9.8) shows that

$$h_{C_p}(v_q) \leq \langle x_q - t_mx_p, v_q\rangle \leq (1 + 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}})h_{C_p}(v_q) \quad \text{for } m = 0, 1, \tag{9.14}$$

and hence

$$|\langle (t_1 - t_0)x_p, v_q\rangle| \leq 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}}h_{C_p}(v_q) < 2\tilde{\gamma}_h\varepsilon^{\frac{1}{6n}}w_p(v_q).$$

Applying (9.12), (9.13), and the analoguous argument to y_q implies that

$$|\langle x_p, v_q\rangle|, |\langle y_p, v_q\rangle| \leq \gamma_3\varepsilon^{\frac{1}{6n}}. \tag{9.15}$$

Let P be the parallepiped

$$P = \{x \in \mathbb{R}^n : \langle x, v_i\rangle \leq \langle x_i, v_i\rangle,\ \langle x, -v_i\rangle \leq \langle y_i, -v_i\rangle,\ i = 1, \ldots, n\},$$

and hence each facet of P contains one of $x_i + C_i$, $y_i + C_i$, $i = 1, \ldots, n$. We claim that

$$\tfrac{1}{4n} P \subset K. \tag{9.16}$$

We suppose that (9.16) does not hold, and seek a contradiction. Possibly reversing the orientation of some of the v_i, we may asssume that

$$z = \frac{1}{4n} \sum_{i=1}^{n} \langle x_i, v_i \rangle \, v_i \notin K. \tag{9.17}$$

In particular, $\|z\| \leq \frac{1}{2\sqrt{n}}$ by (9.9), and there exists $u \in S^{n-1}$ such that

$$\langle u, z \rangle > \langle u, x \rangle \quad \text{for } x \in K. \tag{9.18}$$

There exists v_p such that $|\langle u, v_p \rangle| \geq 1/\sqrt{n}$, and hence (9.9) and (9.15) yield that $\langle u, x_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{6n}}$ if $\langle u, v_p \rangle \geq 1/\sqrt{n}$, and $\langle u, y_p \rangle \geq \frac{1}{\sqrt{n}} - \gamma_4 \varepsilon^{\frac{1}{6n}}$ if $\langle u, v_p \rangle \leq -1/\sqrt{n}$. However $\langle u, z \rangle \leq \|z\| \leq \frac{1}{2\sqrt{n}}$, contradicting (9.17). Therefore we conclude (9.16).

For $i = 1, \ldots, n$, let

$$\Xi_{2i-1} = [o, x_i + C_i] \quad \text{and} \quad \Xi_{2i} = [o, y_i + C_i].$$

Since the basis of the cones $\Xi_1, \ldots, \Xi_{2n}$ lie in different facets of P, the interiors of $\Xi_1, \ldots, \Xi_{2n}$ are pairwise disjoint. By (9.7) and (9.9) we know $\mathrm{V}(\Xi_j) \geq (\frac{1}{2n} - \gamma_5 \varepsilon^{\frac{1}{6}}) \mathrm{V}(K)$, and so we get

$$\mathrm{V}(\Xi) > (1 - 2n\gamma_5 \varepsilon^{\frac{1}{6}}) \mathrm{V}(K) \quad \text{for } \Xi = \textstyle\bigcup_{j=1}^{2n} \Xi_j \subset K.$$

We conclude from (9.16) that

$$\mathrm{V}(P \backslash K) \leq \mathrm{V}(P \backslash \Xi) = (4n)^n V\left((\tfrac{1}{4n} P) \backslash \Xi \right) \leq (4n)^n V\left(K \backslash \Xi \right) \leq \gamma_6 \varepsilon^{\frac{1}{6}} \mathrm{V}(K).$$

Therefore (9.7) yields

$$\mathcal{H}_{n-1}((P \cap v_i^{\perp}) \backslash C_i) \leq \gamma_7 \varepsilon^{\frac{1}{6}} \mathcal{H}_{n-1}(C_i), \ i = 1, \ldots, n,$$

and Lemma 5.1 implies that $(1 - \gamma_8 \varepsilon^{\frac{1}{6n}}) P \subset K$, completing the proof of Theorem 1.9.

References

[1] K.M. Ball, *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$, Studia Math. **88** (1988), 69–84.

[2] K.M. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. **44** (1991), 351–359.

[3] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 335–361.

[4] F. Barthe, *A continuous version of the Brascamp-Lieb inequalities*, in Geometric aspects of functional analysis, Lecture Notes in Math., **1850**, 53–63, Springer, Berlin, 2004.

[5] F. Barthe, D. Cordero-Erausquin, *Invariances in variance estimates*, Proc. Lond. Math. Soc,. **106** (2013), 33–64.

[6] F. Barthe, O. Guedon, S. Mendelson, A. Naor, *A probabilistic approach to the geometry of the l_p^n-ball*, Ann. of Probability, **33** (2005), 480–513.
[7] J. Bennett, A. Carbery, M. Christ, T. Tao, *The BrascampLieb inequalities: finiteness, structure, and extremals*, Geom. funct. anal., **17** (2007), 1343–1415.
[8] K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math., **231** (2012), 1974–1997.
[9] K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The logarithmic Minkowski problem*, J. AMS, **26** (2013), 831–852.
[10] K.J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *Affine images of isotropic measures*, to appear in J. Diff. Geom.
[11] E. Carlen, D. Cordero-Erausquin, *Subadditivity of the entropy and its relation to Brascamp-Lieb type inequalities*, Geom. Funct. Anal., **19** (2009), 373–405.
[12] V. I. Diskant, *Strengthening of an isoperimetric inequality*, Siberian J. Math. **14** (1973), 608–611.
[13] H. Federer, *The Gauss-Green theorem*, Trans. Amer. Math. Soc., **58** (1945), 44–76.
[14] A. Figalli, F. Maggi, A. Pratelli: *A mass transportation approach to quantitative isoperimetric inequalities*, Inv. Mathematicae, **182** (1), (2010), 167–211.
[15] R.J. Gardner, *Geometric Tomography*, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. **58**, Cambridge University Press, Cambridge, 2006.
[16] R.J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.
[17] M. Gromov, V.D. Milman, *Generalization of the spherical isoperimetric inequality for uniformly convex Banach Spaces*, Composito Math. **62** (1987), 263–282.
[18] P.M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften, **336**, Springer, Berlin, 2007.
[19] B. Grünbaum, *Partitions of mass-distributions and of convex bodies by hyperplanes*, Pacific J. Math., **10** (1960), 1257–1261.
[20] O. Guedon, E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct. Anal. **21** (2011), 1043–1068.
[21] Ch. Haberl and L. Parapatits, *The centro-affine Hadwiger theorem*, to appear in J. Amer. Math. Soc. (JAMS).
[22] B. He, G. Leng, K. Li, *Projection problems for symmetric polytopes*, Adv. Math., **207** (2006), 73–90.
[23] M. Henk, A. Schürmann, J.M.Wills, *Ehrhart polynomials and successive minima*, Mathematika, **52** (2005), 1–16.
[24] M. Henk, E. Linke, *Cone-volume measures of polytopes*, Adv. Math. **253** (2014), 50–62.
[25] R. Kannan, L. Lovász, M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), 541–559.
[26] B. Klartag, *A Berry-Esseen type inequality for convex bodies with an unconditional basis*, Probab. Theory Related Fields **145** (2009), 1–33.
[27] B. Klartag, *On nearly radial marginals of high-dimensional probability measures*, J. Eur. Math. Soc., **12** (2010), 723–754.
[28] M. Ludwig, *General affine surface areas*, Adv. Math., **224** (2010), 2346–2360.
[29] M. Ludwig, M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math., **172** (2010), 1223–1271.
[30] E. Lutwak, D. Yang, G. Zhang, *A new affine invariant for polytopes and Schneider's projection problem*, Trans. Amer. Math. Soc., **353** (2001), 1767–1779.
[31] E. Lutwak, D. Yang, G. Zhang, *Volume inequalities for subspaces of L_p*, J. Differential Geom., **68** (2004), 159–184.
[32] E. Lutwak, D. Yang, G. Zhang, *L^p John ellipsoids*, Proc. London Math. Soc., **90** (2005), 497–520.

[33] E. Lutwak, D. Yang, G. Zhang, *Volume inequalities for isotropic measures*, Amer. J. Math., **129** (2007), 1711–1723.

[34] A. Naor, *The surface measure and cone measure on the sphere of l_p^n*, Trans. Amer. Math. Soc., **359** (2007), 1045–1079.

[35] A. Naor, D. Romik, *Projecting the surface measure of the sphere of l_p^n*, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 241–261.

[36] G. Paouris, E. Werner, *Relative entropy of cone measures and L_p centroid bodies*, Proc. London Math. Soc, **104** (2012), 253–286.

[37] W.F. Pfeffer, *The divergence theorem and sets of finite perimeter*, CRC Press, Boca Raton, FL, 2012.

[38] R.T. Rockafellar, *Convex analysis*, Princeton University Press, 1997.

[39] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. **44**, Cambridge University Press, Cambridge, 1993, Second expanded edition, 2014.

[40] A. Stancu, *The discrete planar L_0-Minkowski problem*, Adv. Math., **167** (2002), 160–174.

[41] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem*, Adv. Math. **180** (2003), 290-323.

[42] A. Stancu, *Centro-affine invariants for smooth convex bodies*, Int. Math. Res. Not., (2012), 2289–2320.

[43] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996.

[44] G. Xiong, *Extremum problems for the cone-volume functional of convex polytopes*, Adv. Math. **225** (2010), 3214–3228.

[45] G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math. **262** (2014), 909–931.

[46] G. Zhu, *The centro-affine Minkowski problem for polytopes*, accepted for publication in J Differential Geom.

Alfréed Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reltanoda u. 13-15., H-1053 Budapest, Hungary

E-mail address: `carlos@renyi.hu`

OvG-Universität Magdeburg, Fakultät für Mathematik, Universitätsplatz 2, D-39106 Magdeburg, Germany

E-mail address: `martin.henk@ovgu.de`