

The complexity of recognizing minimally tough graphs

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Abstract: Let t be a positive real number. A graph is called t -tough, if the removal of any cutset S leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t -tough. A graph is minimally t -tough, if the toughness of the graph is t and the deletion of any edge from the graph decreases the toughness. The complexity class DP is the set of all languages that can be expressed as the intersection of a language in NP and a language in coNP. We prove that recognizing minimally t -tough graphs is DP-complete for any positive integer t and for any positive rational number $t \leq 1/2$.

Keywords: 3–6 keywords toughness, complexity, DP-complete

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components and $\alpha(G)$ denote the independence number. For a graph G and a vertex set $V \subseteq V(G)$, let $G[V]$ denote the subgraph of G induced by V .

The complexity class DP was introduced by C. H. Papadimitriou and M. Yannakakis [4].

Definition 1 *A language L is in the class DP if there exist two languages $L_1 \in NP$ and $L_2 \in coNP$ such that $L = L_1 \cap L_2$.*

We mention that $DP \neq NP \cap coNP$, if $NP \neq coNP$. Moreover, $NP \cup coNP \subseteq DP$. A language is called DP-hard if all problems in DP can be reduced to it in polynomial time. A language is DP-complete if it is in DP and it is DP-hard.

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A critical-type DP-complete problem is CRITICALCLIQUE [5], in our proofs we use an equivalent form of it, α -CRITICAL.

CriticalClique

Instance: a graph G and a positive integer k .

Question: is it true that G has no clique of size k , but adding any missing edge e to G , the resulting graph $G + e$ has a clique of size k ?

By taking the complement of the graph, we can obtain α -CRITICAL from CRITICALCLIQUE.

Definition 2 A graph G is called α -critical, if $\alpha(G - e) > \alpha(G)$ for all $e \in E(G)$.

α -Critical

Instance: a graph G and a positive integer k .

Question: is it true that $\alpha(G) < k$, but $\alpha(G - e) \geq k$ for any edge $e \in E(G)$?

Since a graph is clique-critical if and only if its complement is α -critical, α -CRITICAL is also DP-complete.

Corollary 3 α -CRITICAL is DP-complete.

The notion of toughness was introduced by Chvátal [2].

Definition 4 Let t be a positive real number. A graph G is called t -tough, if

$$\omega(G - S) \leq \frac{|S|}{t}$$

for any cutset S of G (i.e. for any S with $\omega(G - S) > 1$). The toughness of G , denoted by $\tau(G)$, is the largest t for which G is t -tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

We say that a cutset $S \subseteq V(G)$ is a tough set if $\omega(G - S) = |S|/\tau(G)$.

For all positive rational number t we can define a separate problem:

t -Tough

Instance: a graph G ,

Question: is it true that $\tau(G) \geq t$?

Bauer et al. proved the following.

Theorem 5 ([1]) For any positive rational number t , t -TOUGH is coNP-complete.

The critical form of this problem is minimally toughness.

Definition 6 A graph G is minimally t -tough, if $\tau(G) = t$ and $\tau(G - e) < t$ for all $e \in E(G)$.

Given t we define:

Min- t -Tough

Instance: a graph G ,

Question: is it true that G is minimally t -tough?

Our main result is the following.

Theorem 7 MIN- t -TOUGH is DP-complete for any positive integer t and for any positive rational number $t \leq 1/2$.

First we prove this theorem for $t = 1$, then we generalize that proof for positive integers, and finally we prove it for any positive rational number $t \leq 1/2$.

2 Preliminaries

In this section we prove some useful lemmas.

Proposition 8 *Let G be a connected noncomplete graph on n vertices. Then $\tau(G) \in \mathbb{Q}^+$, and if $\tau(G) = a/b$, where a, b are positive integers and $(a, b) = 1$, then $1 \leq a, b \leq n - 1$.*

PROOF: By definition,

$$\tau(G) = \min_{\substack{S \subseteq V(G) \\ \text{cutset}}} \frac{|S|}{\omega(G - S)}$$

for a noncomplete graph G . Since G is connected and noncomplete, $1 \leq |S| \leq n - 2$ and since S is a cutset, $2 \leq \omega(G - S) \leq n - 1$. \square

Corollary 9 *Let G and H be two connected noncomplete graphs on n vertices. If $\tau(G) \neq \tau(H)$, then*

$$|\tau(G) - \tau(H)| > \frac{1}{n^2}.$$

Claim 10 *For every positive rational number t , MIN- t -TOUGH \in DP.*

PROOF: For any positive rational number t ,

$$\begin{aligned} \text{MIN-}t\text{-TOUGH} &= \{G \text{ graph} \mid \tau(G) = t \text{ and } \tau(G - e) < t \text{ for all } e \in E(G)\} = \\ &= \{G \text{ graph} \mid \tau(G) \geq t\} \cap \{G \text{ graph} \mid \tau(G) \leq t\} \cap \\ &\quad \cap \{G \text{ graph} \mid \tau(G - e) < t \text{ for all } e \in E(G)\}. \end{aligned}$$

Let

$$L_{1,1} = \{G \text{ graph} \mid \tau(G - e) < t \text{ for all } e \in E(G)\},$$

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) \leq t\}$$

and

$$L_2 = \{G \text{ graph} \mid \tau(G) \geq t\}.$$

$L_2 \in \text{coNP}$, a witness is a cutset $S \subseteq V(G)$ whose removal leaves more than $|S|/t$ components. $L_{1,1} \in \text{NP}$, the witness is a set of cutsets: $S_e \subseteq V(G)$ for each edge e whose removal leaves more than $|S_e|/t$ components.

Now we show that $L_{1,2} \in \text{NP}$, i.e. we can express $L_{1,2}$ in a form of

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) < t + \varepsilon\},$$

which belongs to NP. Let a, b be positive integers such that $t = a/b$ and $(a, b) = 1$, and let G be an arbitrary graph on n vertices. If G is disconnected, then $\tau(G) = 0$, and if G is complete, then $\tau(G) = \infty$, so in both cases G is not minimally t -tough. By Proposition 8, if $1 \leq a, b \leq n - 1$ does not hold, then G is also not minimally t -tough. So we can assume that $t = a/b$, where a, b are positive integers, $(a, b) = 1$ and $1 \leq a, b \leq n - 1$. With this assumption

$$L_{1,2} = \{G \text{ graph} \mid \tau(G) \leq t\} = \left\{ G \text{ graph} \mid \tau(G) < t + \frac{1}{|V(G)|^2} \right\},$$

so $L_{1,2} \in \text{NP}$.

Since $L_{1,1} \cap L_{1,2} \in \text{NP}$, $L_2 \in \text{coNP}$ and $\text{MIN-}t\text{-TOUGH} = (L_{1,1} \cap L_{1,2}) \cap L_2$, we can conclude that $\text{MIN-}t\text{-TOUGH} \in \text{DP}$. \square

Claim 11 Let t be a positive rational number and G a minimally t -tough graph. For every edge e of G ,

1. the edge e is a bridge in G , or
2. there exists a vertex set $S = S(e) \subseteq V(G)$ with

$$\omega(G - S) \leq \frac{|S|}{t} \quad \text{and} \quad \omega((G - e) - S) > \frac{|S|}{t},$$

and the edge e is a bridge in $G - S$.

In the first case, we define $S = S(e) = \emptyset$.

PROOF: Let e be an arbitrary edge of G , which is not a bridge. Since G is minimally t -tough, $\tau(G - e) < t$. So there exists a cutset $S = S(e) \subseteq V(G - e) = V(G)$ in $G - e$ satisfying $\omega((G - e) - S) > |S|/t$. On the other hand, $\tau(G) = t$, so $\omega(G - S) \leq |S|/t$. This is only possible if e connects two components of $(G - e) - S$. \square

Finally we cite a Lemma that our proof relies on.

Lemma 12 (Problem 14 of 8 in [3]) *If we replace a vertex of an α -critical graph with a clique, and connect every neighbor of the original vertex with every vertex in the clique, then the resulting graph is still α -critical.*

3 Recognizing minimally 1-tough graphs

To show that MIN-1-TOUGH is DP-hard, we reduce α -CRITICAL to it.

Theorem 13 MIN-1-TOUGH is DP-complete.

PROOF: In Claim 10 we have already proved that MIN-1-TOUGH \in DP.

Let G be an arbitrary connected graph on the vertices v_1, \dots, v_n . Let G_α be defined as follows. It will be easy to see that it can be constructed from G in polynomial time. For all $i \in [n]$, let

$$V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,\alpha}\}$$

and place a clique on the vertices of V_i . For all $i, j \in [n]$, if $v_i v_j \in E(G)$, then place a complete bipartite graph on $(V_i; V_j)$. For all $i \in [n]$ and for all $j \in [\alpha]$ add the vertex $u_{i,j}$ to the graph and connect it to $v_{i,j}$. Let

$$V = \bigcup_{i=1}^n V_i$$

and

$$U = \{u_{i,j} \mid i \in [n], j \in [\alpha]\}.$$

Add the vertex set

$$W = \{w_1, \dots, w_\alpha\}$$

to the graph and for all $j \in [\alpha]$ connect w_j to $v_{1,j}, \dots, v_{n,j}$.

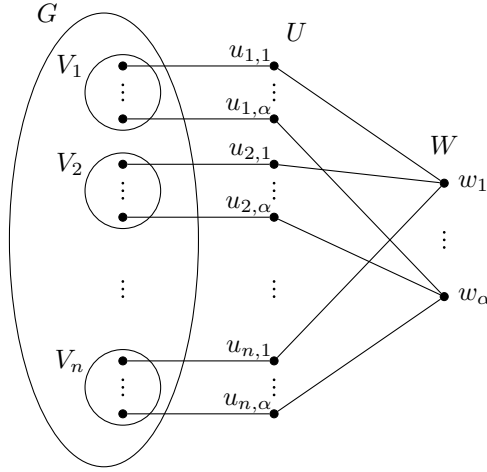


Figure 1: The graph G_α .

We need to prove that G is α -critical with $\alpha(G) = \alpha$ if and only if G_α is minimally 1-tough. First we prove the following lemma.

Lemma 14 *Let G be a graph with $\alpha(G) \leq \alpha$. Then G_α is 1-tough.*

PROOF: Let $S \subseteq V(G_\alpha)$ be a cutset. We show that $\omega(G_\alpha - S) \leq |S|$.

Case 1: $W \subseteq S$. If a vertex of U has only one neighbor in $V(G_\alpha) \setminus S$, then we can assume that this vertex is not in S . Then there are two types of components in $G_\alpha - S$: isolated vertices from U and components containing at least one vertex from V . There are at most $\alpha(G)$ components of the second type and (exactly) $|V \cap S| = |S| - \alpha$ components of the first type. Thus $\omega(G_\alpha - S) \leq |S| - \alpha + \alpha(G) \leq |S|$.

Case 2: $W \not\subseteq S$. First, we make two convenient assumptions for S .

(1) $U \cap S = \emptyset$.

It is easy to see that if $u_{i,j} \in S$, then we can assume that $v_{i,j} \notin S$. Now there are two cases.

Case 2.1: $v_{i,j}$ is not isolated in $G_\alpha - S$. Then we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{v_{i,j}\}$ instead of S .

Case 2.2: $v_{i,j}$ is isolated in $G_\alpha - S$. Since there are no isolated vertices in G , there exists $k \in [n]$ such that $v_i v_k \in E(G)$. Then $v_{k,j} \in S$, so $u_{k,j} \notin S$, which means that w_j is not isolated in $G_\alpha - S$, so we can consider $S' = (S \setminus \{u_{i,j}\}) \cup \{w_j\}$ instead of S .

(2) For all $i \in [n]$, either $V_i \subseteq S$ or $V_i \cap S = \emptyset$.

After the assumption (1), assume that only a proper subset of V_i is contained in S . Let v be an element of this subset. We can consider the cutset $S \setminus \{v\}$ instead of S , since this decreases the number of components by at most one. So we can repeat this procedure until $V_i \cap S = \emptyset$.

So in $G_\alpha - S$ there are isolated vertices from U and one more component containing the remaining vertices of W and V . So there are less than $|V \cap S|$ isolated vertices, thus

$$\omega(G_\alpha - S) \leq |V \cap S| \leq |S|.$$

So G_α is 1-tough. \square

We show that G is α -critical with $\alpha(G) = \alpha$ if and only if G_α is minimally 1-tough.

Let us assume that G is α -critical with $\alpha(G) = \alpha$. So by Lemma 14 G_α is 1-tough. Let $e \in E(G_\alpha)$ be an arbitrary edge. If e has an endpoint in U , then this endpoint has degree 2, so $\tau(G_\alpha - e) < 1$. If e does not have an endpoint in U , then it connects two vertices of V . By Lemma 12 $G_\alpha[V]$ is α -critical, so in $G_\alpha[V] - e$ there exists an independent vertex set I of size $\alpha(G) + 1$. Let $S = (V \setminus I) \cup W$. Then $|S| = (|V| - \alpha(G) - 1) + \alpha = |V| - 1$ and $\omega((G_\alpha - e) - S) = |V|$, so $\tau(G_\alpha - e) < 1$.

Let us assume that G is not α -critical with $\alpha(G) = \alpha$.

Case 1: $\alpha(G) > \alpha$. Let I be an independent vertex set of size $\alpha(G)$ in $G_\alpha[V]$ and let $S = (V \setminus I) \cup W$. Then $|S| = (|V| - \alpha(G)) + \alpha < |V|$ and $\omega(G_\alpha - S) = |V|$, so $\tau(G_\alpha) < 1$, which means that G_α is not minimally 1-tough.

Case 2: $\alpha(G) \leq \alpha$. Since G is not α -critical there exists an edge $e \in E(G)$ such that $\alpha(G - e) \leq \alpha$. By Lemma 14 $(G - e)_\alpha$ is 1-tough, but we can obtain $(G - e)_\alpha$ from G_α by edge-deletion, which means that G_α is not minimally 1-tough. \square

4 Further results

Theorem 15 *For every positive integer t , MIN- t -TOUGH is DP-complete.*

To prove this more general theorem, first we generalize the construction on Figure 1. We follow a similar argument to show that this construction has the required properties. However, due to the more complicated construction, the proof is harder.

The case when $t \leq 1/2$ is also covered in the paper.

Theorem 16 *For every positive rational number $t = a/b \leq 1/2$, MIN- t -TOUGH is DP-complete.*

It is shown that MIN-1-TOUGH can be reduced to this problem. The construction and the proof uses different ideas than the previous proofs.

We were not able to prove the DP-completeness for the remaining t values, but we make the following conjecture.

Conjecture 17 *MIN- t -TOUGH is DP-complete for any positive rational number t .*

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