Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 3, Article 60, 2003

ON ZEROS OF RECIPROCAL POLYNOMIALS OF ODD DEGREE

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Received 12 December, 2002; accepted 30 July, 2003
Communicated by A. Sofo

ABSTRACT. The first author [1] proved that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C}),$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \left[\frac{m}{2}\right]$) are on the unit circle, provided that

$$|A_m| \geq \sum_{k=0}^{m} |A_k - A_m| = \sum_{k=1}^{m-1} |A_k - A_m|.$$

Moreover, the zeros of P_m are near to the $m+1$st roots of unity (except the root 1). A. Schinzel [3] generalized the first part of Lakatos' result for self–inversive polynomials i.e. polynomials

$$P_m(z) = \sum_{k=0}^{m} A_k z^k$$

for which $A_k \in \mathbb{C}$, $A_m \neq 0$ and $\epsilon \bar{A}_k = A_{m-k}$ for all $k = 0, \dots, m$ with a fixed $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$. He proved that all zeros of P_m are on the unit circle, provided that

$$|A_m| \geq \inf_{c,d \in \mathbb{C},\, |d|=1} \sum_{k=0}^{m} |cA_k - d^{m-k} A_m|.$$

If the inequality is strict the zeros are single. The aim of this paper is to show that for real reciprocal polynomials of odd degree Lakatos' result remains valid even if

$$|A_m| \geq \cos^2 \frac{\pi}{2(m+1)} \sum_{k=1}^{m-1} |A_k - A_m|.$$

We conjecture that Schinzel's result can also be extended similarly: all zeros of P_m are on the unit circle if P_m is self-inversive and

$$|A_m| \geq \cos \frac{\pi}{2(m+1)} \inf_{c,d \in \mathbb{C},\, |d|=1} \sum_{k=0}^{m} |cA_k - d^{m-k} A_m|.$$

Key words and phrases: Reciprocal, Semi-reciprocal polynomials, Chebyshev transform, Zeros on the unit circle.

2000 *Mathematics Subject Classification.* Primary 30C15, Secondary 12D10, 42C05.

ISSN (electronic): 1443-5756

The authors had their research supported by Hungarian NFSR grant No.s T 043034 and T 043080 respectively.

143-02

1. INTRODUCTION

Studying the spectral properties of the Coxeter transformation Lakatos [1] found that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^{m} A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients, i.e.

$$A_m \neq 0, \quad A_k \in \mathbb{R}, \text{ and } A_k = A_{m-k} \ \left(k = 0, \dots, \left[\tfrac{m}{2}\right]\right) \tag{1.1}$$

are on the unit circle, provided that

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|. \tag{1.2}$$

She used Chebyshev transformation to prove this result.

The manuscript on this was sent to A. Schinzel for his comments. He generalized the above theorem [3] for self-inversive polynomials by proving that all zeros of the polynomial $P_m(z) = \sum_{k=0}^{m} A_k z^k$ where

$$A_m \neq 0, \quad A_k \in \mathbb{C}, \text{ and } \text{ and } \epsilon \bar{A}_k = A_{m-k} \ (k = 0, \dots, m) \text{ with } \epsilon \in \mathbb{C}, \ |\epsilon| = 1 \tag{1.3}$$

are on the unit circle, provided that

$$|A_m| \geq \inf_{c,d \in \mathbb{C}, \, |d|=1} \sum_{k=0}^{m} |cA_k - d^{m-k} A_m| \tag{1.4}$$

holds.

Schinzel's proof was based on a theorem of Cohn [4] and on the estimate

$$\min_{z \in \mathbb{C}, |z|=1} \left| \sum_{k=1}^{m} k z^{m-k} \right| \geq \frac{m}{2}. \tag{1.5}$$

Learning of Schinzel's generalization, the first author made an attempt to improve her result. Although the method of Chebyshev transformation does not seem to work for self-inversive polynomials, it can be used to obtain information about the location of the zeros on the unit circle. She could prove [1] that condition (1.2) ensures that the distribution of the zeros e^{iu_j} $(j = 1, \dots, m)$ of $P_m(z) = \sum_{k=0}^{m} A_k z^k$ satisfying (1.1), (1.2) is quite regular. They can be arranged such that

$$\left|\epsilon_j - e^{iu_j}\right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m) \tag{1.6}$$

holds, where

$$\epsilon_j = e^{i \frac{j}{m+1} 2\pi} \quad (j = 1, 2, \dots, m)$$

are the $m+1$st roots of unity except 1.
If $m = 2n+1$ is odd then $-1 = e^{iu_{n+1}}$ is always a zero and all zeros of P_{2n+1} are single.
If $m = 2n$ is even, (1.2) holds with equality and

$$\operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n})$$

for all $k = 1, 2, \dots, n$ with $A_k - A_{2n} \neq 0$, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_{2n}. Otherwise all zeros of P_{2n} are single.

The aim of this paper is to show that for polynomials of odd degree both results can be improved. For even degree polynomials this is not possible. Since, for first degree reciprocal or self-inversive polynomials the only zero of the polynomial has modulus one we may assume that the degree $m \geq 2$.

2. THE MAIN RESULT

The theorem of Lakatos can be improved as follows.

Theorem 2.1. *All zeros of the reciprocal polynomial*

$$P_{2n+1}(z) = \sum_{k=0}^{2n+1} A_k z^k \quad (z \in \mathbb{C}) \tag{2.1}$$

of odd degree $2n+1 \geq 3$ *with real coefficients i.e.*

$$A_{2n+1} \neq 0, \quad A_k \in \mathbb{R}, \text{ and } A_k = A_{2n+1-k} \ (k = 0, \dots, n) \tag{2.2}$$

are on the unit circle, provided that

$$|A_{2n+1}| \geq \cos^2 \frac{\pi}{2(2n+2)} \sum_{k=1}^{2n} |A_k - A_{2n+1}|. \tag{2.3}$$

Moreover, if (2.2), (2.3) *hold then all zeros* e^{iu_j} $(j = 1, 2, \dots, 2n+1)$ *of* P_{2n+1} *are single,* $-1 = e^{iu_{n+1}}$ *is always a zero and the zeros can be arranged such that*

$$\left| \epsilon_j - e^{iu_j} \right| < \frac{\pi}{2n+2} \quad (j = 1, \dots, 2n+1), \tag{2.4}$$

where

$$\epsilon_j = e^{i \frac{j}{2n+2} 2\pi} \quad (j = 1, 2, \dots, 2n+1)$$

are the $2n+2$ *nd roots of unity except* 1.

Proof. With the notation

$$l := A_{2n+1} = A_0, a_1 := A_{2n} - l = A_1 - l, \dots, a_n := A_{n+1} - l = A_n - l$$

we have

$$\begin{aligned} h_{2n+1}(z) :&= l(z^{2n+1} + z^{2n} + \dots + z + 1) + \sum_{k=1}^{n} a_k \left(z^{2n+1-k} + z^k \right) \\ &= P_{2n+1}(z) = \sum_{k=0}^{2n+1} A_k z^k \quad (z \in \mathbb{C}). \end{aligned}$$

(2.2) goes over into

$$l \neq 0, l, a_k \in R \ (k = 1, \dots, n)$$

while (2.3) goes over into

$$|l| \geq 2 \cos^2 \frac{\pi}{2(2n+2)} \sum_{k=1}^{n} |a_k|. \tag{2.5}$$

We have $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with

$$\bar{h}_{2n}(z) = l \bar{v}_{2n}(z) + \sum_{k=1}^{n} a_k e_k(z) \bar{w}_{2n-2k}(z)$$

where

$$\bar{v}_{2n}(z) = z^{2n} + z^{2n-2} + \cdots + z^2 + 1$$
$$\bar{w}_{2n-2k}(z) = \frac{z^{2n+1-2k} + 1}{z+1}, \quad e_k(z) = z^k.$$

The Chebyshev transform $\mathcal{T}\bar{h}_{2n}$ of $\bar{h}_{2n}$ was calculated in [1]:

$$\begin{aligned}\mathcal{T}\bar{h}_{2n}(x) &= l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^{n} a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) \\ &= l\, U_n\left(\frac{x}{2}\right) + \sum_{k=1}^{n} a_k \left[U_{n-k}\left(\frac{x}{2}\right) - U_{n-k-1}\left(\frac{x}{2}\right)\right],\end{aligned}$$

where U_n is the Chebyshev polynomial of degree n of the second kind defined by $U_n(\cos x) := \frac{\sin(n+1)x}{\sin x}$ $(n = 1, 2, \dots)$ and $U_{-1}(x) := 0$. Evaluating of $\mathcal{T}\bar{h}_{2n}$ at the points

$$x_j = 2\cos y_j \quad \text{with} \quad y_j = \frac{j + \frac{1}{2}}{2n+2} 2\pi \quad (j = 0, \dots, n)$$

of the open interval $]-2, 2[$ gives that (see [1])

$$\mathcal{T}\bar{h}_{2n}(x_j) = \frac{l(-1)^j}{\sin y_j} + \sum_{k=1}^{n} a_k \frac{\cos \frac{2n-2k+1}{2} y_j}{\cos \frac{y_j}{2}} = \frac{l(-1)^j + 2\sum_{k=1}^{n} a_k \sin \frac{y_j}{2} \cos \frac{2n-2k+1}{2} y_j}{\sin y_j}.$$

We have for $j = 0, \dots, n-1$

$$(0 <) \sin \frac{y_j}{2} < \sin \frac{y_n}{2} = \sin\left(\frac{\pi}{2} - \frac{\pi}{2(2n+2)}\right) = \cos \frac{\pi}{2(2n+2)}, \tag{2.6}$$

while for $j = n$ there is equality here. The absolute value of the factor

$$\cos \frac{2n-2k+1}{2} y_j = \cos \frac{2(n-k)j + j + (n-k) + \frac{1}{2}}{2n+2}\pi \quad (k = 1, \dots, n; j = 0, \dots, n)$$

takes its maximum if the fraction

$$\frac{2(n-k)j + j + (n-k) + \frac{1}{2}}{2n+2}$$

is nearest to an integer. Clearly the nearest possible value of this fraction to an integer is $\frac{1}{2(2n+2)}$ (this value is attained at $j = 0, k = n$). Thus we have shown that

$$\left|\cos \frac{2n-2k+1}{2} y_j\right| \le \cos \frac{\pi}{2(2n+2)} \quad (k = 1, \dots, n; j = 0, \dots, n). \tag{2.7}$$

Let for $j = 0, \dots, n$

$$S_j := 2\left|\sum_{k=1}^{n} a_k \sin \frac{y_j}{2} \cos \frac{2n-2k+1}{2} y_j\right|.$$

Then, for $j = 0, \dots, n-1$ by (2.6), (2.7) we have

$$S_j \le 2\sin\frac{y_j}{2} \sum_{k=1}^{n} |a_k| \cos \frac{\pi}{2(2n+2)} < 2\cos^2 \frac{\pi}{2(2n+2)} \sum_{k=1}^{n} |a_k|$$

unless $\sum_{k=1}^{n} |a_k| = 0$.

Thus, by (2.3) or (2.5) we have

$$S_j < 2\cos^2\frac{\pi}{2(2n+2)}\sum_{k=1}^{n}|a_k| \le |l|$$

and the resulting inequality

$$S_j < |l|$$

remains valid even if $\sum_{k=1}^{n}|a_k| = 0$.

For $j = n$ we have

$$\begin{aligned}
S_n &\le 2\sin\frac{y_n}{2}\sum_{k=1}^{n}|a_k|\left|\cos\frac{2(n-k)n+n+(n-k)+\frac{1}{2}}{2n+2}\pi\right| \\
&= 2\cos\frac{\pi}{2(2n+2)}\sum_{k=1}^{n}|a_k|\left|\cos\left(n-k+\frac{k+\frac{1}{2}}{2n+2}\right)\pi\right| \\
&= 2\cos\frac{\pi}{2(2n+2)}\sum_{k=1}^{n}|a_k|\cos\frac{k+\frac{1}{2}}{2n+2}\pi \\
&< 2\cos^2\frac{\pi}{2(2n+2)}\sum_{k=1}^{n}|a_k|
\end{aligned}$$

unless $\sum_{k=1}^{n}|a_k| = 0$. Thus, by (2.3) or (2.5) we have

$$S_n < 2\cos^2\frac{\pi}{2(2n+2)}\sum_{k=1}^{n}|a_k| \le |l|.$$

and the inequality

$$S_n < |l|$$

remains valid even if $\sum_{k=1}^{n}|a_k| = 0$.

Looking again at the Chebyshev transform we can see that by the inequalities $S_j < |l|$ $(j = 0,\dots,n)$ we have

$$\operatorname{sgn}\left(\mathcal{T}\bar{h}_{2n}(x_j)\right) = \operatorname{sgn} l\ \operatorname{sgn}(-1)^j\ (j = 0,1,\dots,n)$$

therefore $\mathcal{T}\bar{h}_{2n}$ has n different zeros in $]-2,2[$. Writing these zeros in the form $2\cos v_j$ with $0 \le v_1 \le v_2 \le \cdots \le v_n \le \pi$ and applying Lemma 1 of [1] we conclude that all zeros of $\bar{h}_{2n}$ are single, and of the form $e^{\pm i v_j}$, where

$$y_{j-1} < v_j < y_j\ (j = 1,\dots,n). \tag{2.8}$$

Let $u_j := v_j$ for $j = 1,\dots,n$, $u_{n+1} := \pi$ and $u_{n+1+j} := 2\pi - u_{n+1-j}$ $(j = 1,\dots,n)$, then we obtain that all zeros of $P_{2n+1} = h_{2n+1}$ are e^{iu_j} $(j = 1,\dots,2n+1)$ and by (2.8) the condition (2.4) holds. □

3. REMARKS ON SCHINZEL'S THEOREM

Schinzel's result can be generalized as follows.

Theorem 3.1. *Let $P_m(z) = \sum_{k=0}^{m} A_k z^k$ be a self–inversive polynomial of degree m, i.e. let $A_k \in \mathbb{C}$, $A_m \ne 0$ and $\epsilon\bar{A}_k = A_{m-k}$ for all $k = 0,\dots,m$ with a fixed $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$. If*

$$|A_m| \ge \frac{m}{2\mu_m}\inf_{c,d\in\mathbb{C},\,|d|=1}\sum_{k=0}^{m}|cA_k - d^{m-k}A_m|, \tag{3.1}$$

where

$$\mu_m := \min_{|z|\le 1}\left|\sum_{k=1}^{m} k z^{m-k}\right| \tag{3.2}$$

then all zeros of P_m are on the unit circle. If the inequality is strict the zeros are single.

Apart from minor changes Schinzel's proof [3] is valid for this more general result hence we omit the proof.

By Cohn's theorem [4] any self–inversive polynomial P_m and the polynomial

$$z^{m-1}P_m'(z^{-1}) = \sum_{k=1}^{m} kA_k z^{m-k} \quad (z \in \mathbb{C})$$

have the same number of zeros inside the unit circle. Applying this for the polynomial $\sum_{k=0}^{m} z^k$ we obtain that $\sum_{k=1}^{m} kz^{m-k}$ has no zeros inside the unit circle, thus the modulus of the latter takes its positive minimum in the unit disk on the unit circle:

$$\mu_m = \min_{|z|=1}\left|\sum_{k=1}^{m} kz^{m-k}\right| > 0.$$

By some known identities (see [2, Part 6, Problems 16, 18]) for trigonometric sums we easily get that

$$\begin{aligned} D_m(t) :&= \left|\sum_{k=1}^{m} kz^{m-k}\right|_{z=e^{it}} \\ &= \sqrt{\left[\sum_{k=1}^{m} k\cos(m-k)t\right]^2 + \left[\sum_{k=1}^{m} k\sin(m-k)t\right]^2} \\ &= \sqrt{\left[\frac{m}{2} + \frac{1}{2}\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^2\right]^2 + \left[\frac{m\sin t - \sin mt}{4\sin^2\frac{t}{2}}\right]^2}. \end{aligned}$$

From this it follows that $\mu_m = \min\limits_{t\in[0,2\pi]} D_m(t) \ge \frac{m}{2}$ and for even $m = 2n$ we have equality here, since for $t = \pi$

$$D_m(\pi) = \frac{m}{2}.$$

This means that for even m Theorem 3.1 coincides with Schinzel's result.
For odd m however

$$\mu_m > \frac{m}{2}.$$

Let for $t \in [0, 2\pi]$

$$\begin{aligned} x_m(t) &:= \frac{m}{2} + \frac{1}{2}\left(\frac{\sin\frac{mt}{2}}{\sin\frac{t}{2}}\right)^2, \\ y_m(t) &:= \frac{m\sin t - \sin mt}{4\sin^2\frac{t}{2}}, \\ z_m(t) &:= x_m(t) + iy_m(t), \end{aligned}$$

then $D_m^2(t) = |z_m(t)|^2 = x_m(t)^2 + y_m(t)^2$. As $D_m(\pi+t) = D_m(\pi-t)$ it is enough to consider D_m on the interval $[0, \pi]$.

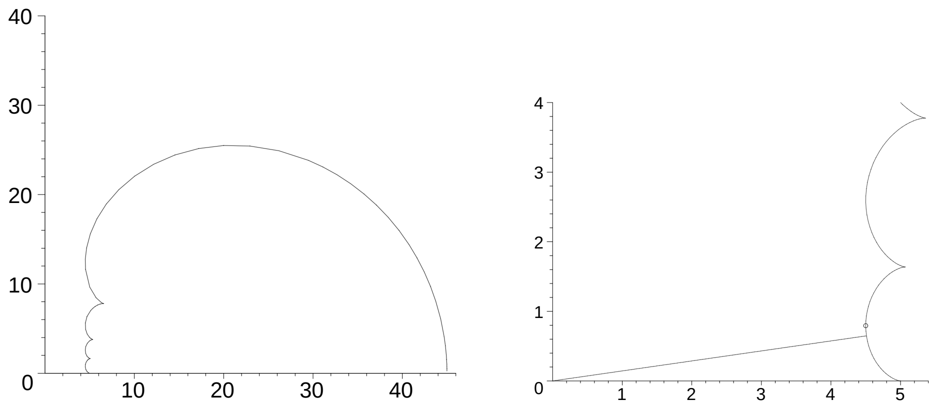

Figure 3.1: Graph of z_9.

The next figure shows the graph of z_9 in the complex plane and an enlargement of the part which is nearest to the origin. In the latter the distance of the origin from the graph of z_9 is also shown. The point of the graph with $t = t_m = \frac{m-1}{m}\pi$ is distinguished by a small circle.

Our numerical experiments give base to the following conjecture.

Conjecture 3.2. *For odd m we have*

$$\mu_m = \min_{t\in[0,2\pi]} D_m(t) \geq \frac{m}{2} \sec \frac{\pi}{2m+2}.$$

A simple calculation shows that for odd $m = 2n+1$

$$D_m(t_m) = \frac{m}{2} \sec \frac{\pi}{2m}.$$

It is clear that μ_m is the *distance of the graph of z_m from the origin.* The minimum of D_m is attained near to t_m. It is relatively easy to show that the minimum is attained in the interval $[t_m, \pi]$. Numerical calculations seem to justify that the minimum point is in the smaller interval

$$\left[t_m, t_m + \frac{2\pi}{m^2}\right].$$

Theorem 3.1 and Conjecture 3.2 give

Conjecture 3.3. *All zeros of the self–inversive polynomial*

$$P_{2n+1}(z) = \sum_{k=0}^{2n+1} A_k z^k \quad (z \in \mathbb{C})$$

of odd degree $2n+1$, i.e.

$$A_{2n+1} \neq 0, \quad A_k \in \mathbb{C}, \text{ and } \epsilon\bar{A}_k = A_{2n+1-k} \ (k = 0, \ldots, 2n+1) \text{ with } \epsilon \in \mathbb{C}, |\epsilon| = 1$$

are on the unit circle, provided that

$$|A_{2n+1}| \geq \cos \frac{\pi}{2(2n+2)} \inf_{c,d\in\mathbb{C},\, |d|=1} \sum_{k=0}^{2n+1} |cA_k - d^{2n+1-k} A_{2n+1}|.$$

holds. If the inequality is strict here then the zeros are single.

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