THE PLANAR L_p-MINKOWSKI PROBLEM FOR $0 < p < 1$

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Abstract. The planar L_p Minkowski problem is solved for $p \in (0,1)$.

1. Introduction

For the notions of the Brunn-Minkowski theory in $\mathbb{R}^n$ used in this paper, see Schneider [64]. We write $\mathcal{H}^m$, $m \leq n$, to denote m-dimensional Hausdorff measure normalized in a way such that it coincides with the Lebesgue measure on $\mathbb{R}^m$. We call a compact convex set K with non-empty interior in $\mathbb{R}^n$ a convex body. For any $x \in \partial K$, we choose an exterior unit normal $\nu_K(x)$ to ∂K at x, which is unique for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$. The surface area measure S_K on S^{n-1} is defined for a Borel set $\omega \subset S^{n-1}$ by

$$S_K(\omega) = \int_{x \in \nu_K^{-1}(\omega)} d\mathcal{H}^{n-1}(x).$$

The classical Minkowski existence theorem, due to Minkowski in the case of polytopes or discrete measures and to Alexandrov for the general case, states that a Borel measure μ on S^{n-1} is the surface area measure of a convex body if and only if the measure of any open hemispehere is positive, and

$$\int_{S^{n-1}} u d\mu(u) = 0.$$

The solution is unique up to translation. If the measure μ has a density function f with respect to $\mathcal{H}^{n-1}$ on S^{n-1}, then even the regularity of the solution is well understood, see Lewy [43], Nirenberg [60], Cheng and Yau [15], Pogorelov [63], and Caffarelli [11].

Lutwak [48] initiated the study of the so called L_p surface area measure for any $p \in \mathbb{R}$. For a convex compact set K in $\mathbb{R}^n$, let h_K be its support function, and hence

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\} \quad \text{for } u \in \mathbb{R}^n$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product. Let $\mathcal{K}_0^n$ denote family of convex bodies in $\mathbb{R}^n$ containing the origin o. If $p \in \mathbb{R}$ and $K \in \mathcal{K}_0^n$, then the L_p-surface area measure is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K.$$

In particular, if $p < 1$ and $\omega \subset S^{n-1}$ Borel, then

$$S_{K,p}(\omega) = \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x)\rangle^{1-p} d\mathcal{H}^{n-1}(x). \tag{1}$$

Here the case $p = 1$ corresponds to the surface area measure S_K, and $p = 0$ to the so called cone volume measure.

The L_p surface area measure has been intensively investigated in the recent decades, see say [1, 4, 12, 25, 26, 28, 29, 34, 45–47, 50–52, 55, 56, 58, 59, 61, 62]. In [48], Lutwak posed the associated L_p Minkowski problem which extends the classical Minkowski problem for $p \geq 1$, which

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case was essentially solved by Chou, Wang [17], Guan, Lin [24] and Hug, Lutwak, Yang, Zhang [38]. In addition, the L_p Minkowski problem for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990's. The L_p Minkowski problem is the classical Minkowski problem when $p = 1$, while the L_p Minkowski problem is the so called logarithmic Minkowski problem when $p = 0$, see say [5, 8–10, 45–47, 58, 59, 61, 65, 66, 72]. The L_p Minkowski problem is interesting for all real p, and have been studied by Lutwak [48], Lutwak and Oliker [49], Chou and Wang [17], Guan and Lin [24], Hug, et al. [38], Böröczky, et al. [8]. Additional references regarding the L_p Minkowski problem and Minkowski-type problems can be found say in [8,14,17,23–27,36–38,41,42,44,48,49,54,57,65,66,73,74]. Applications of the solutions to the L_p Minkowski problem can be found in, e.g., [2,3,16,18,19,30–32,39,40,53,69,71].

L_p-Minkowski problem: For $p \in \mathbb{R}$, what are the necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} to ensure that μ is the L_p surface area measure of a convex body in $\mathbb{R}^n$?

Besides discrete measures corresponding to polytopes, an important special case is when

$$d\mu = f\, d\mathcal{H}^{n-1} \tag{2}$$

for some non-negative measurable function f on S^{n-1}. If $p < 1$ and (2) holds, then the L_p-Minkowski problem amounts to solving the Monge-Ampère type equation

$$h^{1-p} \det(\nabla^2 h + hI) = nf \tag{3}$$

where h is the unknown non-negative function on S^{n-1} to be found (the support function), $\nabla^2 h$ denote the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix.

If $n = 2$, then we may assume that both h and f are non-negative periodic functions on $\mathbb{R}$ with period 2π. In this case the corresponding differential equation is

$$h^{1-p}(h" + h) = 2f. \tag{4}$$

After earlier work by V. Umanskiy [68] and W. Chen [14], equation (4) in the π-periodic case that corresponds to planar origin symmetric convex bodies has been thoroughly investigated by M.Y. Jiang [41] if $p > -2$, and by M.N. Ivaki [40] if $p = -2$ (the "critical case").

Here we concentrate on the case $p \in (0, 1)$. The case when μ has positive density function is handled by Chou, Wang [17]:

Theorem 1.1 (Chou, Wang). *If $p \in (-n, 1)$, $n \geq 2$ and μ is a Borel measure on S^{n-1} satisfying (2) where f is bounded and $\inf_{u \in S^{n-1}} f(u) > 0$, then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$.*

We note that if $p \in (2-n, 1)$, then there exists $K \in \mathcal{K}_0^n$ with $o \in \partial K$ such that $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous $f : S^{n-1} \to \mathbb{R}$ (see Example 1.6).

If $p \in (0, 1)$, then the L_p-Minkowski problem for polytopes have been solved by Zhu [74].

Theorem 1.2 (Zhu). *For $p \in (0, 1)$ and $n \geq 2$, a non-trivial discrete Borel measure μ on S^{n-1} is the L_p-surface area measure of a polytope $P \in \mathcal{K}_0^n$ with $o \in \operatorname{int} P$ if and only if μ is not concentrated on any closed hemisphere.*

Remark If $G \subset O(n)$ is a subgroup such that $\mu(\{Au\}) = \mu(\{u\})$ for any $u \in S^{n-1}$ and $A \in G$, then one may assume that $AP = P$ for any $A \in G$, as we explain in the Appendix.

For $p \in (0, 1)$, C. Haberl, E. Lutwak, D. Yang, G. Zhang [27] solved the L_p-Minkowski problem for even measures, or equivalently, for origin symmetric convex bodies.

Theorem 1.3 (Haberl, Lutwak, Yang, Zhang)**.** *For $p \in (0,1)$ and $n \geq 2$, a non-trivial bounded even Borel measure μ on S^{n-1} is the L_p-surface area measure of an origin symmetric $K \in \mathcal{K}_0^n$ if and only if μ is not concentrated on any great subsphere.*

The main goal of the paper is to solve the planar L_p Minkowski problem in full generality if $p \in (0,1)$. We note that if $n = 2$ and μ satisfies (2), then we may assume that both h and f are non-negative periodic functions on $\mathbb{R}$ with period 2π.

Theorem 1.4. *For $p \in (0,1)$ and a non-trivial bounded Borel measure μ on S^1, μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^2$ if and only if* $\operatorname{supp}\mu$ *does not consist of a pair of oppositive vectors.*

Remark If $G \subset O(2)$ is a finite subgroup such that $\mu(A\omega) = \mu(\omega)$ for any Borel $\omega \subset S^1$ and $A \in G$, then one may assume that $AK = K$ for any $A \in G$.

Corollary 1.5. *For $p \in (0,1)$ and any non-negative 2π-periodic function $f \in L_1([0,2\pi])$, the differential equation (4) has a non-negative 2π-periodic solution.*

Remark If f is even, or is periodic with respect to $2\pi/k$ for an integer $k \geq 2$, then the same can be said about h.

Unfortunately, the method of the proof of Theorem 1.4 does not extend to higher dimensions (see Example 3.8, and the remarks above).

We note that for $p \in (2-n, 1)$ in $\mathbb{R}^n$ (or $p \in (0,1)$ in $\mathbb{R}^2$), even if the function f on the right hand side of (3) or (4) is positive and continuous, then possibly $o \in \partial K$ for the solution K. The following example is based on the example the end of Hug, Lutwak, Yang, Zhang [38], and on examples in the preprint Guan, Lin [24] and in Chou, Wang [17].

Example 1.6. *If $p \in (2-n, 1)$, then there exists $K \in \mathcal{K}_0^n$ with C^2 boundary with $o \in \partial K$ such that $dS_{K,p} = f\, d\mathcal{H}^{n-1}$ for a positive continuous $f : S^{n-1} \to \mathbb{R}$.*

Proof. We fix $v \in S^{n-1}$, set $B^{n-1} = v^\perp \cap B^n$ and for $x \in v^\perp$ and $t \in \mathbb{R}$, we write point $(x,t) = x + tv$. For

$$q = \frac{2(n-1)}{n+p-2} > 2,$$

we consider the C^2 function $g(x) = \|x\|^q$ on B^{n-1}. We define the convex body K in $\mathbb{R}^n$ with C^2 boundary in a way such that $o \in \partial K$ and the graph $\{(x, g(x)) : x \in B^{n-1}\}$ of g above B^{n-1} is a subset of ∂K. We may assume that ∂K has positive Gauß curvature at each $z \in \partial K \backslash \{o\}$.

We observe that K is strictly convex and $-v$ is the exterior unit normal at o, and hence $S_K(\{-v\}) = 0$. If $z \in \partial K$, then we write $\nu(z)$ to denote the exterior unit normal at z, and $\kappa(\nu(z))$ to denote the Gauß curvature at z, therefore even if $\kappa(-v) = 0$, we have

$$dS_K = \kappa^{-1}\, d\mathcal{H}^{n-1}.$$

In turn, we deduce that

$$dS_{K,p} = h_K^{1-p} \kappa^{-1}\, d\mathcal{H}^{n-1}. \tag{5}$$

Let $x \in B^{n-1}$ satisfy $0 < \|x\| < 1$, and let $z = (x, g(x))$, and hence $\kappa(\nu(z)) > 0$. We have

$$\nabla g(x) = q\|x\|^{q-2}x \quad \text{and} \quad \nu(z) = a(x)^{-1}(\nabla g(x), -1)$$

for

$$a(x) = (1 + \|\nabla g(x)\|^2)^{1/2}.$$

In particular, writing $u = \nu(z)$, we have

$$h_K(u) = \langle u, z\rangle = a(x)^{-1}\left(\langle\nabla g(x), x\rangle - g(x)\right) = a(x)^{-1}(q-1)\|x\|^q.$$

In addition,

$$\kappa(u) = a(x)^{-(n+1)}\det(\nabla^2 g(x)) = (q-1)q^{n-1}a(x)^{-(n+1)}\|x\|^{(q-2)(n-1)},$$

therefore the Radon-Nikodym derivative in (5) is

$$h_K(u)^{1-p}\kappa(u)^{-1} = (q-1)^{-p}q^{1-n}a(x)^{n+p}\|x\|^{q(1-p)-(q-2)(n-1)} = (q-1)^{-p}q^{1-n}a(x)^{n+p}.$$

Since $a(x)$ is continuous and positive function of $x\in B^{n-1}$, we deduce that $S_{K,p}$ has a positive and continuous Radon-Nikodym derivative f with respect to $\mathcal{H}^{n-1}$ on S^{n-1}. Q.E.D.

2. Preliminary statements

In this section, we prove some statements that are essential in proving Theorem 1.4. For $v\in S^{n-1}$ and $t\in[0,1)$, let

$$\Omega(v,t) = \{u\in S^{n-1} : \langle u, v\rangle > t\}.$$

In particular, $\Omega(v,0)$ is the open hemi-sphere centered at v.

Lemma 2.1. *If μ is a finite Borel measure on S^{n-1} such that the measure of any open hemi-sphere is positive, then there exists $\delta\in(0,\frac{1}{2})$ such that for any $v\in S^{n-1}$,*

$$\mu\left(\Omega(v,\delta)\right) > \delta.$$

Remark We may also assume that $\mu(S^{n-1}) < 1/\delta$.

Proof. Suppose, to the contrary, that for any $k\in\mathbb{N}$, $k>1$, there exists $u_k\in S^{n-1}$ for which $\mu\left(\Omega\left(u_k,\frac{1}{k}\right)\right)\le\frac{1}{k}$. It follows from the compactness of S^{n-1} that there is a convergent subsequence $\{u_{k_j}\}$ of $\{u_k\}$ to some $u\in S^{n-1}$.

Since $\mu\left(\Omega(u,0)\right) > 0$, there exists $\tau=\cos\alpha$ for $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that $\mu\left(\Omega(u,\tau)\right)>0$. There exists large enough $k_j\in\mathbb{N}$ satisfying $\frac{1}{k_j}<\mu\left(\Omega(u,\tau)\right)$, $\frac{1}{k_j}<\cos\frac{\pi+2\alpha}{4}$ and the angle θ of u_{k_j} and u is at most $\frac{\pi-2\alpha}{4}$. Since

$$\cos(\alpha+\theta)\ge\cos\left(\alpha+\frac{\pi-2\alpha}{4}\right)=\cos\frac{\pi+2\alpha}{4}>\frac{1}{k_j},$$

the spherical triangle inequality yields $\Omega(u,\tau)\subset\Omega\left(u_{k_j},\frac{1}{k_j}\right)$. We deduce that

$$\mu\left(\Omega\left(u_{k_j},\frac{1}{k_j}\right)\right)\ge\mu\left(\Omega(u,\tau)\right)>\frac{1}{k_j},$$

contradicting the definition of u_k, and proving Lemma 2.1. Q.E.D.

Recall that the convex compact sets K_m tend to the convex compact set K in $\mathbb{R}^n$ if

$$\lim_{m\to\infty}\max\{u\in S^{n-1} : \|h_{K_m}(u)-h_K(u)\|\}=0.$$

We also note that the surface area measure can be extended to compact convex sets. Let K be a compact convex set in $\mathbb{R}^n$ (see R. Schneider [64]). If $\dim K\le n-2$, then S_K is the constant zero measure. In addition, if $\dim K=n-1$ and $v\in S^{n-1}$ is normal to aff K, then S_K is concentrated onto $\{\pm v\}$, and $S_K(\{v\})=S_K(\{-v\})=\mathcal{H}^{n-1}(K)$.

Lemma 2.2. *If $\varphi : [0,\infty) \to [0,\infty)$ is continuous, and the sequence of convex compact convex sets K_m with $o \in K_m$ tends to the convex compact set K in $\mathbb{R}^n$, then the measures $\varphi \circ h_{K_m}\, dS_{K_m}$ tend weakly to $\varphi \circ h_K\, dS_K$.*

Proof. According to Theorem 4.2.1 in R. Schneider [64], S_{K_m} tends weakly to S_K. Since $o \in K_m$ for all K_m, we have $o \in K$. There exists $R > 0$ such that $K_m \subset RB^n$ for all m, and hence $h_{K_m}(u) \leq R$ for m. Since φ is uniformly continuous on $[0, R]$, for any continuous function $g : S^{n-1} \to \mathbb{R}$, the function $u \mapsto g(u)\varphi(h_{K_m}(u))$ tends uniformly to $u \mapsto g(u)\varphi(h_K(u))$ on S^{n-1}. Therefore $g(\varphi \circ h_{K_m})\, dS_{K_m}$ tends to $g(\varphi \circ h_K)\, dS_K$. Q.E.D.

Corollary 2.3. *If $p \leq 1$, and a sequence of compact convex sets K_m with $o \in K_m$ tends to the compact convex set K in $\mathbb{R}^n$, then $S_{K_m,p}$ tends weakly to $S_{K,p}$.*

For $u_1, \ldots, u_k \in S^{n-1}$, we set

$$\text{pos}\{u_1, \ldots, u_k\} = \{\lambda_1 u_1 + \ldots + \lambda_k u_k :\ \lambda_1, \ldots, \lambda_k \geq 0\}.$$

Lemma 2.4. *If $x \in \mathbb{R}^n$, $u_1, \ldots, u_k \in S^{n-1}$ and $u \in S^{n-1} \cap \text{pos}\{u_1, \ldots, u_k\}$ satisfy that $\langle u_i, x\rangle \geq 0$ for $i = 1, \ldots, k$, then*

$$\langle u, x\rangle \geq \min\{\langle u_1, x\rangle, \ldots, \langle u_k, x\rangle\}.$$

Proof. We may assume that $\langle u_1, x\rangle \leq \langle u_i, x\rangle$ for $i = 1, \ldots, k$. The convexity of the unit ball yields that there exist $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \ldots + \lambda_k \geq 1$ such that $u = \lambda_1 u_1 + \ldots + \lambda_k u_k$, and hence

$$\langle u, x\rangle = \sum_{i=1}^{k} \lambda_i \langle u_i, x\rangle \geq \left(\sum_{i=1}^{k} \lambda_i\right) \langle u_1, x\rangle \geq \langle u_1, x\rangle.$$

Q.E.D.

For a planar convex body K in $\mathbb{R}^2$ and $x \in \partial K$, we choose an exterior unit normal $\nu_K(x)$ at x, which notation coincide with the earlier defined if x is a smooth point. We say that $x_1, x_2 \in \partial K$ are opposite points if there exists an exterior normal $u \in S^1$ at x_1 such that $-u$ is exterior normal at $x_2 \in \partial K$. If $x_1, x_2 \in \partial K$ are not opposite, then we write $\sigma(K, x_1, x_2)$ to denote the arc of ∂K connecting x_1 and x_2 not containing opposite points. It is possible that $x_1 = x_2$. We observe that if $x \in \sigma(K, x_1, x_2) \backslash \{x_1, x_2\}$, then

$$\nu_K(x) \in \text{pos}\{\nu_K(x_1), \nu_K(x_2)\}. \tag{6}$$

Claim 2.5. *For $p < 1$, a planar convex body K in $\mathbb{R}^2$ and non-opposite $x_1, x_2 \in \partial K$, if $\langle x_1, \nu_K(x_2)\rangle > 0$ and $\langle x_2 - x_1, u\rangle > 0$ for $u \in S^1$, then*

$$\min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}^{1-p} \cdot \langle x_2 - x_1, u\rangle \leq \int_{S^1} h_K^{1-p}\, dS_K.$$

Proof. If $x \in \sigma(K, x_1, x_2)$ is a smooth point, then (6) and Lemma 2.4 yield

$$\langle x, \nu_K(x)\rangle \geq \langle x_1, \nu_K(x)\rangle \geq \min\{\langle x_1, \nu_K(x_1)\rangle, \langle x_1, \nu_K(x_2)\rangle\} = \min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}.$$

Therefore

$$\begin{aligned}\int_{S^1} h_K^{1-p}\, dS_K &= \int_{\partial K} \langle x, \nu_K(x)\rangle^{1-p}\, d\mathcal{H}^1(x) > \int_{\sigma(K,x_1,x_2)} \langle x, \nu_K(x)\rangle^{1-p}\, d\mathcal{H}^1(x)\\ &\geq \min\{h_K(\nu_K(x_1)), \langle x_1, \nu_K(x_2)\rangle\}^{1-p} \cdot \mathcal{H}^1(\sigma(K, x_1, x_2)),\end{aligned}$$

and finally Claim 2.5 follows from $\mathcal{H}^1(\sigma(K, x_1, x_2)) \geq \langle x_2 - x_1, u\rangle$. Q.E.D.

3. Theorem 1.4 if the measure of any open semicircle is positive

Let $p\in(0,1)$, let μ be a finite Borel measure on S^1 such that the measure of any open semicircle is positive, and let $\delta\in(0,\frac{1}{2})$ be the constant of Lemma 2.1 for μ also satisfying $\mu(S^1)<1/\delta$.

We construct a sequence $\{\mu_m\}$ of discrete Borel measures on S^1 tending weakly to μ such that the μ_m measure of any open semicircle is positive for each m. It is the easiest to construct the sequence by identifying $\mathbb{R}^2$ with $\mathbb{C}$. For $m\geq 3$, we write $u_{jm}=e^{j2\pi/m}$ for $j=1,\ldots,m$, and we define μ_m be the measure having the support $\{u_{1m},\ldots,u_{mm}\}$ with

$$\mu_m(\{u_{jm}\})=\frac{1}{m^2}+\mu\left(\{e^{it}:\,(j-1)2\pi<t\leq j2\pi\}\right)\quad\text{for } j=1,\ldots,m.$$

According to Theorem 1.2 due to Zhu [74], there exists a polygon P_m with $o\in\operatorname{int}P_m$ such that $d\mu_m=h_{P_m}^{1-p}\,dS_{P_m}$ for each m. It follows from Lemma 2.2 that we may assume that

$$\int_{S^1}h_{P_m}^{1-p}\,dS_{P_m}<1/\delta. \tag{7}$$

Proposition 3.1. *$\{P_m\}$ is bounded.*

Proof. We assume that $d_m=\operatorname{diam}P_m$ tends to infinity, and seek a contradiction. Choose $y_m,z_m\in P_m$ such that $\|z_m-y_m\|=d_m$ and $\|z_m\|\geq\|y_m\|$. Let $v_m=(z_m-y_m)/\|z_m-y_m\|$, and let $w_m\in S^1$ be orthogonal to v_m. We observe that v_m and $-v_m$ are exterior normals at z_m and y_m, respectively. It follows that $\langle z_m,v_m\rangle\geq d_m/2$. By possibly taking subsequences, we may assume that v_m tends to $\tilde{v}\in S^1$. It follows from Lemma 2.1 and Lemma 2.2 that if m is large, then

$$\int_{\Omega(-v_m,\delta/2)}h_{P_m}^{1-p}\,dS_{P_m}>\delta/2. \tag{8}$$

We prove Proposition 3.1 based on the series of auxiliary statements Lemma 3.2 to Lemma 3.7.

Let $a_m,b_m\in\partial P_m$ such that $\langle a_m-b_m,w_m\rangle>0$ and $\langle a_m,v_m\rangle=\langle b_m,v_m\rangle=d_m/4$. We also deduce that $[a_m,b_m]\cap\operatorname{int}P_m\neq\emptyset$ for the segment $[a_m,b_m]$.

Lemma 3.2. *There exists $c_1>0$ depending on μ and p such that if m is large, then*

$$h_{P_m}(\nu_{P_m}(a_m))\leq c_1d_m^{\frac{-1}{1-p}}\quad\text{and}\quad h_{P_m}(\nu_{P_m}(b_m))\leq c_1d_m^{\frac{-1}{1-p}}. \tag{9}$$

Proof. Since $\langle z_m-a_m,v_m\rangle\geq d_m/4$ and $\langle z_m-b_m,v_m\rangle\geq d_m/4$, (7) and Claim 2.5 with $x_1=a_m$, $x_2=z_m$ and $v=v_m$ yield (9). Q.E.D.

Our intermediate goal, from Lemma 3.3 to Lemma 3.6 is to show that $\nu_{P_m}(a_m)$ and $\nu_{P_m}(b_m)$ point essentially to the same direction as w_m and $-w_m$, respectively, or in other words,

$$\lim_{m\to\infty}\langle\nu_{P_m}(a_m),v_m\rangle=\lim_{m\to\infty}\langle\nu_{P_m}(b_m),v_m\rangle=0.$$

We frequently use the fact that

$$\langle\nu_{P_m}(x_0),x_0-x\rangle\geq 0 \tag{10}$$

In particular, $\langle\nu_{P_m}(a_m),w_m\rangle>0$ and $\langle\nu_{P_m}(b_m),-w_m\rangle>0$ as $\langle\nu_{P_m}(a_m),a_m-b_m\rangle>0$ and $\langle\nu_{P_m}(b_m),b_m-a_m\rangle>0$, respectively, by (10) and $[a_m,b_m]\cap\operatorname{int}P_m\neq\emptyset$.

Lemma 3.3. *For any P_m, we have*

$$\frac{|\langle\nu_{P_m}(a_m),v_m\rangle|}{\langle\nu_{P_m}(a_m),w_m\rangle}\leq\frac{\langle a_m-z_m,w_m\rangle}{d_m/4}\quad\text{and}\quad\frac{|\langle\nu_{P_m}(b_m),v_m\rangle|}{\langle\nu_{P_m}(b_m),-w_m\rangle}\leq\frac{\langle b_m-z_m,-w_m\rangle}{d_m/4}. \tag{11}$$

Proof. It is enough to verify the statement about $\nu_{P_m}(a_m)$ where the definition of a_m implies $\langle a_m - z_m, v_m\rangle \leq -d_m/4$. If $\langle \nu_{P_m}(a_m), v_m\rangle \geq 0$, then

$$\begin{aligned} 0 &\leq \langle \nu_{P_m}(a_m), a_m - z_m\rangle = \langle \nu_{P_m}(a_m), v_m\rangle\langle a_m - z_m, v_m\rangle + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - z_m, w_m\rangle \\ &\leq -\langle \nu_{P_m}(a_m), v_m\rangle(d_m/4) + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - z_m, w_m\rangle \end{aligned}$$

yields (11). If $\langle \nu_{P_m}(a_m), -v_m\rangle \geq 0$, then using $\langle a_m - y_m, -v_m\rangle \leq -d_m/4$ and $\langle a_m - y_m, w_m\rangle = \langle a_m - z_m, w_m\rangle$, we deduce

$$\begin{aligned} 0 &\leq \langle \nu_{P_m}(a_m), a_m - y_m\rangle = \langle \nu_{P_m}(a_m), -v_m\rangle\langle a_m - y_m, -v_m\rangle + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - y_m, w_m\rangle \\ &\leq -\langle \nu_{P_m}(a_m), -v_m\rangle(d_m/4) + \langle \nu_{P_m}(a_m), w_m\rangle\langle a_m - y_m, w_m\rangle, \end{aligned}$$

and in turn we have (11). Q.E.D.

Corollary 3.4. *For any P_m, we have*

$$\langle \nu_{P_m}(a_m), w_m\rangle \geq \frac{1}{5} \quad \textit{and} \quad \langle \nu_{P_m}(b_m), -w_m\rangle \geq \frac{1}{5}. \tag{12}$$

Proof. It is enough to verify the statement about $\nu_{P_m}(a_m)$. Let $\gamma_m = \angle(\nu_{P_m}(a_m), w_m)$. Since $\langle a_m - z_m, w_m\rangle \leq d_m$ and $\langle b_m - z_m, -w_m\rangle \leq d_m$ follow from $\|a_m - b_m\| \leq d_m$, we conclude from (11) that $\tan\gamma_m \leq 4$. We deduce that

$$\langle \nu_{P_m}(a_m), w_m\rangle = \cos\gamma_m = (1+\tan^2\gamma_m)^{-1/2} \geq \frac{1}{\sqrt{17}} > \frac{1}{5}. \text{ Q.E.D.}$$

Possibly interchanging w_m with $-w_m$, and the role of a_m and b_m, we may assume that $\langle y_m, w_m\rangle = \langle z_m, w_m\rangle \geq 0$. We have $z_m = t_m v_m + r_m w_m$ and $y_m = s_m(-v_m) + r_m w_m$ for $t_m \geq s_m \geq 0$ and $r_m \geq 0$ where $t_m + s_m = d_m$. In particular, $t_m \geq d_m/2$.

Lemma 3.5. *There exist $c_2, c_3, c_4 > 0$ depending on μ and p such that if m is large, then*

$$r_m \leq c_2 d_m^{\frac{-1}{1-p}} \tag{13}$$

$$\langle v_m, \nu_{P_m}(a_m)\rangle \leq c_3 d_m^{\frac{p-2}{1-p}} \tag{14}$$

$$\langle v_m, \nu_{P_m}(b_m)\rangle \leq c_4 d_m^{\frac{p-2}{1-p}}. \tag{15}$$

Proof. If $\langle v_m, \nu_{P_m}(a_m)\rangle \geq 0$, then (9) implies

$$r_m\langle w_m, \nu_{P_m}(a_m)\rangle + t_m\langle v_m, \nu_{P_m}(a_m)\rangle = \langle z_m, \nu_{P_m}(a_m)\rangle \leq \langle a_m, \nu_{P_m}(a_m)\rangle \leq c_1 d_m^{\frac{-1}{1-p}},$$

which in turn yields (13) by (12) in this case, and in addition, yields (14) by $t_m \geq d_m/2$. Similarly, if $\langle -v_m, \nu(a_m)\rangle \geq 0$, then we have

$$r_m\langle w_m, \nu_{P_m}(a_m)\rangle + s_m\langle -v_m, \nu_{P_m}(a_m)\rangle = \langle y_m, \nu_{P_m}(a_m)\rangle \leq \langle a_m, \nu_{P_m}(a_m)\rangle \leq c_1 d_m^{\frac{-1}{1-p}},$$

and we conclude (13) using again (12). Finally, if $\langle v_m, \nu_{P_m}(b_m)\rangle \geq 0$, then combining $0 < \langle -w_m, \nu_{P_m}(b_m)\rangle \leq 1$, (13) and

$$-r_m\langle -w_m, \nu_{P_m}(b_m)\rangle + t_m\langle v_m, \nu_{P_m}(b_m)\rangle = \langle z_m, \nu_{P_m}(b_m)\rangle \leq \langle b_m, \nu_{P_m}(b_m)\rangle \leq c_1 d_m^{\frac{-1}{1-p}}$$

implies

$$t_m\langle v_m, \nu_{P_m}(b_m)\rangle \leq (c_1 + c_2) d_m^{\frac{-1}{1-p}},$$

and in turn we conclude (15) by $t_m \geq d_m/2$. Q.E.D.

Lemma 3.6. *There exist $c_5, c_6 > 0$ depending on μ and p such that if m is large, then*

$$\langle v_m, \nu_{P_m}(a_m)\rangle \geq -c_5 d_m^{\frac{p-1}{3-3p+p^2}-1} \tag{16}$$

$$\langle v_m, \nu_{P_m}(b_m)\rangle \geq -c_6 d_m^{\frac{p-1}{3-3p+p^2}-1}. \tag{17}$$

Proof. According to (11), it is sufficient to prove that there exist $c_7, c_8 > 0$ depending on μ and p such that

$$\alpha_m = \langle a_m - z_m, w_m\rangle \leq c_7 d_m^{\frac{p-1}{3-3p+p^2}} \quad \text{provided } \langle v_m, \nu_{P_m}(a_m)\rangle < 0, \tag{18}$$

$$\beta_m = \langle b_m - z_m, -w_m\rangle \leq c_8 d_m^{\frac{p-1}{3-3p+p^2}} \quad \text{provided } \langle v_m, \nu_{P_m}(b_m)\rangle < 0. \tag{19}$$

For (18), $\|a_m - z_m\| \leq d_m$ and $|\langle a_m - z_m, v_m\rangle| \geq d_m/4$ yield $\alpha_m \leq \frac{\sqrt{15}}{4} d_m$, and hence

$$\eta_m = \left(\frac{\alpha_m}{d_m}\right)^{\frac{1-p}{2-p}} \leq \left(\frac{\sqrt{15}}{4}\right)^{\frac{1-p}{2-p}} < 1.$$

The constant η_m is chosen in a way such that the calculations in Case 1 and in Case 2 lead to the same estimate up to a constant factor.

We consider the vector $e_m \in S^1$ such that $\langle e_m, v_m\rangle = \eta_m$ and $\langle e_m, w_m\rangle > 0$, and hence there exists $c_9 > 0$ depending on p such that

$$\langle e_m, w_m\rangle \geq c_9.$$

There exists $a'_m \in \sigma(P_m, a_m, z_m)$ such that w_m is an exterior unit normal, and there exists $\tilde{a}_m \in \sigma(P_m, a'_m, z_m)$ such that e_m is an exterior unit normal at $\tilde{a}_m$. In particular, we may assume that $\nu_{P_m}(a'_m) = w_m$ and $\nu_{P_m}(\tilde{a}_m) = e_m$, and we have

$$\langle a'_m, w_m\rangle \geq \langle a'_m - z_m, w_m\rangle = h_{P_m}(w_m) - \langle z_m, w_m\rangle \geq \langle a_m - z_m, w_m\rangle = \alpha_m.$$

We distinguish two cases.

Case 1 $\langle \tilde{a}_m - z_m, w_m\rangle < \alpha_m/2$

We want to apply Claim 2.5 with $x_1 = a'_m$ $x_2 = \tilde{a}_m$ and $u = v_m$. Since both of $\langle a'_m, w_m\rangle$ and $\langle e_m, w_m\rangle$ are positive, and $\langle a'_m, v_m\rangle \geq d_m/4$, $\langle e_m, v_m\rangle = \eta_m$ and $d_m \geq \alpha_m$, we deduce that

$$\langle a'_m, e_m\rangle = \langle a'_m, v_m\rangle\langle e_m, v_m\rangle + \langle a'_m, w_m\rangle\langle e_m, w_m\rangle \geq (d_m/4)\eta_m = \tfrac{1}{4}\, \alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}} \geq \frac{\alpha_m}{4}.$$

In addition, $h_{P_m}(w_m) \geq \alpha_m$, thus $\min\{h_{P_m}(w_m), \langle a'_m, e_m\rangle\} \geq \frac{\alpha_m}{4}$. Since $\langle \tilde{a}_m - a'_m, w_m\rangle < -\alpha_m/2$ by the condition in Case 1, we have

$$0 \leq \langle \tilde{a}_m - a'_m, e_m\rangle = \langle \tilde{a}_m - a'_m, v_m\rangle\langle e_m, v_m\rangle + \langle \tilde{a}_m - a'_m, w_m\rangle\langle e_m, w_m\rangle \leq \langle \tilde{a}_m - a'_m, v_m\rangle\eta_m - \frac{c_9\alpha_m}{2},$$

and hence

$$\langle \tilde{a}_m - a'_m, v_m\rangle \geq \frac{c_9\alpha_m}{2\eta_m} = \frac{c_9}{2}\, \alpha_m^{\frac{1}{2-p}} d_m^{\frac{1-p}{2-p}}.$$

Therefore (7) and Claim 2.5 with $x_1 = a'_m$, $x_2 = \tilde{a}_m$ and $u = v_m$ imply

$$\left(\frac{\alpha_m}{4}\right)^{1-p} \cdot \frac{c_9}{2}\, \alpha_m^{\frac{1}{2-p}} d_m^{\frac{1-p}{2-p}} < \frac{1}{\delta},$$

and in turn we conclude (18).

Case 2 $\langle \tilde{a}_m - z_m, w_m\rangle \geq \alpha_m/2$

Now $\langle z_m, e_m\rangle \geq (d_m/4)\eta_m$ by $\langle z_m, w_m\rangle \geq 0$, thus $h_{P_m}(v_m) \geq d_m/2$ yields

$$\min\{h_{P_m}(v_m), \langle z_m, e_m\rangle\} \geq (d_m/4)\eta_m = \frac{1}{4}\,\alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}}.$$

Therefore (7) and Claim 2.5 with $x_1 = z_m$, $x_2 = \tilde{a}_m$ and $u = w_m$ yield

$$\left(\frac{1}{4}\,\alpha_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}}\right)^{1-p} \cdot \frac{\alpha_m}{2} < \frac{1}{\delta},$$

and we finally conclude (18).

Next we turn to (19) where the argument is similar to the argument for (18). The difference between the proofs of (19) and (18) is that now $\langle z_m, -w_m\rangle < 0$. However, $\langle z_m, -w_m\rangle = -r_m > -c_2 d_m^{\frac{-1}{1-p}}$ according to (13). If

$$\beta_m < d_m^{\frac{p-1}{3-3p+p^2}},$$

then (19) readily holds. Therefore, we assume that

$$\beta_m \geq d_m^{\frac{p-1}{3-3p+p^2}}.$$

Since $\frac{-1}{1-p} < \frac{p-1}{3-3p+p^2}$, we may assume that m is large enough to ensure that

$$\beta_m \geq d_m^{\frac{p-1}{3-3p+p^2}} > 4c_2 d_m^{\frac{-1}{1-p}} \geq 4r_m,$$

In particular, if m is large, then

$$\langle b_m, -w_m\rangle \geq \frac{3\beta_m}{4}. \tag{20}$$

Since $\|b_m - z_m\| \leq d_m$ and $|\langle b_m - z_m, v_m\rangle| \geq d_m/4$ yield $\beta_m \leq \frac{\sqrt{15}}{4}\, d_m$, we have

$$\theta_m = \left(\frac{\beta_m}{d_m}\right)^{\frac{1-p}{2-p}} \leq \left(\frac{\sqrt{15}}{4}\right)^{\frac{1-p}{2-p}} < 1.$$

We consider the vector $f_m \in S^1$ such that $\langle f_m, v_m\rangle = \theta_m$ and $\langle f_m, -w_m\rangle > 0$, and hence for the $c_9 > 0$ above depending on p, we have

$$\langle f_m, -w_m\rangle \geq c_9.$$

There exists $b'_m \in \sigma(P_m, b_m, z_m)$ such that $-w_m$ is an exterior unit normal, and there exists $\tilde{b}_m \in \sigma(P_m, b'_m, z_m)$ such that f_m is an exterior unit normal at $\tilde{b}_m$. In particular, we may assume that $\nu_{P_m}(b'_m) = -w_m$ and $\nu_{P_m}(\tilde{b}_m) = f_m$, and we have

$$\langle b'_m - z_m, -w_m\rangle \geq \beta_m.$$

Again, we distinguish two cases.

Case 1' $\langle \tilde{b}_m - z_m, -w_m\rangle < \beta_m/2$

In this case, we are going to apply Claim 2.5 with $x_1 = b'_m$, $x_2 = \tilde{b}_m$ and $u = v_m$. Since both of $\langle b'_m, -w_m\rangle$ and $\langle f_m, -w_m\rangle$ are positive, and $\langle b'_m, v_m\rangle \geq d_m/4$, $\langle f_m, v_m\rangle = \theta_m$ and $d_m \geq \beta_m$, we deduce that

$$\langle b'_m, f_m\rangle = \langle b'_m, v_m\rangle\langle f_m, v_m\rangle + \langle b'_m, -w_m\rangle\langle f_m, -w_m\rangle \geq (d_m/4)\theta_m = \tfrac{1}{4}\,\beta_m^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}} \geq \frac{\beta_m}{4}.$$

In addition, $h_{P_m}(-w_m) \geq 3\beta_m/4$ by (20), thus $\min\{h_{P_m}(-w_m), \langle b'_m, f_m\rangle\} \geq \frac{\beta_m}{4}$. Since $\langle \tilde{b}_m - b'_m, -w_m\rangle < -\beta_m/2$ by the condition in Case 1', we have

$$0 \leq \langle \tilde{b}_m - b'_m, f_m\rangle = \langle \tilde{b}_m - b'_m, v_m\rangle\langle f_m, v_m\rangle + \langle \tilde{b}_m - b'_m, -w_m\rangle\langle f_m, -w_m\rangle \leq \langle \tilde{b}_m - b'_m, v_m\rangle\theta_m - \frac{c_9\beta_m}{2},$$

and hence

$$\langle \tilde{b}_m - b'_m, v_m\rangle \geq \frac{c_9\beta_m}{2\theta_m} = \frac{c_9}{2}\,\beta_m^{\frac{1}{2-p}}\,d_m^{\frac{1-p}{2-p}}.$$

Therefore (7) and Claim 2.5 with $x_1 = b'_m$, $x_2 = \tilde{b}_m$ and $u = v_m$ imply

$$\left(\frac{\beta_m}{4}\right)^{1-p} \cdot \frac{c_9}{2}\,\beta_m^{\frac{1}{2-p}}\,d_m^{\frac{1-p}{2-p}} < \frac{1}{\delta},$$

and in turn we conclude (19).

Case 2' $\langle \tilde{b}_m - z_m, -w_m\rangle \geq \beta_m/2$

In this case, (13) implies

$$\langle z_m, f_m\rangle = \langle z_m, v_m\rangle\langle f_m, v_m\rangle + \langle z_m, -w_m\rangle\langle f_m, -w_m\rangle \geq (d_m/4)\theta_m - c_2 d_m^{\frac{-1}{1-p}}.$$

Here, if m is large, then

$$d_m\theta_m = d_m\left(\frac{\beta_m}{d_m}\right)^{\frac{1-p}{2-p}} = \beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}} \geq \left(4c_2 d_m^{\frac{-1}{1-p}}\right)^{\frac{1-p}{2-p}} d_m^{\frac{1}{2-p}} = (4c_2)^{\frac{1-p}{2-p}} > 8c_2 d_m^{\frac{-1}{1-p}},$$

thus $\langle z_m, f_m\rangle \geq (d_m/8)\eta_m$. It follows from $h_{P_m}(v_m) \geq d_m/2$ that

$$\min\{h_{P_m}(v_m), \langle z_m, f_m\rangle\} \geq (d_m/8)\theta_m = \frac{1}{8}\,\beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}}.$$

Therefore (7) and Claim 2.5 with $x_1 = z_m$, $x_2 = \tilde{b}_m$ and $u = -w_m$ yield

$$\left(\frac{1}{8}\,\beta_m^{\frac{1-p}{2-p}}\,d_m^{\frac{1}{2-p}}\right)^{1-p} \cdot \frac{\beta_m}{2} < \frac{1}{\delta},$$

and we finally conclude (19), and in turn Lemma 3.6. Q.E.D.

In order to finish the proof of Proposition 3.1, let $a^*_m \in \partial P_m$ maximize $\langle a^*_m, w_m\rangle$ under the condition that the $\gamma_m \in S^1$ with $\langle \gamma_m, -v_m\rangle = \delta/2$ and $\langle \gamma_m, w_m\rangle > 0$ is an exterior unit normal at a^*_m, and let $b^*_m \in \partial P_m$ maximize $\langle b^*_m, -w_m\rangle$ under the condition that the $\xi_m \in S^1$ with $\langle \xi_m, -v_m\rangle = \delta/2$ and $\langle \xi_m, -w_m\rangle > 0$ is an exterior unit normal at b^*_m.

Lemma 3.7. *There exist $c_{10}, c_{11} > 0$ depending on μ and p such that if m is large, then*

$$\langle a^*_m - y_m, w_m\rangle \leq c_{10} d_m^{\frac{-1}{1-p}} \tag{21}$$

$$\langle b^*_m - y_m, -w_m\rangle \leq c_{11} d_m^{\frac{-1}{1-p}}. \tag{22}$$

Proof. For (21), $\langle y_m, w_m\rangle = r_m \geq 0$ yields

$$\langle a^*_m - y_m, w_m\rangle \leq \langle a^*_m, w_m\rangle. \tag{23}$$

Since $\langle a^*_m - y_m, v_m\rangle \geq 0$ and $\langle a^*_m - y_m, w_m\rangle \geq 0$, we have

$$0 \leq \langle a^*_m - y_m, \gamma_m\rangle = \langle a^*_m - y_m, v_m\rangle\langle \gamma_m, v_m\rangle + \langle a^*_m - y_m, w_m\rangle\langle \gamma_m, w_m\rangle \leq \frac{-\delta}{2}\langle a^*_m - y_m, v_m\rangle + \langle a^*_m - y_m, w_m\rangle.$$

In turn (23) implies

$$\langle a_m^* - y_m, v_m\rangle \le \frac{2}{\delta}\langle a_m^* - y_m, w_m\rangle \le \frac{2}{\delta}\langle a_m^*, w_m\rangle. \tag{24}$$

It follows from (9) and (12) that

$$\begin{aligned} c_1 d_m^{\frac{-1}{1-p}} &\ge h_{P_m}(\nu_{P_m}(a_m)) \ge \langle a_m^*, \nu_{P_m}(a_m)\rangle = \langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle + \langle a_m^*, w_m\rangle\langle \nu_{P_m}(a_m), w_m\rangle \\ &\ge \langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle + \langle a_m^*, w_m\rangle/5. \end{aligned} \tag{25}$$

The rest of the argument is divided into three cases according to the signs of $\langle a_m^*, v_m\rangle$ and $\langle \nu_{P_m}(a_m), v_m\rangle$.

Case 1 $\langle a_m^*, v_m\rangle\langle \nu_{P_m}(a_m), v_m\rangle \ge 0$
In this case (25) and (23) yield (21) directly.

Case 2 $\langle a_m^*, v_m\rangle > 0$ and $\langle \nu_{P_m}(a_m), v_m\rangle < 0$
In this case $\langle y_m, v_m\rangle \le 0$ and (24) imply $\langle a_m^*, v_m\rangle \le \frac{2}{\delta}\langle a_m^*, w_m\rangle$. Since $|\langle \nu_{P_m}(a_m), v_m\rangle| < \frac{\delta}{20}$ for large m according to (16), we conclude from (25) that

$$c_1 d_m^{\frac{-1}{1-p}} \ge -\frac{2}{\delta}\langle a_m^*, w_m\rangle \cdot \frac{\delta}{20} + \frac{\langle a_m^*, w_m\rangle}{5} = \frac{\langle a_m^*, w_m\rangle}{10},$$

proving (21) by (23).

Case 3 $\langle a_m^*, v_m\rangle < 0$ and $\langle \nu_{P_m}(a_m), v_m\rangle > 0$
In this case, we have $\langle a_m^*, v_m\rangle \ge -d_m$ on the one hand, and (14) implies $\langle \nu_{P_m}(a_m), v_m\rangle < c_3 d_m^{\frac{p-2}{1-p}}$ on the other hand, therefore (25) yields

$$c_1 d_m^{\frac{-1}{1-p}} \ge -d_m c_3 d_m^{\frac{p-2}{1-p}} + \frac{\langle a_m^*, w_m\rangle}{5} = -c_3 d_m^{\frac{-1}{1-p}} + \frac{\langle a_m^*, w_m\rangle}{5},$$

completing the proof of (21) by (23).

For (22), we may assume that

$$\langle b_m^* - y_m, -w_m\rangle \ge 2c_2 d_m^{\frac{-1}{1-p}},$$

otherwise (22) readily holds with $c_{11} = 2c_2$. Since $\langle b_m^* - y_m, -w_m\rangle \le \langle b_m^*, -w_m\rangle + c_2 d_m^{\frac{-1}{1-p}}$ by (13), we have

$$\langle b_m^* - y_m, -w_m\rangle \le 2\langle b_m^*, -w_m\rangle. \tag{26}$$

Therefore using (26) in place of $\langle a_m^* - y_m, w_m\rangle \le \langle a_m^*, w_m\rangle$, (22) can be proved similarly to (21), completing the proof of Lemma 3.7. Q.E.D.

Finally, to prove Proposition 3.1, we observe that combining Lemma 3.7 with the definition of a_m^* and b_m^* yields that if m is large, then

$$\mathcal{H}^1(\sigma(P_m, a_m^*, b_m^*)) \le \frac{2}{\delta}\langle a_m^* - b_m^*, w_m\rangle \le \frac{2(c_{10}+c_{11})}{\delta} \cdot d_m^{\frac{-1}{1-p}}.$$

It follows from applying first (8), then $\langle x, \nu_{P_m}(x)\rangle \le d_m$ for $x \in \partial K$ that if m is large, then

$$\begin{aligned} \frac{\delta}{2} &< \int_{\Omega(-v_m,\delta/2)} h_{P_m}^{1-p}\, dS_{P_m} = \int_{\sigma(P_m,a_m^*,b_m^*)} \langle x, \nu_{P_m}(x)\rangle^{1-p}\, d\mathcal{H}^1(x) \\ &\le d_m^{1-p} \cdot \frac{2(c_{10}+c_{11})}{\delta} \cdot d_m^{\frac{-1}{1-p}} = \frac{2(c_{10}+c_{11})}{\delta} \cdot d_m^{\frac{p(p-2)}{1-p}}, \end{aligned}$$

which is absurd as $\frac{p(p-2)}{1-p} < 0$ and d_m tends to infinity. This contradiction verifies Proposition 3.1. Q.E.D.

Proof of Theorem 1.4 if the measure of any open semicircle is positive Since $\{P_m\}$ is bounded and each P_m contains the origin according to Proposition 3.1, the Blaschke selection theorem provides a subsequence $\{P_{m'}\}$ tending to a compact convex set K with $o \in K$. It follows from Corollary 2.3 that $S_{P_{m'},p}$ tends weakly to $S_{K,p}$. However, $\mu_{m'} = S_{P_{m'},p}$ tends weakly to μ by construction. Therefore $\mu = S_{K,p}$. Since any open semi-circle of S^1 has positive μ measure, we conclude that $\mathrm{int}\, K \neq \emptyset$.

Finally, to prove the Remark after Theorem 1.4, let $G \subset O(2)$ be a finite subgroup such that $\mu(A\omega) = \mu(\omega)$ for any Borel $\omega \subset S^1$ and $A \in G$. The idea is that for large m, we subdivide S^1 into arcs of length less than $2\pi/m$ in a way such that the subdivision is symmetric with respect to G and each endpoint has μ measure 0.

We fix a regular l-gon Q, $l \ge 3$ whose vertices lie on S^1 such that G is a subgroup of the symmetry group of Q. In addition, we consider the set Σ of atoms of μ; namely, the set of all $u \in S^1$ such that $\mu(\{u\}) > 0$. In particular, Σ is countable.

For $m \ge 2$, let Q_m be a regular polygon with lm vertices such that all vertices of Q are vertices of Q_m, and let G_m be the symmetry group of Q_m. We observe that G_m contains rotations by angle $\frac{2\pi}{lm}$. We write Σ_m to denote the set obtained from repeated applications of the elements of G_m to the elements of Σ, and hence Σ_m is countable, as well. For a fixed $x_0 \in S^1\backslash\Sigma_m$, we consider the orbit $G_m x_0 = \{Ax_0 : A \in G_m\}$, and let $\mathcal{I}_m$ be the set of open arcs of S^1 that are the components of $S^1\backslash G_m x_0$. We observe that $G_m x_0$ is disjoint from Σ_m, and hence $\mu(\sigma) = \mu(\mathrm{cl}\,\sigma)$ for $\sigma \in \mathcal{I}_m$.

Now we define μ_m. It is concentrated on the set of midpoints of all $\sigma \in \mathcal{I}_m$, and the μ_m measure of the midpoints of a $\sigma \in \mathcal{I}_m$ is $\mu(\sigma)$. In particular, μ_m is invariant under G_m, and hence μ_m is invariant under G. Since the length of each arc in $\mathcal{I}_m$ is at most $\frac{2\pi}{lm}$, we deduce that μ_u tends weakly to μ.

According to the Remark after Theorem 1.2 due to Zhu [74], we may assume that each P_m is invariant under G. Now the argument above shows that some subsequence of $\{P_m\}$ tends to a convex body K satisfying $S_{K,p} = \mu$, and readily K is invariant under G. Q.E.D.

Unfortunately, the proof of Theorem 1.4 we present does not extend to higher dimensions. What we actually prove in this section (see Proposition 3.1) is the following statement: If $0 < p < 1$, μ is a bounded Borel measures on S^1 such that the μ measure of any open semi-circle is positive, and $P_m \in \mathcal{K}_o^2$ is a sequence of convex bodies such that $S_{P_m,p}$ tends weakly to μ, then the sequence $\{P_m\}$ is bounded. The following Example 3.8 shows that this statement already fails in $n = 3$ dimension. For $x_1, \ldots, x_k \in \mathbb{R}^3$, we write $[x_1, \ldots, x_k]$ to denote their convex hull.

Example 3.8. *For $p \in (0,1)$, there exist a measure μ on S^2 such that any open hemisphere has positive measure and an unbounded sequence of polytopes $\{P_m\}$ in $\mathbb{R}^3$ such that $o \in \mathrm{int} P_m$ and $S_{P_m,p}$ tends weakly to μ.*

Proof. We define

$$u_0 = (1,0,0),\ u_1 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),\ u_2 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right),\ u^+ = (0,0,1),\ u^- = (0,0,-1),$$

and the discrete measure μ with $\operatorname{supp}\mu = \{u_0, u_1, u_2, u^+, u^-\}$ and

$$\mu(\{u_0\}) = 8,\ \mu(\{u_1\}) = \mu(\{u_2\}) = 2^{\frac{p}{2}},\ \mu(\{u^+\}) = \mu(\{u^-\}) = 3,$$

and hence any open hemisphere has positive measure.

For $m \geq 2$ and $a = a_m = m^{-(2-p)}$, let

$$v_{1,m} = (0,m,0),\ v_{1,m}^+ = (m,2m,a),\ v_{1,m}^- = (m,2m,-a),$$

$$v_{2,m} = (0,-m,0),\ v_{2,m}^+ = (m,-2m,a),\ v_{2,m}^- = (m,-2m,-a),$$

and let $\widetilde{P}_m$ be their convex hull. The exterior unit normals of the faces $F_{0,m} = [v_{i,m}^+, v_{i,m}^-]_{i=1,2}$, $F_{1,m} = [v_{1,m}, v_{1,m}^+, v_{1,m}^-]$ and $F_{2,m} = [v_{2,m}, v_{2,m}^+, v_{2,m}^-]$ are u_0, u_1, u_2, respectively, which vectors are independent of m. In addition, $\widetilde{P}_m$ has two more facets, $F_m^+ = [v_{1,m}, v_{2,m}, v_{1,m}^+, v_{2,m}^+]$ and $F_m^- = [v_{1,m}, v_{2,m}, v_{1,m}^-, v_{2,m}^-]$ whose exterior unit normals are

$$u_m^+ = \left(\frac{-a_m}{\sqrt{a_m^2+m^2}}, 0, \frac{m}{\sqrt{a_m^2+m^2}}\right) \text{ and } u_m^- = \left(\frac{-a_m}{\sqrt{a_m^2+m^2}}, 0, \frac{-m}{\sqrt{a_m^2+m^2}}\right),$$

which satisfy $\lim_{m\to\infty} u_m^+ = u^+$ and $\lim_{m\to\infty} u_m^- = u^-$.

For $i = 1,2$, we have $h_{\widetilde{P}_m}(u_0) = m$, $h_{\widetilde{P}_m}(u_1) = h_{\widetilde{P}_m}(u_2) = \frac{m}{\sqrt{2}}$ and $h_{\widetilde{P}_m}(u_m^+) = h_{\widetilde{P}_m}(u_m^-) = 0$, therefore

$$\begin{aligned} S_{\widetilde{P}_m,p}(\{u_0\}) &= h_{\widetilde{P}_m}(u_0)^{1-p}\mathcal{H}^2(F_{0,m}) = m^{1-p}8ma_m = 8 \\ S_{\widetilde{P}_m,p}(\{u_i\}) &= h_{\widetilde{P}_m}(u_i)^{1-p}\mathcal{H}^2(F_{i,m}) = \left(\frac{m}{\sqrt{2}}\right)^{1-p}\sqrt{2}ma_m = 2^{\frac{p}{2}} \quad \text{for } i = 1,2, \end{aligned}$$

Now we translate $\widetilde{P}_m$ in order to alter $S_{\widetilde{P}_m,p}(\{u_m^+\})$. We define $t_m > 0$ in a way such that $P_m = \widetilde{P}_m - t_m u_0$ satisfies

$$h_{P_m}(u_m^+) = m^{\frac{-2}{1-p}}.$$

It follows that

$$m^{\frac{-2}{1-p}} = h_{P_m}(u_m^+) = t_m\langle u_m^+, u_0\rangle = \frac{t_m a_m}{\sqrt{m^2+a_m^2}} > \frac{t_m}{2m^{3-p}}.$$

We observe that $r = 3 - p - \frac{2}{1-p} < 3 - 2 = 1$ if $p \in (0,1)$, and hence $\lim_{m\to\infty} t_m/m = 0$. We deduce that

$$\begin{aligned} \lim_{m\to\infty} S_{P_m,p}(\{u_0\}) &= 8 \\ \lim_{m\to\infty} S_{P_m,p}(\{u_i\}) &= 2^{\frac{p}{2}} \quad \text{for } i = 1,2, \\ \lim_{m\to\infty} S_{P_m,p}(\{u_m^+\}) &= \lim_{m\to\infty} h_{P_m}(u_m^+)^{1-p}\mathcal{H}^2(F_m^+) = \lim_{m\to\infty} m^{-2}3m\sqrt{m^2+a_m^2} = 3. \end{aligned}$$

Therefore $S_{P_m,p}$ tends weakly to μ. Q.E.D.

4. Theorem 1.4 if the measure is concentrated on a closed semi-circle

First we show that the L_p surface area measure of a convex body K containing the origin can't be supported on two antipodal points.

Lemma 4.1. *If $K \in \mathcal{K}_0^2$, then* $\operatorname{supp} S_{K,p}$ *is not a pair of antipodal points.*

Proof. We suppose that $\operatorname{supp} S_{K,p} = \{v, -v\}$ for some $v \in S^1$, and seek a contradiction. Let $w \in S^1$ be orthogonal to v.

If $o \in \operatorname{int} K$ then $\operatorname{supp} S_{K,p} = \operatorname{supp} S_K$, which is not contained in any closed semi-circle. Therefore $o \in \partial K$, and let C be the exterior normal cone at o; namely, $C \cap S^1 = \{u \in S^1 : h_K(u) = 0\}$. Since $\operatorname{supp} S_{K,p} = \{v, -v\}$, we have $h_K(v) > 0$ and $h_K(-v) > 0$, and hence $v, -v \notin C$. Thus we may assume possibly after replacing w with $-w$ that $C \cap S^1 \subset \Omega(-w, 0)$. It follows that $h_K(u) > 0$ for $u \in \Omega(w, 0)$, and since $S_K(\Omega(w, 0)) > 0$, it also follows that

$$S_{K,p}(\Omega(w,0)) = \int_{\Omega(w,0)} h_K^{1-p}\, dS_K > 0.$$

This contradicts $\operatorname{supp} S_{K,p} = \{v, -v\}$, and proves Lemma 4.1. Q.E.D.

Let μ be a non-trivial measure on S^1 that is concentrated on a closed semi-circle σ of S^1 connecting $v, -v \in S^1$ such that $\operatorname{supp} \mu$ is not a pair of antipodal points. We may assume that for the $w \in \sigma$ orthogonal to v, we have either $\operatorname{supp} \mu = \{w\}$, or

$$w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu). \tag{27}$$

Case 1 $\operatorname{supp} \mu = \{w\}$

Let $w_1, w_2 \in S^1$ such that $w_1 + w_2 = -w$, and let K_0 be the regular triangle

$$K_0 = \{x \in \mathbb{R}^2 : \langle x, w_1 \rangle \le 0,\ \langle x, w_2 \rangle \le 0,\ \langle x, w \rangle \le 1\}.$$

There exists $\lambda > 0$ such that $\lambda S_{K_0,p}(\{w\}) = \mu(\{w\})$, and hence $S_{\lambda_0 K_0,p} = \mu$ for $\lambda_0 = \lambda^{\frac{1}{2-p}}$.

Case 2 $w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu)$

Let A be the reflection through the line $\operatorname{lin} v$. We define a measure $\tilde{\mu}$ on S^1 by

$$\tilde{\mu}(\omega) = \mu(\omega) + \mu(A\omega) \ \text{ for Borel sets } \omega \subset S^1.$$

We observe that $\tilde{\mu}$ is invariant under A,

$$\begin{aligned} \tilde{\mu}(\omega) &= \mu(\omega) \ \text{ if } \omega \subset \Omega(w,0), \\ \tilde{\mu}(\{v\}) &= 2\mu(\{v\}), \\ \tilde{\mu}(\{-v\}) &= 2\mu(\{-v\}). \end{aligned}$$

It follows from $w \in \operatorname{int} \operatorname{pos}(\operatorname{supp} \mu)$ that no closed semi-circle contains $\operatorname{supp} \tilde{\mu}$. We deduce from the previous section that there exists a convex body $\widetilde{K}$ invariant under A such that $S_{\widetilde{K},p} = \tilde{\mu}$.

We claim that

$$S_{K,p} = \mu \ \text{ for } K = \{x \in \widetilde{K} : \langle x, w \rangle \ge 0\}. \tag{28}$$

For any convex body M and $u \in S^1$, we write $F(M, u) = \{x \in M : \langle x, u \rangle = h_M(u)\}$ to denote the face of M with exterior unit normal u, and for any $x, y \in \mathbb{R}^2$, we write $[x, y]$ to denote the convex hull of x and y, which is a segment if $x \neq y$. Since $\widetilde{K}$ is invariant under A, there exist $t, s \ge 0$ such that $tv, -sv \in \partial \widetilde{K}$, and the exterior normals at tv and $-sv$ are v and $-v$, respectively. In addition, $\mathcal{H}^1(F(\widetilde{K}, v)) = 2\,\mathcal{H}^1(F(K, v))$, $\mathcal{H}^1(F(\widetilde{K}, -v)) = 2\,\mathcal{H}^1(F(K, -v))$ and $F(K, -w) = [tv, -sv]$.

To prove (28), first we observe that by definition, we have

$$\mu(\{v\}) = \frac{\tilde{\mu}(\{v\})}{2} = \frac{h_{\widetilde{K}}(v)^{1-p} \cdot \mathcal{H}^1(F(\widetilde{K}, v))}{2} = h_K(v)^{1-p} \cdot \mathcal{H}^1(F(K, v)) = S_{K,p}(\{v\}),$$

and similarly $\mu(\{-v\}) = S_{K,p}(\{-v\})$. Next (1) yields that

$$S_{K,p}(\Omega(-w, 0)) = \int_{[tv,-sv]} \langle x, w\rangle^{1-p} \, d\mathcal{H}^1(x) = 0 = \mu(\Omega(-w, 0)).$$

Finally, if $\omega \subset \Omega(w, 0)$, then $\nu_{\widetilde{K}}^{-1}(\omega) = \nu_K^{-1}(\omega)$, which yields

$$\mu(\omega) = \tilde{\mu}(\omega) = S_{\widetilde{K},p}(\omega) = S_{K,p}(\omega),$$

and in turn (28).

Therefore all we are left to do is to check the symmetries of μ. Actually the only possible symmetry is the reflection B through $\operatorname{lin} w$. In this case, $\tilde{\mu}$ is also invariant under B, and hence we may assume that $\widetilde{K}$ is also invariant under B. We conclude that K is invariant under B, completing the proof of Theorem 1.4. Q.E.D.

5. Appendix

Let $p \in (0, 1)$, let μ be a discrete measure on S^{n-1} such that any open hemi-sphere has positive measure, and let $G \subset O(n)$ is a subgroup such that $\mu(\{Au\}) = \mu(\{u\})$ for any $u \in S^{n-1}$ and $A \in G$. We review the proof of Theorem 1.2 due to Zhu [74] to show that for the polytope P with $o \in \operatorname{int} P$ and $S_{P,p} = \mu$, one may even assume that $AP = P$ for any $A \in G$.

We set $\operatorname{supp} \mu = \{u_1, \ldots, u_N\}$ and $\mu(\{u_i\}) = \alpha_i$ for $i = 1, \ldots, N$, and we write

$$\mathcal{P}^G(u_1, \ldots, u_N)$$

to denote the family of n-dimensional polytopes whose exterior unit normals are among $u_1, \ldots, u_N$ and are G invariant. In particular, if $P \in \mathcal{P}^G(u_1, \ldots, u_N)$ and $A \in G$, then $h_P(Au_i) = h_P(u_i)$ for $i = 1, \ldots, N$.

In order to find the a polytope $P_0 \in \mathcal{P}^G(u_1, \ldots, u_N)$ with $S_{P_0,p} = \mu$, following Zhu [74], we consider

$$\Phi_P(\xi) = \int_{S^{n-1}} h_{P-\xi}^p \, d\mu = \sum_{i=1}^{N} \alpha_i (h_P(u_i) - \langle \xi, u_i\rangle)^p$$

for $P \in \mathcal{P}^G(u_1, \ldots, u_N)$ and $\xi \in P$, and show that the extremal problem

$$\inf \left\{ \sup_{\xi \in P} \Phi_P(\xi) : \ P \in \mathcal{P}^G(u_1, \ldots, u_N) \text{ and } V(P) = 1 \right\}$$

has a solution that is a dilated copy of P_0.

According to Lemma 3.1 and Lemma 3.2 in [74], if $P \in \mathcal{P}^G(u_1, \ldots, u_N)$, then there exists a unique $\xi(P) \in \operatorname{int} P$ such that

$$\sup_{\xi \in P} \Phi_P(\xi) = \Phi_P(\xi(P)).$$

The uniqueness of $\xi(P)$ yields that

$$A\xi(P) = \xi(P) \ \text{ for } A \in G.$$

We deduce from Lemma 3.3 in [74] that $\xi(P)$ is a continuous function of P.

Let $\mathcal{P}_N^G(u_1,\ldots,u_N)$ be the family of all $P\in\mathcal{P}^G(u_1,\ldots,u_N)$ with N facets. Based on Lemma 3.4 and Lemma 3.5 in [74], slightly modifying the argument for Lemma 3.6 in [74], we deduce the existence of $\widetilde{P}\in\mathcal{P}_N^G(u_1,\ldots,u_N)$ with $V(\widetilde{P})=1$ such that

$$\Phi_{\widetilde{P}}(\xi(\widetilde{P}))=\inf\left\{\Phi_P(\xi(P)):\,P\in\mathcal{P}^G(u_1,\ldots,u_N)\text{ and }V(P)=1\right\}.$$

The only change in the argument in the argument for Lemma 3.6 in [74] is making the definition of P_δ G invariant. So supposing that $\dim F(\widetilde{P},u_{i_0})\leq n-2$, let $I\subset\{1,\ldots,N\}$ be defined by

$$\{Au_{i_0}:\,A\in G\}=\{u_i:\,i\in I\}.$$

Therefore for small $\delta>0$, we set

$$P_\delta=\{x\in P:\,\langle x,u_i\rangle\leq h_{\widetilde{P}}(u_i)-\delta\ \text{ for } i\in I\}.$$

The rest of the argument for Lemma 3.6 in [74] carries over.

Finally, in the proof of Theorem 4.1 in [74], the only necessary change is that for the $\delta_1,\ldots,\delta_N\in\mathbb{R}$ we assume that for any $A\in G$ and $i\in\{1,\ldots,N\}$, if $u_j=Au_i$, then $\delta_j=\delta_i$.

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