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One Possible Analytical Approximation of the Critical Point of the Three-Dimensional Ising Model

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In 1925, Ising developed a model for systems containing two-state particles /1/ which was solved by Onsager in 1944 for two-dimensional systems /2/. No general three-dimensional solution is available so far, although such a solution could give a basis for solving several complex problems of statistical mechanics /3/.

The present note suggests a simple analytical function for calculation of the critical point, based on symmetry principles which fits well numerical extrapolations. The function gives values close to computer approximations for the isotropic and also anisotropic three-dimensional case. The maximum deviation is 1.3% for the isotropic and 4.2% for anisotropic case. Our analytical formula reproduces the two-dimensional solution of Onsager fully, in the case of degeneration towards any of the axes.

The two-dimensional dual transformation described by Kramers and Wannier /4/ and used also by other authors /5/ is as follows:

$$\sinh 2\Phi_x \sinh 2\Theta = 1$$

or, in an equivalent way:

$$\sinh 2\Phi_y \sinh 2\Theta' = 1 \quad ,$$

where $\Phi_i = J_i/kT$; J_i ($i = x, y, z$) is the bond strength of nearest neighbours in the direction i, k is the Boltzmann constant, and T is the temperature and Θ as well as Θ' new variables corresponding to the dual transformation /3/ and defined by the above written conditions /4/. Thus, Onsager's solution containing also the dual transformation will be

$$\cosh 2\Phi_x \cosh 2\Theta - \sinh 2\Phi_x \sinh 2\Theta \cos\mu = 1$$

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for the determination of the largest eigenvalue, resulting in the critical point in the case of $\mu = 0$:

$$\cosh 2\Phi_x \cosh 2\Theta - \sinh 2\Phi_x \sinh 2\Theta = 1 \ . \qquad (1)$$

In the isotropic case ($\Phi_x = \Phi_y = \Phi_c$; $\Phi_z = 0$), this will lead us to the conditions

$$\cosh 2\Phi_c^{(2d)} = \sqrt{2}\,; \ \sinh 2\Phi_c^{(2d)} = 1\,;$$

that is

$$\Phi_c^{(2d)} = 0.4406868 \ .$$

The dual transformation should be incorporated into the three-dimensional solution, for each pair of two-dimensional planes. Thus, e.g. if the z-axis is selected as the base line, the duals of planes zx and zy will belong to it, and the three-dimensional equation analogue to (1) will be

$$\cosh 2\Phi_x \cosh 2\Phi_y \cosh 2\Theta - \sinh 2\Phi_x \sinh 2\Theta - \sinh 2\Phi_y \sinh 2\Theta = 1 \ .$$

Of course, this cannot be regarded as the total solution, since the above equation represents a model desintegrated into two-dimensional planes; planes zx and zy should be connected by planes zy. If we have selected the z-axis as the basis, the solution of the planes zy in this system can be given by a function of appropriate symmetry as follows:

$$\cosh 2(\Phi_x + \Phi_y)\cosh 2\Theta - \sinh 2(\Phi_x + \Phi_y)\sinh 2\Theta \ .$$

Consequently, the function for symmetric solution on the basis of the z-axis is as follows:

$$\cosh 2\Phi_x \cosh 2\Phi_y \cosh 2\Theta - \sinh 2\Phi_x \sinh 2\Theta - \sinh 2\Phi_y \sinh 2\Theta +$$
$$+ \sinh 2\Phi_x \sinh 2\Phi_y \sinh 2\Phi_z \left[\cosh 2(\Phi_x + \Phi_y)\cosh 2\Theta - \cosh 2(\Phi_x + \Phi_y)\sinh 2\Theta\right] =$$
$$= 1 \ ,$$

or, transformed to values of $\Phi_x; \Phi_y; \Phi_z$

$$\cosh 2\Phi_x \cosh 2\Phi_y \cosh 2\Phi_z - (\sinh 2\Phi_x + \sinh 2\Phi_y + \sinh 2\Phi_z) +$$
$$+ \sinh 2\Phi_x \sinh 2\Phi_y \sinh 2\Phi_z \left[\cosh 2(\Phi_x + \Phi_y)\cosh 2\Phi_z - \sinh 2(\Phi_x + \Phi_y)\right] = 0 \ .$$

Since any of the axes can serve as basis, i.e. they are symmetric with respect to each other /6/, it follows that

Table 1

	No.	anisotropy J_z/J_x	anisotropy J_z/J_y	anisotropy J_y/J_x	Φ_c numerical approx. /7/	Φ_c present work	deviation (%)
3-dim.	1	1	1	1	0.2286	0.2257	1.3
quasi 2-dim.	2	0.1	0.1	1	0.3716	0.3565	4.1
	3	0.02	0.02	1	0.4170	0.4026	3.5
	4	0.005	0.005	1	0.4323	0.4275	1.1
	5	0	0	1	0.4407	0.4407	0
quasi 1-dim.	6	0.1	1	0.1	0.7728	0.7402	4.2
	7	0.02	1	0.02	1.3210	1.2642	4.3
	8	0.005	1	0.005	1.8416	1.7857	3.0

$$\cosh 2\Phi_x \cosh 2\Phi_y \cosh 2\Phi_z - (\sinh 2\Phi_x + \sinh 2\Phi_y + \sinh 2\Phi_z) + \frac{1}{3}\sinh 2\Phi_x \sinh 2\Phi_y \sinh 2\Phi_z \Big[\cosh 2(\Phi_x + \Phi_y)\cosh 2\Phi_z + \cosh 2(\Phi_x + \Phi_z)\cosh 2\Phi_y + \cosh 2(\Phi_y + \Phi_z)\cosh 2\Phi_x - \sinh 2(\Phi_x + \Phi_y) - \sinh 2(\Phi_x + \Phi_z) - \sinh 2(\Phi_y + \Phi_z)\Big] = 0 \ . \quad (2)$$

The numerical investigation of (2) is very promising. The value of the critical point in the isotropic three-dimensional case is

$$\Phi_c^{(3d)} = 0.22565 .$$

Its deviation from the values obtained by numerical extrapolations is as low as 1.3% /7/. For two-dimensional cases the result is equal to those obtained by the Onsager method. These and a few other typical cases have been summarized in Table 1.

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