Percolation on the stationary distributions of the voter model

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Abstract

The voter model on $\mathbb{Z}^d$ is a particle system that serves as a rough model for changes of opinions among social agents or, alternatively, competition between biological species occupying space. When $d \geq 3$, the set of (extremal) stationary distributions is a family of measures μ_α, for α between 0 and 1. A configuration sampled from μ_α is a strongly correlated field of 0's and 1's on $\mathbb{Z}^d$ in which the density of 1's is α. We consider such a configuration as a site percolation model on $\mathbb{Z}^d$. We prove that if $d \geq 5$, the probability of existence of an infinite percolation cluster of 1's exhibits a phase transition in α. If the voter model is allowed to have sufficiently spread-out interactions, we prove the same result for $d \geq 3$.

1   Introduction

1.1   Model and results

Given integers $d \geq 1$ and $R \geq 1$, the voter model with range R on the d-dimensional lattice $\mathbb{Z}^d$ is a Markov process, denoted here by $(\xi_t)_{t \geq 0}$, with configuration space $\{0,1\}^{\mathbb{Z}^d}$ and stochastic dynamics described informally as follows. Each vertex (or site) x of $\mathbb{Z}^d$ updates its current state $\xi_t(x) \in \{0,1\}$ at rate one by copying the state $\xi_t(y)$ of a vertex y that is chosen uniformly among all vertices at (ℓ^1-norm) distance at most R from x.

In Section 3 we give the formal definition of the model and recall some of its relevant properties. In this Introduction, we will only very briefly present the concepts that are needed to state our main results.

The voter model was introduced independently by Clifford and Sudbury in [CS73] and Holley and Liggett in [HL75]. In the interpretation of the latter pair of authors, each site of $\mathbb{Z}^d$ represents a voter which can have one of two possible opinions (corresponding to the states 0 and 1). The model thus represents the evolution of the opinions among the population. Clifford and Sudbury gave a biological interpretation for the model: there are two competing species, denoted 0 and 1, and each site is a region of space that can be occupied by an individual of one of the two species.

The set of stationary distributions of the voter model on $\mathbb{Z}^d$ has been thoroughly studied; the following is a summary of known results. For fixed $d \geq 3$, $R \geq 1$ and $\alpha \in [0,1]$, one defines a probability measure μ_α on $\{0,1\}^{\mathbb{Z}^d}$ as the distributional limit (which is shown to exist), as time is taken to infinity, of the voter model with the random initial configuration in which the states of all sites are independent and Bernoulli(α). μ_α is then stationary for the voter model dynamics. Moreover, it is shown that the set of stationary distributions for the voter model dynamics that are *extremal* – i.e., that

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cannot be expressed as non-trivial convex combinations of other stationary distributions – is precisely the family

$$\{\, \mu_\alpha : \alpha \in [0,1]\,\}.$$

We note that this property of the voter model is rather delicate and small perturbations of the dynamics can result in an interacting particle system which has only one non-trivial stationary distribution, see [CP14].

The measures μ_α can be obtained in a more constructive way with the aid of coalescing random walks. A realization of a system of coalescing random walks with range R on $\mathbb{Z}^d$ induces a partition of $\mathbb{Z}^d$ into *coalescence classes*: we say that x and y are in the same class if the walkers started at x and y are eventually joined. We then assign 0's or 1's to the coalescence classes independently with probabilities $1-\alpha$ and α, respectively, and the resulting configuration $\xi \in \{0,1\}^{\mathbb{Z}^d}$ has law μ_α. (Again, the sentences in this paragraph will be given a precise meaning in Section 3).

With the aid of this construction, it is not difficult to show that each μ_α is invariant and ergodic with respect to translations of $\mathbb{Z}^d$ (see [Li85, Theorem 2.5 of Chapter V, Corollary 4.14 of Chapter I]) and satisfies $\mu_\alpha[\,\xi(0) = 1\,] = \alpha$, so that α is equal to the density of 1's. Moreover, the family $\{\mu_\alpha\}$ is stochastically increasing: in the partial order on $\{0,1\}^{\mathbb{Z}^d}$ induced by the order $0 < 1$ on the coordinates, we have that μ_α is stochastically dominated by $\mu_{\alpha'}$ when $\alpha < \alpha'$.

The objective of this paper is to show that the measures μ_α exhibit a non-trivial percolation phase transition. Loosely speaking, we want to show that if α is close to zero then the set of 1's only contains finite connected components and if α is close to one then the set of 1's contains an infinite component. Let us explain this concept more precisely. We define the event Perc $\subseteq \{0,1\}^{\mathbb{Z}^d}$ which consists of those voter configurations ξ for which the subgraph of the nearest-neighbour lattice $\mathbb{Z}^d$ spanned by the set of occupied sites $\{x : \xi(x) = 1\}$ has an infinite connected component. By ergodicity, $\mu_\alpha(\text{Perc})$ is either 0 or 1. If Perc occurs, we say that the set $\{x : \xi(x) = 1\}$ percolates. We can then define α_c as the supremum of all the values of α for which $\mu_\alpha(\text{Perc}) = 0$. By the stochastic ordering mentioned in the previous paragraph, $\mu_\alpha(\text{Perc})$ is non-decreasing in α. Thus for any $\alpha < \alpha_c$ we have $\mu_\alpha(\text{Perc}) = 0$ and for any $\alpha > \alpha_c$ we have $\mu_\alpha(\text{Perc}) = 1$. Our aim is to show that the family of measures $\{\mu_\alpha : 0 \le \alpha \le 1\}$ exhibits a non-trivial percolation percolation phase transition, i.e., that $0 < \alpha_c < 1$. Our main results are

Theorem 1.1. *If $d \ge 5$ and $R \ge 1$, then the family of stationary distributions of the voter model exhibits a non-trivial percolation phase transition.*

Theorem 1.2. *If $d = 3$ or 4 then there exists $R_0 = R_0(d) \in \mathbb{N}$ such that if $R \ge R_0$ then the family of stationary distributions of the voter model exhibits a non-trivial percolation phase transition.*

1.2 Context

Although it may at first seem intuitively clear that, similarly to the case of Bernoulli percolation, ξ should be non-percolative if α is close to zero, this statement is not obvious. As the dynamics of the voter model favours that voters synchronize their opinions, the measures μ_α present long-range dependences. In fact, it follows from (3.7) below that for any $\alpha \in (0,1)$, the configuration ξ under the law μ_α has covariances given by

$$c(\alpha, d, R) \cdot |x - y|^{2-d} \le \text{Cov}_{\mu_\alpha}(\xi(x), \xi(y)) \le C(\alpha, d, R) \cdot |x - y|^{2-d}, \qquad x \ne y \in \mathbb{Z}^d. \quad (1.1)$$

It is *a priori* possible that percolation models with strong correlations present no phase transition. It is easy to build artificial examples, but let us recall an example that arises

"naturally". The random interlacement set $\mathcal{I}^u$ at level $u > 0$, introduced in [Sz10] is a random subset of $\mathbb{Z}^d$: (a) the law of $\mathcal{I}^u$ is stochastically dominated by the law of $\mathcal{I}^{u'}$ when $u < u'$, (b) the correlations of $\mathcal{I}^u$ decay like (1.1) (see [Sz10, (1.68)]) and (c) the density of $\mathcal{I}^u$ can be taken arbitrarily small by making u small (see [Sz10, (1.58)]), yet the set $\mathcal{I}^u$ is connected for any $u > 0$, (see [Sz10, (2.21)]).

On the other hand, in case one attempts to prove that phase transition *does* occur, then the slowly decaying correlations (1.1) pose a challenge, as many of the well-known tools that are used for Bernoulli percolation are not applicable. Additionally, since general criteria are lacking and (as mentioned above) phase transition may in principle fail to occur, one needs to envisage strategies of proof that are model-specific. The proof of non-degeneracy of the percolation threshold has been carried out for the vacant set $\mathcal{V}^u = \mathbb{Z}^d \backslash \mathcal{I}^u$ of random interlacements in [Sz10, S10] and the excursion sets of the Gaussian free field in [BLM87] (for $d = 3$) and [RS13] ($d \geq 3$). Both of these percolation models exhibit a decay of correlations described by (1.1).

In the case of the voter model, the question of percolation has been considered before, in [LS86], [BLM87], [LM06] and [Ma07]. The main focus of these works is on the case where $d = 3$ and $R = 1$. Through simulations and numerical studies, the first, third and fourth of these references argue that there should be a non-trivial phase transition and that the predictions of [HW83, W84] regarding the critical behaviour of percolation models with correlations described by (1.1) should be correct. However, the problem of finding a rigorous proof of the non-triviality of the percolation phase transition of the stationary state of the voter model remained open. This problem is (partially) settled by our Theorems 1.1 and 1.2.

Another investigation of geometric properties of the stationary distribution of the voter model has recently been carried out in [HMN15]. The object of interest there is the voter model on a finite rhombus of the triangular lattice; the boundary of the rhombus, composed of four segments, is frozen so that two adjacent segments are always in state 0 and the other two in state 1. In this finite setting, there is only one stationary distribution, which can be constructed with the aid of coalescing random walks and the resulting coalescence classes, similarly to the μ_α's on $\mathbb{Z}^d$. The authors study the volume of the coalescing classes and the interface curve that appears as a consequence of the opposing boundaries.

Questions regarding percolation of the stationary distributions of interacting particle systems other than the voter model have also been investigated. It is proved in [LS06] that the upper invariant measure $\overline{\nu}_\lambda$ of the contact process with infection rate λ on $\mathbb{Z}^d$, $d \geq 2$ percolates if $\lambda \geq 6.25$. To the best of our knowledge, Question 2 of Section 8 of [LS06] is still open, i.e., it is not known whether there exists $\lambda > \lambda_c$ for which $\overline{\nu}_\lambda$ is non-percolative. However, it is proved in [vdB11] that for $d = 2$ the percolation phase transition of $\overline{\nu}_\lambda$ is sharp. This result is extended to more complex versions of the contact process in [vdBBH15]. Let us note here that the stationary distribution μ_α of the voter model is rather different from the upper invariant measure $\overline{\nu}_\lambda$ of the contact process, e.g., $\mathrm{Cov}_{\overline{\nu}_\lambda}(\xi(x), \xi(y))$ decays exponentially as $|x - y| \to \infty$ for any value of λ (see, e.g., [vdB11, Lemma 2.2]), as opposed to the polynomial decay exhibited by μ_α in (1.1).

Let us also point out that the scaling limit of the voter model is super-Brownian motion (see [CDP00, BClG01]), and, despite the fact that continuum scaling limits do not explicitly appear in the calculations that we are about to present, our intuition was guided by the question of the disconnecedness of the support of super-Brownian motion, as we discuss in Remark 7.1.

1.3 Ideas and structure of proof

Let us now explain how the paper is organized and also the contents of each section.

In Section 2, we give a notation summary and also collect some facts regarding martingales and random walks that are needed in the rest of the exposition.

Section 3 contains an introduction to the voter model on $\mathbb{Z}^d$, including its graphical construction, duality properties and the construction of the extremal stationary distributions using a family of coalescing random walks.

We begin to prove our main results in Section 4. Our goal is to show (see (4.2)) that for sufficiently small values of α, the probability that a large annulus is crossed by a $*$-connected path of 1's in ξ is smaller than a stretched exponential function of the radius of the annulus. The condition (4.2) is then shown to imply $0 < \alpha_c < 1$. It is self-evident that if (4.2) holds, then there is no percolation for small enough α. We also show, through a classical argument using planar duality, that (4.2) implies that if α is close enough to 1, then there is percolation.

We were able to establish (4.2) for the two sets of assumptions that appear in our main theorems (namely: first for $d \geq 5$, $R \geq 1$ and second for $d \geq 3$ and R large enough). We prove both cases using a renormalization scheme inspired by Sections 2 and 3 of [Sz12], which involves embeddings of binary trees into $\mathbb{Z}^d$ that are "spread-out on all scales". In Section 4.2, we present this renormalization scheme and some of its properties.

In Section 5 we establish (4.2) for $d \geq 3$ and R large, and in Section 6 we establish it for $d \geq 5$ and $R \geq 1$. For simplicity of notation, Section 6 only treats explicitly $d \geq 5$ and $R = 1$ (i.e., the case of nearest neighbour interactions), but it will be easy to see that the proof given there applies for any value of R. In fact, the proof of Section 6 could also be adapted to cover the case of $d \geq 3$ and R large enough, so that Section 5 is (strictly speaking) redundant. We have nevertheless chosen to include it for three reasons: first, because it is quite short; second, because the method might find other applications; and third, the contents of Section 5 may be helpful for the reader to grasp the more involved arguments of Section 6.

A common point in the proofs of Section 5 and 6 is the need to provide an upper bound for probabilities of the form

$$\mu_\alpha \left[\xi(x) = 1 \text{ for all } x \in \mathcal{X} \right] \tag{1.2}$$

for certain finite sets $\mathcal{X} \subseteq \mathbb{Z}^d$ that appear at the "bottom" scale of the renormalization construction. An immediate consequence (as we will explain in Section 3, up to equation (3.6)) of the construction of μ_α through "coalescence classes" is that (1.2) is equal to $\mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X})} \right]$, where $\mathcal{N}_\infty(\mathcal{X})$ is the (random) terminal number of random walkers in a system of coalescing random walks started from the configuration in which there is one walker in each vertex of $\mathcal{X}$. Hence, in order to give a good upper bound for (1.2), one needs to argue that $\mathcal{N}_\infty(\mathcal{X})$ is comparable to $|\mathcal{X}|$ (the cardinality of $\mathcal{X}$). It is worth noting that $\alpha^{|\mathcal{X}|}$ is the probability of the event in (1.2) for independent, Bernoulli(α) percolation.

Our renormalization construction ensures that the set $\mathcal{X}$ under consideration here is "sparse on all scales". Hence, one expects that walkers started from the vertices of $\mathcal{X}$ tend to avoid other walkers, and the amount of loss due to coalescence, $|\mathcal{X}| - \mathcal{N}_\infty(\mathcal{X})$, is far from $|\mathcal{X}|$ with overwhelming probability. In order to make this precise, we use different strategies in Sections 5 and 6. Both of these techniques are novel.

- ($d \geq 3$, $R \gg 1$) In Section 5, we replace the system of coalescing random walks with a system of *annihilating* random walks and observe that annihilation events are "negatively correlated". This allows us to derive a useful explicit bound on (1.2) which is particularly effective if the range R of the walkers is big enough to guarantee that the expected number of annihilations is sufficiently small.

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- ($d \geq 5$, $R = 1$) The proof of Section 6 involves two important ideas. First, it turns out that under some carefully constructed circumstances one can run the walkers for some period of time independently from each other (i.e., without coalescence), which allows them to "wander away" from each other before they start to coalesce. Second, we reveal the paths of random walkers one by one and pre-emptively throw away those future walks that are too likely to coalesce with the ones already revealed. We can then control

  (a) the number of walkers that we throw away and

  (b) the number of coalescences occurring between the remaining walkers

  in such a way that the sum of these two numbers (which is greater than or equal to $|\mathcal{X}| - \mathcal{N}_\infty(\mathcal{X})$) is not too big compared to $|\mathcal{X}|$.

To state the obvious, Theorems 1.1 and 1.2 leave open the cases of dimension 3 and 4 and range R small, even though, as mentioned above, simulations and numerical work suggest that non-trivial phase transition should also occur in these cases. In our final Section 7, we give an heuristic explanation to the ineffectiveness of the method of Section 6 in treating $d = 3, 4$ and $R = 1$ (see Remark 7.3). In Remarks 7.2 and 7.4 we explain why the tricks of Section 5 are insufficient to prove Theorem 1.1, so that we could not do without the more involved method of Section 6. In Remark 7.1 we heuristically explain how voter model percolation is related to the question of disconnectedness of the closed support of super-Brownian motion.

2 Notation and preliminary facts

2.1 Summary of notation

Given a set or event A, we denote by $\mathbb{1}_A$ its indicator function and by $|A|$ its cardinality.

Given a vertex $x \in \mathbb{Z}^d$, we denote by $|x|$ its ℓ^∞ norm and by $|x|_1$ its ℓ^1 norm. We then write

$$
\begin{aligned}
B(L) &= \{x \in \mathbb{Z}^d : |x| \leq L\}, & B(x, L) &= x + B(L); \\
B_1(L) &= \{x \in \mathbb{Z}^d : |x|_1 \leq L\}, & B_1(x, L) &= x + B_1(L); \\
S(L) &= \{x \in \mathbb{Z}^d : |x| = L\}, & S(x, L) &= x + S(L).
\end{aligned}
\tag{2.1}
$$

If for $x, y \in \mathbb{Z}^d$ we have $|x - y|_1 = 1$, then these points are said to be neighbors, and we abbreviate this by $x \sim y$. They are $*$-neighbors if $|x - y| = 1$. For sets $A, B \subset \mathbb{Z}^d$, $\mathrm{dist}(A, B) = \min\{|x - y| : x \in A, \ y \in B\}$. The expression $A \subset\subset \mathbb{Z}^d$ indicates that A is a finite subset of $\mathbb{Z}^d$.

A nearest-neighbor path in $\mathbb{Z}^d$ is a (finite or infinite) sequence $\gamma(0), \gamma(1), \ldots$ so that $\gamma(i+1) \sim \gamma(i)$ for each i. A $*$-connected path is a sequence $\gamma(0), \gamma(1), \ldots$ so that $\gamma(i+1)$ and $\gamma(i)$ are $*$-neighbors for each i. We denote by $\{\gamma\}$ the set $\{\gamma(0), \gamma(1), \ldots\}$.

Definition 2.1. *Let $\xi \in \{0, 1\}^{\mathbb{Z}^d}$ and let A and B denote two disjoint subsets of $\mathbb{Z}^d$.*

(a) *We say A and B are connected by an open path in ξ (and write $A \overset{\xi}{\leftrightarrow} B$) if there exists a nearest-neighbor path $\gamma(0), \ldots, \gamma(n)$ such that $\gamma(0)$ is the neighbor of a point of A, $\gamma(n)$ is a neighbor of a point of B and $\xi(\gamma(i)) = 1$ for each i.*

(b) *Similarly, we write $A \overset{*\xi}{\longleftrightarrow} B$ if there exists a $*$-connected path $\gamma(0), \ldots, \gamma(n)$ so that $\gamma(0)$ is the $*$-neighbor of a point of A, $\gamma(n)$ is the $*$-neighbor of a point of B and $\xi(\gamma(i)) = 1$ for each i.*

2.2 Martingale facts

We will need a concentration inequality involving continuous-time martingales. We start recalling two definitions. Consider a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$.

Definition 2.2. *A process $(X_t)_{t \geq 0}$ is predictable with respect to $(\mathcal{F}_t)$ if*

$$X_t \in \mathcal{F}_{t-} = \sigma\left(\cup_{s<t}\mathcal{F}_s\right) \qquad \text{for all } t.$$

Note that if (X_t) is continuous and adapted to $(\mathcal{F}_t)$, then it is predictable with respect to $(\mathcal{F}_t)$.

Definition 2.3. *Let $(N_t)_{t \geq 0}$ be a square-integrable càdlàg martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$. The predictable quadratic variation of (N_t) is the predictable process $(\langle N \rangle_t)_{t \geq 0}$ such that $(N_t^2 - \langle N \rangle_t)_{t \geq 0}$ is a martingale with respect to $(\mathcal{F}_t)$.*

The almost sure uniqueness of the predictable quadratic variation follows from Doob-Meyer-Doléans decomposition ([Kal, Theorem 25.5]) applied to the submartingale (N_t^2). Note that $\langle N \rangle_t$ is a non-decreasing function of t. We refer the reader to [Kal, Proposition 26.1] for elementary properties of $\langle N \rangle$. The result we will need, which follows from [Kal, Theorem 26.17], is:

Theorem 2.4. *Let $S \in [0, +\infty]$. Let (N_t) be a square-integrable càdlàg martingale with $\langle N \rangle_S \leq \sigma^2$ almost surely for some $\sigma^2 \in (0, +\infty)$. Assume that the jumps of N are almost surely bounded by $\Delta \in (0, \sigma]$. Then we have*

$$\mathbb{P}\left(\max_{t \in [0,S]} N_t - N_0 \geq r\right) \leq \exp\left(-\frac{1}{2}\frac{r}{\Delta}\ln\left(1 + \frac{r\Delta}{\sigma^2}\right)\right), \qquad r \geq 0. \qquad (2.2)$$

Note that [Kal, Theorem 26.17] is only stated for the $\sigma = 1$ case; however, our version (2.2) follows from an application of that theorem to the martingale N_t/σ.

2.3 Random walk facts

Definition 2.5. *Given $R \in \mathbb{N}_+$, we say that $(X_t^z)_{t \geq 0}$ is an R-spread-out random walk on $\mathbb{Z}^d$ starting at $z \in \mathbb{Z}^d$ if $X_0^z = z$ and $(X_t^z)_{t \geq 0}$ is a continuous-time càdlàg Markov process on $\mathbb{Z}^d$ with infinitesimal generator*

$$(\mathcal{L}f)(x) = \sum_{\substack{y \in \mathbb{Z}^d: \\ 0 < |x-y|_1 \leq R}} \frac{f(y) - f(x)}{|B_1(R)| - 1},$$

where $f : \mathbb{Z}^d \to \mathbb{R}$. When $R = 1$, then we call (X_t^z) a (continuous-time) nearest-neighbour simple random walk on $\mathbb{Z}^d$.

In words: the holding times between jumps are i.i.d. with $\mathrm{Exp}(1)$ distribution and if a jump occurs at time t and $X_{t-}^z = x$ then X_t^z is uniformly distributed on $B_1(x, R) \setminus \{x\}$. If $R = 1$, then X_t^z is uniformly distributed on the set of nearest neighbours of x.

Let us formulate a useful corollary of Theorem 2.4 about random walks:

Corollary 2.6. *Let X_t denote a d-dimensional continuous-time nearest-neighbour simple random walk with jump rate 1 starting at the origin. Then for any $S, r \geq 0$ we have*

$$\mathbb{P}\left[\max_{0 \leq t \leq S} |X_t| > r\right] \leq 2d \exp\left(-\frac{1}{2}r\ln\left(1 + \frac{d \cdot r}{S}\right)\right). \qquad (2.3)$$

Proof. The d coordinates of X_t are 1-dimensional simple random walks with jump rate $1/d$, hence after a union bound we only need to apply (2.2) with $\sigma^2 = S/d$ and $\Delta = 1$ to achieve (2.3). $\qquad\square$

Let us define the transition kernel and the Green function of R-spread-out random walk on $\mathbb{Z}^d$ by

$$p_{R,t}(x, y) = \mathbb{P}\left[X^z_{s+t} = y \,|\, X^z_s = x \right], \qquad g_R(x, y) = \int_0^\infty p_{R,t}(x, y)\,\mathrm{d}t. \qquad (2.4)$$

If $R = 1$ then we drop the R from the subscript and simply denote $p_t(x, y)$ and $g(x, y)$. We have

$$p_{R,t}(x, y) = p_{R,t}(y, x), \quad p_{R,t}(x, y) = p_{R,t}(y - x, 0),$$
$$g_R(x, y) = g_R(y, x), \quad g_R(x, y) = g_R(y - x, 0), \quad g_R(x, x) \geq 1. \qquad (2.5)$$

It follows from the Chapman-Kolmogorov equations for $p_{R,t}(\cdot, \cdot)$ that we have

$$\sum_{y \in \mathbb{Z}^d} p_{R,T}(x, y) \cdot g_R(y, z) = \int_T^\infty p_{R,t}(x, z)\,\mathrm{d}t. \qquad (2.6)$$

It follows from the Local Central Limit Theorem (see [L96, Section 1.2]) that for any $d \geq 3$ there exist constants $c = c(d, R) > 0$ and $C = C(d, R) < +\infty$ such that

$$\frac{\int_T^\infty p_{R,t}(x, y)\,\mathrm{d}t}{(|x - y| \vee \sqrt{T} + 1)^{2-d}} \in [c, C], \qquad x, y \in \mathbb{Z}^d, \quad T \geq 0. \qquad (2.7)$$

It follows from the strong Markov property of random walks that we have

$$P[\,\exists\, t \geq 0\,:\, X^x_t = y\,] = \frac{g_R(x, y)}{g_R(y, y)} \overset{(2.5)}{\leq} g_R(x, y). \qquad (2.8)$$

The distributions of the increments of our random walks are symmetric, therefore if the random walks (X^x_t) and (X^y_t) are independent, then

$$(X^y_t - X^x_t)_{t \geq 0} \quad \text{has the same law as} \quad \left(X^{y-x}_{2t}\right)_{t \geq 0}. \qquad (2.9)$$

Let us define

$$h_R(x, y) = P[\,\exists\, t \geq 0\,:\, X^x_t = X^y_t\,], \qquad x, y \in \mathbb{Z}^d \qquad (2.10)$$

the probability that two independent R-spread-out random walks started from x and y ever meet. We have

$$h_R(x, y) \overset{(2.9),(2.8),(2.5)}{=} \frac{g_R(x, y)}{g_R(0, 0)} \leq g_R(x, y). \qquad (2.11)$$

In Section 5 we will make use of the following claim about spread-out random walks:

Claim 2.7. *Given $d \geq 3$, there exists $f : \mathbb{N} \to \mathbb{R}_+$ such that*

$$\forall\, R \in \mathbb{N},\ x \neq y \in \mathbb{Z}^d\,:\quad h_R(x, y) \leq f(R) \cdot |x - y|^{2-d}, \qquad \lim_{R \to \infty} f(R) = 0. \qquad (2.12)$$

Remark 2.8. *Before proving Claim 2.7, we first observe that, for fixed R, the bound $\sup_{x \neq y \in \mathbb{Z}^d} h_R(x, y) \cdot |x - y|^{d-2} < \infty$ already follows from (2.4), (2.7) and (2.11). The bound (2.12) is more informative, as it gives us asymptotic information as $R \to \infty$.*

Proof of Claim 2.7. The bound (2.12) follows from (2.11), [HvdHS03, Proposition 1.6] and the observation that the Green function of a continuous-time random walk with jump rate 1 is identical to the Green function of the corresponding discrete-time random walk. To see how the mentioned result in [HvdHS03] is applied, first note that their parameter L translates to our parameter R and their expression $S_1(x)$ is equal to our $g_R(0, x)$. Then, by letting their parameters α and μ both be equal to 1, their equation (1.36) yields that there exists $C < \infty$ such that (in our notation):

$$g_R(0, x) \leq C R^{-1} |x|^{2-d} \qquad \text{for } R \text{ large enough and all } x \in \mathbb{Z}^d, \ x \neq 0.$$

Claim 2.7 readily follows by (2.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will also make use of the following bound on the difference of Green function values of nearest neighbour sites: there exists a $C = C(d)$ such that

$$|g(x, y) - g(x, y + e)| \leq C \cdot (|x - y| + 1)^{1-d}, \qquad x, y \in \mathbb{Z}^d, \ e \sim 0. \qquad (2.13)$$

This bound follows from the much stronger [L96, Theorem 1.5.5].

The following heat kernel bound follows from the Local Central Limit Theorem: there exist $C = C(d) < +\infty$ and $c = c(d) > 0$ such that

$$p_t(x, y) \leq C t^{-\frac{d}{2}} \exp\left(-c \frac{|x - y|^2}{t}\right), \qquad x, y \in \mathbb{Z}^d, \ t \geq 1. \qquad (2.14)$$

In Section 6.7 we will make use of the following bound.

Claim 2.9. *There exists $C = C(d)$ such that*

$$\sum_{w \in \mathbb{Z}^d} p_t(y, w)|g(w, v) - g(w, v + e)| \leq C t^{\frac{1}{2} - \frac{d}{2}}, \qquad y, v \in \mathbb{Z}^d, \ e \sim 0, \ t \geq 1. \qquad (2.15)$$

Proof. By (2.5) we may assume $y = 0$ without loss of generality.

$$\sum_{w \in \mathbb{Z}^d} p_t(0, w)|g(w, v) - g(w, v + e)| \overset{(2.13),(2.14)}{\leq} C' t^{-\frac{d}{2}} \sum_{w \in \mathbb{Z}^d} \exp\left(-c \frac{|w|^2}{t}\right) (|v - w| + 1)^{1-d}$$

$$\overset{(*)}{\leq} C' t^{-\frac{d}{2}} \sum_{w \in \mathbb{Z}^d} \exp\left(-c \frac{|w|^2}{t}\right) (|w| + 1)^{1-d}$$

$$\leq C'' t^{-\frac{d}{2}} \sum_{n=0}^{\infty} \exp\left(-c \frac{n^2}{t}\right) (n + 1)^{1-d} \cdot n^{d-1} \leq C t^{\frac{1}{2} - \frac{d}{2}},$$

where $(*)$ follows from the *rearrangement inequality* [HLP52, Section 10.2, Theorem 368]. $\qquad\square$

3 Voter model: graphical construction, duality, stationary distributions

In this section we define the voter model on $\mathbb{Z}^d$ and present some well-known facts about it. We refer the reader to [Li85] for an introduction to the voter model and proofs of all the statements that we make in this section.

Fix d, $R \in \mathbb{N}$. The voter model on $\mathbb{Z}^d$ with range R, denoted by $(\xi_t)_{t \geq 0}$, is the Markov process with state space $\{0, 1\}^{\mathbb{Z}^d}$ and infinitesimal generator given by

$$(\mathcal{L}f)(\xi) = \sum_{\substack{x, y \in \mathbb{Z}^d: \\ 0 < |x - y|_1 \leq R}} \frac{f(\xi^{y \to x}) - f(\xi)}{|B_1(R)| - 1}, \qquad (3.1)$$

where $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is any function that only depends on finitely many coordinates, $\xi \in \{0,1\}^{\mathbb{Z}^d}$ and

$$\xi^{y \to x}(z) = \begin{cases} \xi(z) & \text{if } z \neq x, \\ \xi(y) & \text{if } z = x. \end{cases}$$

In words, each site $x \in \mathbb{Z}^d$ updates its state $\xi(x)$ with rate 1 by uniformly choosing a site $y \in B_1(x, R) \setminus \{x\}$ and adopting the state $\xi(y)$ of y. In case $R = 1$, we say that the model is nearest-neighbour.

Given $\xi \in \{0,1\}^{\mathbb{Z}^d}$, we denote by P_ξ a probability measure under which $(\xi_t)_{t \geq 0}$ is defined and satisfies $P_\xi[\xi_0 = \xi] = 1$. Likewise, given a probability distribution ν on $\{0,1\}^{\mathbb{Z}^d}$, we write $P_\nu = \int P_\xi \, \mathsf{d}\nu(\xi)$.

The process (ξ_t) satisfies a duality relation with respect to a system of coalescing random walks. We will now explain what is meant by this – or rather, we will give a particularly simple formulation of duality that will be sufficient for our purposes.

For each $x, y \in \mathbb{Z}^d$ with $0 < |x - y|_1 \leq R$, let $(D_t^{(x,y)})_{t \geq 0}$ be a Poisson process with rate $(|B_1(R)| - 1)^{-1}$ on $[0, \infty)$, so that $D_0^{(x,y)} = 0$ and $D_t^{(x,y)} - D_{t-}^{(x,y)}$ is equal to 0 or 1 for all t. One pictures $D_t^{(x,y)} - D_{t-}^{(x,y)} = 1$ as an arrow pointing from x to y at time t. We denote by $\mathbb{P}$ a probability measure under which all these processes are defined and are independent. For each $x \in \mathbb{Z}^d$, we then define (on this same probability space) $(Y_t^x)_{t \geq 0}$ as the unique $\mathbb{Z}^d$-valued process which is right-continuous with left limits and satisfies

$$Y_0^x = x, \qquad Y_t^x = Y_{t-}^x + \sum_{z \in B_1(R)} z \cdot \left(D_t^{(Y_{t-}^x, Y_{t-}^x + z)} - D_{t-}^{(Y_{t-}^x, Y_{t-}^x + z)} \right). \tag{3.2}$$

One pictures Y_t^x as it moves along the time axis and follows the arrows that it encounters. The collection of processes $\{(Y_t^x)_{t \geq 0} : x \in \mathbb{Z}^d\}$ is what we refer to as a system of coalescing random walks. This terminology makes sense because, as is clear from the above definition, each $(Y_t^x)_{t \geq 0}$ is a continuous-time random walk on $\mathbb{Z}^d$ with rate 1 which jumps to a uniformly distributed location in $Y_{t-}^x + (B_1(R) \setminus \{0\})$ and moreover, these walks move independently until they meet, after which they coalesce and remain together.

Now, for any fixed $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $A \subset\subset \mathbb{Z}^d$ and $t \geq 0$, we have

$$\text{law of } (\xi(Y_t^x) : x \in A) \text{ under } \mathbb{P} \quad = \quad \text{law of } (\xi_t(x) : x \in A) \text{ under } P_\xi. \tag{3.3}$$

See Figure 1 for an illustration.

As a consequence of (3.3), we obtain the following *duality equation* for the voter model: for any $A \subset\subset \mathbb{Z}^d$, $t \geq 0$ and probability measure ν on $\{0,1\}^{\mathbb{Z}^d}$,

$$P_\nu[\xi_t(x) = 1 \text{ for all } x \in A] = \int \mathbb{P}[\{Y_t^x : x \in A\} \subset \{y : \xi(y) = 1\}] \, \mathsf{d}\nu(\xi). \tag{3.4}$$

Note that by inclusion-exclusion the equation (3.4) characterizes the distribution of ξ_t for the process started with distribution ν. Of particular interest is the case when ν is equal to

$$\pi_\alpha := (\alpha \delta_{\{1\}} + (1 - \alpha)\delta_{\{0\}})^{\otimes \mathbb{Z}^d},$$

the product measure of Bernoulli(α) on $\mathbb{Z}^d$, for $\alpha \in [0, 1]$. In order to discuss this case, let us introduce some notation. For $A \subset \mathbb{Z}^d$, we let

$$\mathcal{N}_t(A) = |\{Y_t^x : x \in A\}|, \quad t \geq 0 \quad \text{and} \quad \mathcal{N}_\infty(A) = \lim_{t \to \infty} \mathcal{N}_t(A); \tag{3.5}$$

the limit exists because $\mathcal{N}_t(A)$ decreases with t. Denoting by $\mathbb{E}$ the expectation operator associated with $\mathbb{P}$, we can then rewrite (3.4) as

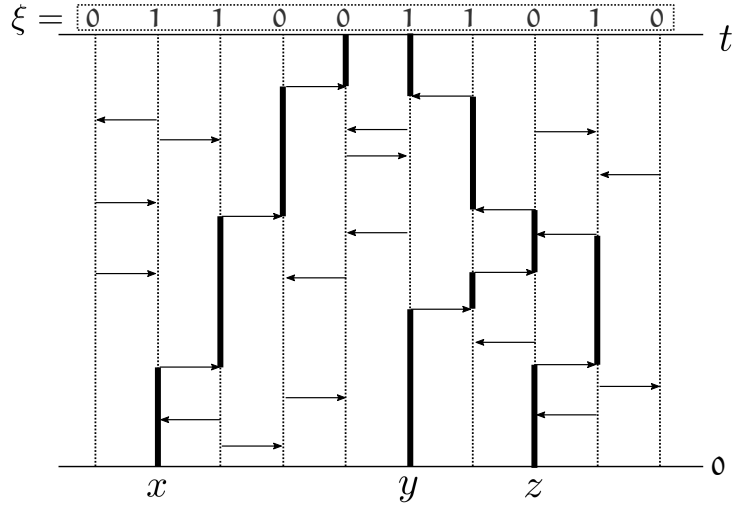$$P_{\pi_\alpha}[\xi_t(x) = 1 \text{ for all } x \in A] = \mathbb{E}\left[\alpha^{\mathcal{N}_t(A)}\right].$$

Figure 1: Illustration of the system of coalescing random walks to which the voter model is dual. The horizontal axis represents space (which is one-dimensional in this picture) and the vertical axis represents time; arrows are plotted as described in the text. On top, a configuration $\xi \in \{0,1\}^{\mathbb{Z}^d}$ is represented. The thick vertical paths represent the trajectories of $(Y_s^x), (Y_s^y), (Y_s^z)$ for $0 \le s \le t$. In the situation illustrated we have $\xi(Y_t^x) = 0$ and $\xi(Y_t^y) = \xi(Y_t^z) = 1$.

By taking the limit on the right-hand side as $t \to \infty$, we can conclude that, under P_{π_α}, as $t \to \infty$, ξ_t converges in distribution to a measure μ_α on $\{0,1\}^{\mathbb{Z}^d}$ characterized by

$$\mu_\alpha\left[\xi(x) = 1 \text{ for all } x \in A\right] = \mathbb{E}\left[\alpha^{\mathcal{N}_\infty(A)}\right] \tag{3.6}$$

for every finite $A \subset \mathbb{Z}^d$. The measures μ_α are invariant and ergodic with respect to translations on $\mathbb{Z}^d$ and satisfy

$$\mu_\alpha[\xi(x) = 1] = \alpha, \qquad \text{Corr}_{\mu_\alpha}\left(\xi(x), \xi(y)\right) \overset{(2.10)}{=} h_R(x,y), \qquad x, y \in \mathbb{Z}^d, \tag{3.7}$$

thus (1.1) indeed holds by (2.7) and (2.11). We also note that

$$\mu_\alpha\left[\xi(x) = 1 \text{ for all } x \in A\right] \overset{(3.5),(3.6)}{\ge} \mathbb{E}\left[\alpha^{\mathcal{N}_0(A)}\right] = \alpha^{|A|}. \tag{3.8}$$

As the measures μ_α are obtained as distributional limits of (ξ_t), they are also stationary with respect to the dynamics of the voter model. In fact, in Section V.1 of [Li85] it is shown that

- if $d \ge 3$, then the set of extremal stationary distributions of the voter model is equal to $\{\mu_\alpha : \alpha \in [0,1]\}$. Here, a measure is said to be extremal if it cannot be written as a nontrivial convex combination of other stationary distributions.

- if $d = 1$ or 2, then there are only two extremal stationary distributions, namely the point masses on the constant configurations $\xi \equiv 1$ and $\xi \equiv 0$. (If $d = 1$ or 2, by recurrence of the random walk we have $\mathcal{N}_\infty(A) = 1$ almost surely for any finite and non-empty A. We can then see from (3.6) that μ_α is a convex combination with weight α of the point masses on the constant configurations).

Finally, we give a useful construction, jointly on the same probability space, of the system of coalescing random walks and for each $\alpha \in [0, 1]$, a random $\xi^{(\alpha)} \in \{0, 1\}^{\mathbb{Z}^d}$ distributed as μ_α. To this end, we take the probability space in which the aforementioned measure $\mathbb{P}$ and the processes $\left((Y_t^x)_{t \geq 0} : x \in \mathbb{Z}^d \right)$ are defined, and enlarge it so that a sequence of random variables $\mathcal{U}_n$, $n \in \mathbb{N}$, all independent and uniformly distributed on $[0, 1]$, are also defined (and are independent of the Y_t^x's). Next, fix an arbitrary enumeration $x_1, x_2, \ldots$ of $\mathbb{Z}^d$. For any $n \geq 1$, define the random variables

$$\eta(n) = \min\{m \ : \ m \leq n \text{ and } Y_t^{x_m} = Y_t^{x_n} \text{ for some } t \geq 0\} \tag{3.9}$$

and then set

$$\xi^{(\alpha)}(x_n) = \mathbb{1}_{\{\mathcal{U}_{\eta(n)} \leq \alpha\}}, \qquad n \in \mathbb{N}, \ \alpha \in [0, 1]. \tag{3.10}$$

It is then straightforward to check that $\xi^{(\alpha)}$ has law μ_α, as defined in (3.6), and moreover it satisfies

$$\xi^{(\alpha)}(x) = \xi^{(\alpha)}(y) \quad \text{if} \quad Y_t^x = Y_t^y \text{ for some } t.$$

Moreover, it follows from this construction that if $\alpha \leq \alpha'$, then $\xi^{(\alpha)}(x) \leq \xi^{(\alpha')}(x)$ for each $x \in \mathbb{Z}^d$, therefore μ_α is stochastically dominated by $\mu_{\alpha'}$, that is, if $f : \{0, 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is increasing (with respect to the partial order on $\{0, 1\}^{\mathbb{Z}^d}$ that is induced by the order $0 < 1$ on the coordinates), then

$$\int f \, \mathrm{d}\mu_\alpha \leq \int f \, \mathrm{d}\mu_{\alpha'}. \tag{3.11}$$

We will also need the following consequence of the joint construction:

$$\text{the law of } 1 - \xi \text{ under } \mu_\alpha \text{ is the same as law of } \xi \text{ under } \mu_{1-\alpha}. \tag{3.12}$$

4 First facts about voter model percolation, $d \geq 3$

In this section we collect the definitions and facts that are common to the proofs of Theorems 1.1 and 1.2. Throughout this section, we fix $d \geq 3$, $R \geq 1$ (see (3.1)) and $\alpha \in [0, 1]$. ξ denotes an element of $\{0, 1\}^{\mathbb{Z}^d}$ and μ_α denotes the extremal stationary distribution of the voter model with density α, as described in Section 3.

In Section 4.1 we state the key inequality (4.2) and deduce Theorems 1.1 and 1.2 from it. In Section 4.2 we set up the multi-scale renormalization scheme that we will employ to prove (4.2).

4.1 A sufficient condition for percolation phase transition

We will show that there exist $\alpha_0 > 0$ and a sequence $(L_N)_{N \geq 0}$ of form

$$L_N = L \cdot \ell^N, \tag{4.1}$$

where $\ell \geq 6$, $L \geq 1$ such that

$$\mu_{\alpha_0} \left[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c \right] \leq 2^{-2^N}, \qquad N \geq 1. \tag{4.2}$$

In words: the probability under μ_{α_0} that an annulus with inner radius $L_N - 2$ and outer radius $2L_N$ is crossed by a $*$-connected path of 1's in ξ is less than or equal to 2^{-2^N}. Note that (4.2) implies that the crossing probability of the annulus $B(2M) \setminus B(M)$ decays as a stretched exponential function of M as $M \to \infty$:

$$\mu_{\alpha_0} \left[B(M) \overset{*\xi}{\longleftrightarrow} B(2M)^c \right] \leq Ce^{-M^\kappa} \quad \text{for some } C < +\infty \text{ and } \kappa > 0.$$

We will prove (4.2) for $d \geq 3$ and $R \gg 1$ in Section 5 and for $d \geq 5$ and $R = 1$ in Section 6. Let us now deduce the main results of this paper from (4.2).

Proof of Theorems 1.1 and 1.2. As we have already discussed in Section 3, the measure μ_α is invariant and ergodic under spatial shifts of $\mathbb{Z}^d$. Therefore the probability under μ_α of the event

$$\text{Perc} = \{\ \{x : \xi(x) = 1\} \text{ has an infinite connected component}\ \} \tag{4.3}$$

can only be zero or one for any α. Also, since the event in (4.3) is increasing, by (3.11) there indeed exists $0 \le \alpha_c \le 1$ such that $\mu_\alpha(\text{Perc}) = 0$ for any $\alpha < \alpha_c$ and $\mu_\alpha(\text{Perc}) = 1$ for any $\alpha > \alpha_c$. Our aim is to to prove that $0 < \alpha_c < 1$.

Let us now explain how (4.2) implies

$$\alpha_0 \le \alpha_c \le 1 - \alpha_0.$$

As soon as we prove these inequalities, the statements of Theorems 1.1 and 1.2 will follow.

First, we will prove $\alpha_0 \le \alpha_c$ by showing that $\mu_{\alpha_0}(\text{Perc}) = 0$. Denote by $\{x \xleftrightarrow{\xi} \infty\}$ the event that there exists a nearest-neighbor path $\gamma(0), \gamma(1), \ldots$, such that $\gamma(0) = x$, $\lim_{n\to\infty} |\gamma(n)| = \infty$ and $\xi(\gamma(n)) = 1$ for each n. Since

$$\text{Perc} = \bigcup_{x \in \mathbb{Z}^d} \{x \xleftrightarrow{\xi} \infty\}$$

and μ_{α_0} is invariant under translations of $\mathbb{Z}^d$, it is enough to prove that $\mu_{\alpha_0}[0 \xleftrightarrow{\xi} \infty] = 0$. This follows from (4.2) and the inclusions

$$\{0 \xleftrightarrow{\xi} \infty\} \subseteq \left\{ B(L_N - 2) \xleftrightarrow{\xi} B(2L_N)^c \right\} \overset{(*)}{\subseteq} \left\{ B(L_N - 2) \xleftrightarrow{*\xi} B(2L_N)^c \right\}, \quad N \ge 1,$$

where $(*)$ holds since a nearest-neighbour path is also a $*$-path, see Definition 2.1.

Now we will prove $\alpha_c \le 1 - \alpha_0$ using a variant of the classical Peierls argument, see [Pei36] and [Gr99, Section 1.4]. Let us define the plane $\mathcal{P} \subset \mathbb{Z}^d$ by

$$\mathcal{P} = \{\ x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_i = 0 \text{ for all } i \ge 3\ \}.$$

For $\xi \in \{0,1\}^{\mathbb{Z}^d}$, we denote by $\bar\xi$ the restriction of ξ to $\mathcal{P}$. For any $N \ge 1$ we define the events

$E_N = \{\ B(L_N) \cap \mathcal{P} \text{ is not connected to } \infty \text{ by a nearest neighbour path of 1's in } \bar\xi\ \}$,
$F_N = \{\ B(L_N) \cap \mathcal{P} \text{ is surrounded by a } *\text{-connected cycle } \widetilde{\gamma} \text{ of 0's in } \bar\xi\ \}$.

By planar duality (see Definition 4, Definition 7 and Corollary 2.2 of [K82]), we have

$$E_N = F_N.$$

If F_N occurs, denote by $\widetilde{M}$ the smallest integer such that $B(L_{\widetilde{M}+1}) \cap \{\widetilde{\gamma}\} \ne \emptyset$. By the definition of F_N we have $\widetilde{M} \ge N$.

If F_N occurs, we can pick an $\widetilde{x} \in (L_{\widetilde{M}} \cdot \mathbb{Z}^d) \cap \mathcal{P}$ satisfying $|\widetilde{x}| \le L_{\widetilde{M}+1}$ such that $B(\widetilde{x}, L_{\widetilde{M}} - 1) \cap \{\widetilde{\gamma}\} \ne \emptyset$. By the definition of $\widetilde{M}$ the cycle $\widetilde{\gamma}$ surrounds $B(L_{\widetilde{M}}) \cap \mathcal{P}$, therefore by Definition 2.1(b) the annulus $B(\widetilde{x}, 2L_{\widetilde{M}}) \setminus B(\widetilde{x}, L_{\widetilde{M}} - 2)$ is crossed by $\widetilde{\gamma}$. Thus for any $N \in \mathbb{N}$ we can bound

$$\mu_{1-\alpha_0}[\text{Perc}^c] \le \mu_{1-\alpha_0}[E_N] = \mu_{1-\alpha_0}[F_N]$$

$$\le \sum_{M=N}^{\infty} \sum_{\substack{x \in L_M \cdot \mathbb{Z}^d \\ |x| \le L_{M+1}}} \mu_{1-\alpha_0}\left[B(x, L_M - 2) \xleftrightarrow{*(1-\xi)} B(x, 2L_M)^c \right]$$

$$\overset{(3.12),(4.1),(4.2)}{\le} \sum_{M=N}^{\infty} (2\ell + 1)^d \cdot 2^{-2^M},$$

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from which $\mu_{1-\alpha_0}[\text{Perc}] = 1$ follows by letting $N \to \infty$. This implies $\alpha_c \leq 1 - \alpha_0$. The proof of Theorems 1.1 and 1.2 is complete, given (4.2). $\qquad\square$

4.2 Renormalization scheme for percolation, $d \geq 3$

We are going to use multi-scale renormalization. Similar methods have been successfully employed to prove the percolation phase transition of the vacant set of random interlacements (see [S10, Sz10]) and the excursion set of the Gaussian free field (see [RS13]). We will borrow the renormalization scheme of [Ra15], which is in turn a variant of the method developed in Sections 2 and 3 of [Sz12].

Let us fix $d \geq 3$. We let ℓ and L be two integers describing the scales of renormalization:

$$L_N = L \cdot \ell^N, \qquad N \geq 0. \tag{4.4}$$

Using these scales we define the renormalized lattices

$$\mathcal{L}_N = L_N \cdot \mathbb{Z}^d, \qquad N \geq 0. \tag{4.5}$$

Remark 4.1. *The basic idea behind the proof of* (4.2) *is as follows. Denote by $p(N)$ the probability of the crossing event that appears on the left-hand side of* (4.2). *The crossing of an annulus of scale L_N implies that two annuli of scale L_{N-1} that are far enough from each other are also crossed (see Figure 2 below), so one naively hopes to upper bound $p(N)$ in terms of $p(N-1)^2$ and thus prove* (4.2) *by induction on N. To make this idea rigorous, one needs to take into account the combinatorial term that counts the number of choices of the smaller annuli, and, more importantly, the strong positive correlation between the two crossing events on the smaller scale.*

We start our proof of (4.2) *by repeating the above sketched renormalization step until we reach the bottom scale L_0. We encode the choices of the centers of these annulli as embeddings of the binary tree T_N of depth N into $\mathbb{Z}^d$ (see Definition 4.2) – this way the proof of* (4.2) *boils down to bounding the probability of the joint occurrence of 2^N instances of a simple bottom-level event, indexed by the leaves of T_N (see Lemma 4.4).*

Let $T_{(k)} = \{1,2\}^k$ for $k \geq 0$ (in particular, $T_{(0)} = \{\emptyset\}$) and then let

$$T_N = \cup_{k=0}^N T_{(k)}$$

be the binary tree of height N. If $0 \leq k < N$ and $m = (\eta_1, \ldots, \eta_k) \in T_{(k)}$, we let

$$m_1 = (\eta_1, \ldots, \eta_k, 1), \qquad m_2 = (\eta_1, \ldots, \eta_k, 2) \tag{4.6}$$

be the two children of m in $T_{(k+1)}$.

Definition 4.2. $\mathcal{T} : T_N \to \mathbb{Z}^d$ *is a proper embedding of T_N if*

1. *$\mathcal{T}(\{\emptyset\}) = 0$;*

2. *for all $0 \leq k \leq N$ and $m \in T_{(k)}$ we have $\mathcal{T}(m) \in \mathcal{L}_{N-k}$;*

3. *for all $0 \leq k < N$ and $m \in T_{(k)}$ we have*

$$|\mathcal{T}(m_1) - \mathcal{T}(m)| = L_{N-k}, \qquad |\mathcal{T}(m_2) - \mathcal{T}(m)| = 2L_{N-k}. \tag{4.7}$$

We denote by Λ_N the set of proper embeddings of T_N into $\mathbb{Z}^d$.

We now collect a few facts from [Ra15] about these embeddings. Although the lemmas in [Ra15] that correspond to our Lemmas 4.3, 4.4 and 4.5 below are stated for $\ell = 6$, their statements hold true (and have the same proof) for any integer $\ell \geq 6$.

Lemma 4.3.

$$|\Lambda_N| = \left[((4\ell + 1)^d - (4\ell - 1)^d) \cdot ((2\ell + 1)^d - (2\ell - 1)^d) \right]^{2^N - 1}. \qquad (4.8)$$

This follows from [Ra15, Lemma 3.2]. Informally, given $\mathcal{T}(m)$, there are $(2\ell+1)^d - (2\ell-1)^d$ ways to choose $\mathcal{T}(m_1)$ and $(4\ell + 1)^d - (4\ell - 1)^d$ ways to choose $\mathcal{T}(m_2)$.

Next is the statement that, given a crossing of the L_N-scale annulus $B(2L_N)\backslash B(L_N)$, we can find a proper embedding $\mathcal{T} \in \Lambda_N$ so that all L_0-scale annuli $B(\mathcal{T}(m), 2L_0) \setminus B(\mathcal{T}(m), L_0) : m \in T_{(N)}$ are crossed. Recall the notion of $S(x, L)$ from (2.1).
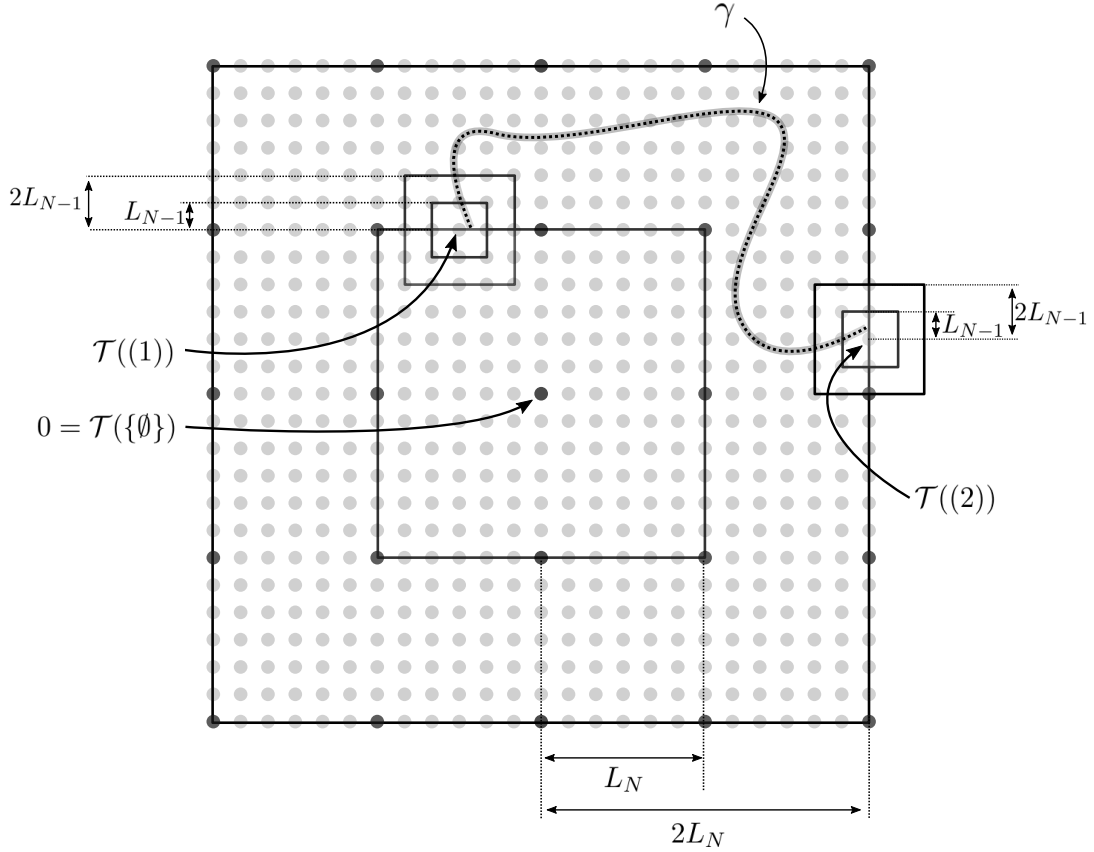

Figure 2: Illustration of the relation between the proper embedding $\mathcal{T}$ (see Definition 4.2) and the path γ that appears in Lemma 4.4. The light grey circles and dark grey circles represent points of the lattices $\mathcal{L}_{N-1}$ and $\mathcal{L}_N$, respectively.

Lemma 4.4. *If γ is a $*$-connected path in $\mathbb{Z}^d$ with*

$$\{\gamma\} \cap S(L_N - 1) \neq \varnothing, \quad \{\gamma\} \cap S(2L_N) \neq \varnothing,$$

then there exists $\mathcal{T} \in \Lambda_N$ such that

$$\{\gamma\} \cap S(\mathcal{T}(m), L_0 - 1) \neq \varnothing \ \text{and} \ \{\gamma\} \cap S(\mathcal{T}(m), 2L_0) \neq \varnothing \quad \text{for all } m \in T_{(N)}. \qquad (4.9)$$

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This is [Ra15, Lemma 3.3] (in fact, the statement given here corresponds to equation (3.7) in the proof of that lemma). Informally, one recursively constructs a proper embedding $\mathcal{T}$: if γ crosses an annulus of scale L_{N-k} centered at some $\mathcal{T}(m) \in \mathcal{L}_{N-k}$ for $m \in T_{(k)}$, $0 \leq k < N$, then two "children" annuli of scale L_{N-k-1} centered at some $\mathcal{T}(m_1), \mathcal{T}(m_2) \in \mathcal{L}_{N-k-1}$ satisfying (4.7) will also be crossed by γ, see Figure 2.

Finally, given a proper embedding $\mathcal{T} \in \Lambda_N$, the set of images of the leaves $\{\mathcal{T}(m) : m \in T_{(N)}\}$ is "spread-out on all scales".

Lemma 4.5. *For any $\mathcal{T} \in \Lambda_N$ and any $m_0 \in T_{(N)}$, we have*

$$\left| \left\{ m \in T_{(N)} : \mathrm{dist}\left(B(\mathcal{T}(m_0), 2L),\ B(\mathcal{T}(m), 2L) \right) \leq \ell^k L/2 \right\} \right| \leq 2^{k-1}, \quad k \geq 1. \quad (4.10)$$

This is a consequence of our assumption $\ell \geq 6$ and [Ra15, Lemma 3.4]. In particular, by choosing $k = 1$ in (4.10) we obtain that the sets $B(\mathcal{T}(m), 2L)$ for $m \in T_{(N)}$ are disjoint.

5   Spread-out model, $d \geq 3$

In this section we work with the voter model with range R, thus we will denote the stationary distribution (see (3.6)) with density α by $\mu_{R,\alpha}$. The goal of this section is to prove Theorem 1.2. More specifically, we will show that (4.2) holds for any $d \geq 3$ if $R \geq R_0(d)$ for some large R_0 and some $\alpha_0 = \alpha_0(d) > 0$.

Recall the notion of $h_R(x, y)$ from (2.10). The key result in our proof of Theorem 1.2 is the following decorrelation inequality which serves as a partial converse to (3.8):

Lemma 5.1. *For any $\mathcal{X} = \{x_1, \ldots, x_{|\mathcal{X}|}\} \subset \mathbb{Z}^d$ we have*

$$\mu_{R,\alpha}\left[\xi(x) = 1 \text{ for all } x \in \mathcal{X} \right] \leq \alpha^{|\mathcal{X}|} \prod_{1 \leq i < j \leq |\mathcal{X}|} \left(1 + h_R(x_i, x_j) \left(\alpha^{-2} - 1 \right) \right). \quad (5.1)$$

Before we prove Lemma 5.1, let us see how it allows us to conclude.

Proof of (4.2) for $d \geq 3$ and $R \gg 1$. We use the renormalization scheme described in Section 4.2. In this proof we choose $\ell = 6$ and $L = 1$ in (4.4). Given $\mathcal{T} \in \Lambda_N$, we denote

$$\mathcal{X}_{\mathcal{T}} = \bigcup_{m \in T_{(N)}} \{\mathcal{T}(m)\} \overset{(2.1)}{=} \bigcup_{m \in T_{(N)}} S(\mathcal{T}(m), L_0 - 1). \quad (5.2)$$

By Lemma 4.3, we have

$$|\Lambda_N| \leq \widehat{C}^{2^N}, \text{ where } \widehat{C} = ((4 \cdot 6 + 1)^d - (4 \cdot 6 - 1)^d) \cdot ((2 \cdot 6 + 1)^d - (2 \cdot 6 - 1)^d).$$

Combining Definition 2.1 and Lemma 4.4 in a union bound, we get, for any N,

$$\mu_{R,\alpha}\left[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c \right] \leq \widehat{C}^{2^N} \max_{\mathcal{T} \in \Lambda_N} \mu_{R,\alpha}\left[\xi(x) = 1 \text{ for all } x \in \mathcal{X}_{\mathcal{T}} \right]. \quad (5.3)$$

Now we fix some N and $\mathcal{T} \in \Lambda_N$ with the aim of bounding the probability on the right-hand side of (5.3). Note that by Lemma 4.5 we have $|\mathcal{X}_{\mathcal{T}}| = 2^N$. Let us denote $\mathcal{X}_{\mathcal{T}} = \{x_1, \ldots, x_{2^N}\}$. We have

$$\mu_{R,\alpha}\left[\xi(x) = 1 \text{ for all } x \in \mathcal{X}_{\mathcal{T}} \right] \overset{(5.1)}{\leq}$$

$$\alpha^{2^N} \prod_{1 \leq i < j \leq 2^N} \left(1 + h_R(x_i, x_j)\alpha^{-2} \right) \leq \alpha^{2^N} \prod_{i=1}^{2^N - 1} \exp\left(\alpha^{-2} \sum_{j > i} h_R(x_i, x_j) \right). \quad (5.4)$$

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For any $1 \le i < 2^N$ let us bound

$$\sum_{j>i} h_R(x_i, x_j) \overset{(2.12)}{\le} \sum_{j>i} f(R) \cdot |x_i - x_j|^{2-d} \overset{(4.10)}{\le} \sum_{k=1}^{\infty} (2^{k-1}-1) \cdot \left(f(R) \cdot \left(6^k/2\right)^{2-d} \right). \quad (5.5)$$

By $\lim_{R \to \infty} f(R) = 0$ (see (2.12)) and (5.5), for any $\alpha > 0$ we can choose $R = R(\alpha)$ big enough so that for any $1 \le i < 2^N$ we have

$$\alpha^{-2} \sum_{j>i} h_R(x_i, x_j) \le \ln(2). \quad (5.6)$$

Letting $\alpha_0 = \frac{1}{4}\widehat{C}^{-1}$ and $R = R(\alpha_0)$ we obtain the desired (4.2):

$$\mu_{R,\alpha_0} \left[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c \right] \overset{(5.3),(5.4)}{\le} \widehat{C}^{2^N} \alpha_0^{2^N} \prod_{i=1}^{2^N-1} \exp\left(\alpha_0^{-2} \sum_{j>i} h_R(x_i, x_j) \right)$$

$$\overset{(5.6)}{\le} \widehat{C}^{2^N} \left(\frac{1}{4}\widehat{C}^{-1} \right)^{2^N} 2^{2^N} = 2^{-2^N}.$$

$\square$

The rest of this section is devoted to the proof of Lemma 5.1.

Recall the graphical construction of coalescing random walks Y_t^x, $x \in \mathbb{Z}^d$, $t \in \mathbb{R}_+$ defined on the probability space of the Poisson point processes $(D_t^{(x,y)})_{t \ge 0}$ from Section 3. Given $\mathcal{X} \subset\subset \mathbb{Z}^d$, define $\mathcal{X}_t = \{Y_t^x : x \in \mathcal{X}\}$, so that $\mathcal{N}_t(\mathcal{X}) = |\mathcal{X}_t|$. If $D_t^{(x,y)} - D_{t-}^{(x,y)} = 1$ for some $x \in \mathcal{X}_{t-}$, $y \in \mathbb{Z}^d$ and $t \in \mathbb{R}_+$, then the graphical construction (3.2) of coalescing random walks implies

$$\mathcal{X}_t = (\mathcal{X}_{t-} \setminus \{x\}) \cup \{y\}. \quad (5.7)$$

Let us introduce another set-valued stochastic process $\mathcal{X}_t'$, *annihilating random walks*, also defined on the probability space of the Poisson point processes $(D_t^{(x,y)})_{t \ge 0}$. Starting also from $\mathcal{X}_0' := \mathcal{X}$ these particles also perform independent R-spread-out continuous-time random walks until one of the walkers tries to jump on a site occupied by another walker, in which case both of them disappear immediately. The formal definition is as follows. If $D_t^{(x,y)} - D_{t-}^{(x,y)} = 1$ for some $x \in \mathcal{X}_{t-}'$, $y \in \mathbb{Z}^d$ and $t \in \mathbb{R}_+$, then

$$\mathcal{X}_t' = \left(\mathcal{X}_{t-}' \setminus \{x\} \right) \Delta \{y\}, \quad (5.8)$$

where $A \Delta B$ denotes the symmetric difference of the sets A and B.

Similarly to (3.5), let us denote $\mathcal{N}_t'(\mathcal{X}) = |\mathcal{X}_t'|$ and $\mathcal{N}_\infty'(\mathcal{X}) = \lim_{t \to \infty} \mathcal{N}_t'(\mathcal{X})$.

Remark 5.2. *Annihilating random walks were introduced in [EN74] (in a discrete-time version) and studied in the 70s and 80s; see for instance [Sc76], [Gi78], [BG80] and [Ar83]. We also mention that, as explained in Example 4.16 in Chapter III of [Li85], there is a duality relation between the voter model and annihilating random walks, which is of a different nature from the duality between the voter model and coalescing random walks. However, our use of annihilating walks is unrelated to this duality.*

Our intuitive reason for switching from coalescing to annihilating walks is the following: $\mathbb{E}\left[\alpha^{\mathcal{N}_\infty'(\mathcal{X})}\right]$ is easier to bound than $\mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X})}\right]$, because in the case of coalescing walks, one "ill-behaved" walker can "run around" and cause many coalescence events, but in the case of annihilating random walks, an "ill-behaved" walker will self-destruct at the moment of the first collision.

Lemma 5.3. *For any $\mathcal{X} \subset\subset \mathbb{Z}^d$, $\alpha \in [0,1]$, $R \in \mathbb{N}$ and $t \geq 0$ we have*

$$\mathbb{E}\left[\alpha^{\mathcal{N}_t(\mathcal{X})}\right] \leq \mathbb{E}\left[\alpha^{\mathcal{N}'_t(\mathcal{X})}\right]. \tag{5.9}$$

Proof. As soon as we show $\mathcal{X}'_t \subseteq \mathcal{X}_t$, the inequality (5.9) will immediately follow.

Let us assume that $D_t^{(x,y)} - D_{t-}^{(x,y)} = 1$ for some $x \in \mathcal{X}_{t-}$ and that $\mathcal{X}'_{t-} \subseteq \mathcal{X}_{t-}$ holds. One can readily check using (5.7) and (5.8) that we also have $\mathcal{X}'_t \subseteq \mathcal{X}_t$ by considering the cases

(a) $x \in \mathcal{X}_{t-} \setminus \mathcal{X}'_{t-}$, (b) $x \in \mathcal{X}'_{t-}$, $y \notin \mathcal{X}'_{t-}$, (c) $x \in \mathcal{X}'_{t-}$, $y \in \mathcal{X}'_{t-}$

separately. Since $\mathcal{X}_0 = \mathcal{X}'_0 = \mathcal{X}$, the inclusion $\mathcal{X}'_t \subseteq \mathcal{X}_t$ for all $t \in \mathbb{R}_+$ follows by induction. $\square$

Let us now give an alternative construction of $\mathcal{X}'_t$ on a different probability space. Recall the notation $\mathcal{X} = \{x_1, \ldots, x_{|\mathcal{X}|}\}$. Let X_t^i, $1 \leq i \leq |\mathcal{X}|$ denote independent R-spread-out random walks with $X_0^i = x_i$. For $1 \leq i < j \leq |\mathcal{X}|$, we denote

$$\tau(i,j) = \inf\{\, t \,:\, X_t^i = X_t^j \,\}. \tag{5.10}$$

We also define the set-valued stochastic process $\mathcal{I}_t \subseteq [1, \ldots, |\mathcal{X}|]$ and the stopping times $T_0, T_1, T_2, \ldots$ by letting $T_0 = 0$ and $\mathcal{I}_{T_0} = [1, \ldots, |\mathcal{X}|]$, and then inductively for $k \geq 1$ by

$$T_k := \inf\{\, \tau(i,j) \,:\, i < j, \quad i, j \in \mathcal{I}_{T_{k-1}} \,\}, \quad T_k = \tau(i^*, j^*), \quad \mathcal{I}_{T_k} = \mathcal{I}_{T_{k-1}} \setminus \{i^*, j^*\}.$$

In words, T_k is the time of the k'th annihilation and $\mathcal{I}_{T_k}$ is the set of indices of those walkers that are still alive after the k'th annihilation. Of course if $T_k = +\infty$ for some $k \geq 1$ then we stop our inductive definition. We define $\mathcal{I}_t = \mathcal{I}_{T_{k-1}}$ for any $T_{k-1} \leq t < T_k$.

Claim 5.4. *The set-valued process $\mathcal{X}'_t = \{X_t^i \,:\, i \in \mathcal{I}_t\}$ has the same law as the annihilating walks described in* (5.8).

The proof of this claim is straightforward and we omit it. From now on we will use this new definition of annihilating walks. For $1 \leq i < j \leq |\mathcal{X}|$ we also define the indicators

$$\eta_{i,j} = \mathbb{1}\left[\,\tau(i,j) < +\infty, \ \tau(i,j) = T_k \text{ for some } k\,\right], \tag{5.11}$$

thus $\eta_{i,j}$ is the indicator that the walkers indexed by i and j annihilate each other before any other walker annihilates either of them. Let us define

$$\mathcal{A}_\infty(\mathcal{X}) = \sum_{i=1}^{|\mathcal{X}|} \sum_{j=i+1}^{|\mathcal{X}|} \eta_{i,j} \tag{5.12}$$

the total number of annihilations that ever occurred. Now we have

$$\mathcal{N}'_\infty(\mathcal{X}) = |\mathcal{X}| - 2\mathcal{A}_\infty(\mathcal{X}), \tag{5.13}$$

since each annihilation event kills two walkers. By (3.6), Lemma 5.3 and (5.13) we only need to prove

$$\mathbb{E}\left(\alpha^{-2\mathcal{A}_\infty(\mathcal{X})}\right) \leq \prod_{1 \leq i < j \leq |\mathcal{X}|} \left(1 + h_R(x_i, x_j)\left(\alpha^{-2} - 1\right)\right), \qquad 0 < \alpha \leq 1 \tag{5.14}$$

in order to complete the proof of Lemma 5.1. Let us introduce auxiliary Bernoulli random variables $\eta_{i,j}^*$, $1 \leq i < j \leq |\mathcal{X}|$ such that they are independent and (recalling the definition of h_R from (2.10))

$$\mathbb{P}[\eta_{i,j}^* = 1] = 1 - \mathbb{P}[\eta_{i,j}^* = 0] = h_R(x_i, x_j) \stackrel{(5.10)}{=} \mathbb{P}[\tau(i,j) < +\infty]. \tag{5.15}$$

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Similarly to (5.12), let us define

$$\mathcal{A}_\infty^*(\mathcal{X}) = \sum_{i=1}^{|\mathcal{X}|} \sum_{j=i+1}^{|\mathcal{X}|} \eta_{i,j}^*. \tag{5.16}$$

Now the right-hand side of (5.14) is equal to $\mathbb{E}\left(\alpha^{-2\mathcal{A}_\infty^*(\mathcal{X})}\right)$, thus in order to prove (5.14) we only need to show that for any $\lambda \geq 0$ we have

$$\mathbb{E}\left[e^{\lambda \mathcal{A}_\infty(\mathcal{X})}\right] \leq \mathbb{E}\left[e^{\lambda \mathcal{A}_\infty^*(\mathcal{X})}\right]. \tag{5.17}$$

By taking the Taylor expansion of the above exponential functions about $\lambda = 0$, we see that we only need to prove

$$\mathbb{E}\left[(\mathcal{A}_\infty(\mathcal{X}))^k\right] \leq \mathbb{E}\left[(\mathcal{A}_\infty^*(\mathcal{X}))^k\right], \qquad k \geq 0.$$

in order to achieve (5.17). By expanding the k'th power of the sums in the definitions of $\mathcal{A}_\infty(\mathcal{X})$ (see (5.12)) and $\mathcal{A}_\infty^*(\mathcal{X})$ (see (5.16)), we see that we only need to prove that the annihilation events are negatively correlated, i.e., that

$$\mathbb{P}\left[\eta_{i_1,j_1} = \cdots = \eta_{i_k,j_k} = 1\right] \leq \mathbb{P}\left[\eta_{i_1,j_1}^* = \cdots = \eta_{i_k,j_k}^* = 1\right] \tag{5.18}$$

holds for any $k \geq 1$ and any $1 \leq i_l < j_l \leq |\mathcal{X}|$, $1 \leq l \leq k$. First, we may assume that the the list of pairs $\{i_1,j_1\}, \ldots, \{i_k,j_k\}$ does not contain the same pair more than once, because we can throw out such duplicates and reduce the value of k without changing the probabilities on either side of (5.18). Second, we may also assume that the sets $\{i_1,j_1\}, \ldots, \{i_k,j_k\}$ are disjoint, because if some of these sets have non-empty intersection, then the left-hand side of (5.18) is equal to zero by the definition of the indicators $\eta_{i,j}$ (see (5.11)): a walker can only be annihilated once. Now if the sets $\{i_1,j_1\}, \ldots, \{i_k,j_k\}$ are disjoint, then

$$\mathbb{P}\left[\eta_{i_1,j_1} = \cdots = \eta_{i_k,j_k} = 1\right] \overset{(5.11)}{\leq} \mathbb{P}\left[\tau(i_1,j_1) < +\infty, \ldots, \tau(i_k,j_k) < +\infty\right] \overset{(*)}{=}$$
$$\prod_{l=1}^k \mathbb{P}\left[\tau(i_l,j_l) < +\infty\right] \overset{(5.15)}{=} \prod_{l=1}^k \mathbb{P}\left[\eta_{i_l,j_l}^* = 1\right] \overset{(**)}{=} \mathbb{P}\left[\eta_{i_1,j_1}^* = \cdots = \eta_{i_k,j_k}^* = 1\right],$$

where $(*)$ holds because the walkers X_t^i, $1 \leq i \leq |\mathcal{X}|$ are independent and the sets $\{i_1,j_1\}, \ldots, \{i_k,j_k\}$ are disjoint, and $(**)$ holds because $\eta_{i,j}^*$, $1 \leq i < j \leq |\mathcal{X}|$ are independent. The proof of (5.18) and Lemma 5.1 is complete.

6 Nearest-neighbour model, $d \geq 5$

The goal of this section is to prove Theorem 1.1. More specifically, we will show that (4.2) holds for any $d \geq 5$ and $R = 1$ and some $\alpha_0 = \alpha_0(d) > 0$. Note that the same proof would work for any $R \geq 1$; the only reason we stick to the classical nearest-neighbour case is to ease notation. We also note that a slight generalization of the method presented in this section would yield a proof of both Theorem 1.1 and Theorem 1.2, however we chose to also present in Section 5 a relatively short argument which only proves Theorem 1.2.

We use the graphical construction of $\xi^{(\alpha)}$ distributed as μ_α (see (3.10)). However, we will often drop the dependence on α from our notation, especially if a particular calculation works for any $\alpha \in (0,1)$.

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We will use the renormalization scheme of Section 4.2. In order to specify the value of ℓ in (4.4) we fix the exponents

$$\varepsilon = \frac{1}{4d}, \qquad \delta = \frac{\varepsilon}{d}. \tag{6.1}$$

The reasons for the choice of ε and δ are discussed in Remark 6.1 and Remark 6.10.

The following choice of ℓ in (4.4) will be suitable for our purposes:

$$\ell = 3^{1/\delta}. \tag{6.2}$$

This choice of ℓ will be used in Section 6.4 to guarantee the convergence of certain geometric series which are similar in flavour to (5.5).

The choice of a large enough L in (4.4) will be specified later in Section 6.4. In Remark 7.2 we explain why $L = 1$ is an insufficient choice in the $R = 1$ case.

Choosing ℓ as in (6.2) we have

$$|\Lambda_N| \overset{(4.8)}{\leq} C^{2^N} \quad \text{for some} \quad C = C(d). \tag{6.3}$$

Combining Definition 2.1, (6.3) and Lemma 4.4 in a union bound, we get, for any N,

$$\mathbb{P}[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c]$$
$$\leq C^{2^N} \max_{\mathcal{T} \in \Lambda_N} \mathbb{P}\left[\bigcap_{m \in T_{(N)}} \{B(\mathcal{T}(m), L) \overset{*\xi}{\longleftrightarrow} B(\mathcal{T}(m), 2L)^c\}\right]. \tag{6.4}$$

We will take a closer look at the crossing events that occur on the right-hand side of (6.4) in Claim 6.2 below. We discuss an open question related to crossing events in the low-dimensional setting in Remark 7.1.

We now fix N and a proper embedding $\mathcal{T} \in \Lambda_N$ with the aim of bounding the probability on the right-hand side in (6.4) (see (6.14) below). We recall the graphical construction (3.2) of the coalescing random walks $(Y_t^x)_{t \geq 0, x \in \mathbb{Z}^d}$, the construction (3.10) of the configuration $(\xi(x))_{x \in \mathbb{Z}^d}$ as well as the definition of ε from (6.1) and let

$$T = L^{2-\varepsilon} \tag{6.5}$$

and, for $x, y \in \mathbb{Z}^d$, we define the events

$$E_x = \left\{\max_{0 \leq t \leq T} |Y_t^x - x| > \frac{1}{4}L\right\}, \tag{6.6}$$

$$E_{x,y} = E_x^c \cap E_y^c \cap \{Y_t^x \neq Y_t^y, \, 0 \leq t \leq T\}, \tag{6.7}$$

$$F_{x,y} = E_{x,y} \cap \{\xi(x) = \xi(y) = 1\}. \tag{6.8}$$

Remark 6.1. *We defined $T \ll L^2$ in (6.5) because we want $\mathbb{P}[E_x] \ll 1$, see (6.16) and (6.40). We note that instead of defining ε as in (6.1), we could in fact take ε as any positive constant which is small enough so that*

$$(2 - \varepsilon)(1 - \frac{d}{2} + \varepsilon) < 2 - d + \frac{1}{4}$$

holds, see (6.47) below.

Claim 6.2. *For any $z \in L\mathbb{Z}^d \overset{(4.5)}{=} \mathcal{L}_0$, the following inclusion holds:*

$$\{B(z, L) \overset{*\xi}{\longleftrightarrow} B(z, 2L)^c\} \subseteq \left(\bigcup_{x \in B(z, 2L)} E_x\right) \cup \left(\bigcup_{\substack{x,y \in B(z,2L) \\ |x-y|=1}} F_{x,y}\right). \tag{6.9}$$

Proof. Assume that the event on the left-hand side occurs. Then there exists a $*$-connected path $\gamma(1),,\ldots,\gamma(k)$ with $|\gamma(1)| = L+1$, $|\gamma(k)| = 2L$ and $\xi(\gamma(i)) = 1$ for each i. For one such path, define

$$i^* = \max\{i \leq k : Y_t^{\gamma(i)} = Y_t^{\gamma(1)} \text{ for some } t \leq T\}.$$

If $i^* = k$, then $E_{\gamma(1)} \cup E_{\gamma(k)}$ occurs, since $|\gamma(1) - \gamma(k)| > L/2$. If $i^* < k$, then the walks $(Y_t^{\gamma(i^*)})$ and $(Y_t^{\gamma(i^*+1)})$ do not meet before time T, so either $E_{\gamma(i^*)} \cup E_{\gamma(i^*+1)}$ or $F_{\gamma(i^*),\gamma(i^*+1)}$ occurs. $\qquad\square$

With (6.9) in mind, given $\mathcal{T} \in \Lambda_N$ we choose two sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{Z}^d$.

Definition 6.3. *The pair* $(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}, \mathcal{Y} \subset \cup_{m \in T_{(N)}} B(\mathcal{T}(m), 2L)$ *is called admissible if*

(i) *for any* $m \in T_{(N)}$, $(|B(\mathcal{T}(m), 2L) \cap \mathcal{X}|, |B(\mathcal{T}(m), 2L) \cap \mathcal{Y}|)$ *is either* $(2,0)$ *or* $(0,1)$;

(ii) *if* $B(\mathcal{T}(m), 2L) \cap \mathcal{X} = \{x, y\}$, *then* $|x - y| = 1$.

The set of all admissible pairs $(\mathcal{X}, \mathcal{Y})$ *associated to* $\mathcal{T}$ *is denoted* $\mathcal{P}_{\mathcal{T}}$.

Lemma 6.4. *Given* $\mathcal{T} \in \Lambda_N$,

1. *For any* $(\mathcal{X}, \mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}$ *we have*

$$\frac{1}{2}|\mathcal{X}| + |\mathcal{Y}| = 2^N. \tag{6.10}$$

2. *There exists* $C = C(d)$ *such that the number of admissible pairs can be bounded by*

$$|\mathcal{P}_{\mathcal{T}}| \leq (CL^d)^{2^N}. \tag{6.11}$$

3. *We have*

$$\bigcap_{m \in T_{(m)}} \{B(\mathcal{T}(m), L) \xleftrightarrow{*\xi} B(\mathcal{T}(m), 2L)^c\} \subset \bigcup_{(\mathcal{X},\mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}} \left(\bigcap_{\substack{x,z \in \mathcal{X} \\ |x-z|=1}} F_{x,z} \right) \cap \left(\bigcap_{y \in \mathcal{Y}} E_y \right). \tag{6.12}$$

4. *For every* $x \in \mathcal{X}$, *we have*

$$\left| \mathcal{X} \cap B(x, \ell^k L/2) \right| \leq 2^k, \qquad k \geq 1. \tag{6.13}$$

Proof. Given an admissible pair $(\mathcal{X}, \mathcal{Y})$ associated to $\mathcal{T}$, define

$$\mathcal{A}_{(\mathcal{X},\mathcal{Y})} = \{m \in T_{(N)} : (|B(\mathcal{T}(m), 2L) \cap \mathcal{X}|, |B(\mathcal{T}(m), 2L) \cap \mathcal{Y}|) = (2,0)\},$$

so that, by Definition 6.3 (i), we have

$$T_{(N)} \backslash \mathcal{A}_{(\mathcal{X},\mathcal{Y})} = \{m \in T_{(N)} : (|B(\mathcal{T}(m), 2L) \cap \mathcal{X}|, |B(\mathcal{T}(m), 2L) \cap \mathcal{Y}|) = (0,1)\}$$

and thus (6.10) holds:

$$2^N = |T_{(N)}| = |\mathcal{A}_{(\mathcal{X},\mathcal{Y})}| + |T_{(N)} \backslash \mathcal{A}_{(\mathcal{X},\mathcal{Y})}| = \frac{1}{2}|\mathcal{X}| + |\mathcal{Y}|.$$

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Additionally, by Definition 6.3, the pair $(\mathcal{X}, \mathcal{Y})$ is determined when we choose $\mathcal{A}_{(\mathcal{X},\mathcal{Y})}$ and then, for each $m \in \mathcal{A}_{(\mathcal{X},\mathcal{Y})}$, we choose two $*$-connected vertices in $B(\mathcal{T}(m), 2L)$ and for each $m \in T_{(N)} \backslash \mathcal{A}_{(\mathcal{X},\mathcal{Y})}$, we choose one vertex in $B(\mathcal{T}(m), 2L)$. Thus (6.11) indeed holds:

$$|\mathcal{P}_{\mathcal{T}}| \leq \sum_{A \subseteq T_{(N)}} \left(|B(2L)| \cdot 3^d \right)^{|A|} \cdot |B(2L)|^{2^N - |A|} \leq (CL^d)^{2^N}.$$

The inclusion (6.12) is a consequence of (6.9) and Definition 6.3.

The bound (6.13) follows from Lemma 4.5 and the fact that for each $m \in T_{(N)}$ we have $|\mathcal{X} \cap B(\mathcal{T}(m), 2L)| \leq 2$ by Definition 6.3(i).

$\square$

Putting together (6.4), (6.11) and (6.12), we obtain

$$\mathbb{P}[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c]$$

$$\leq \left(CL^d \right)^{2^N} \max_{\substack{\mathcal{T} \in \Lambda_N \\ (\mathcal{X},\mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}}} \mathbb{P}\left[\left(\bigcap_{\substack{x,z \in \mathcal{X} \\ |x-z|=1}} F_{x,z} \right) \cap \left(\bigcap_{y \in \mathcal{Y}} E_y \right) \right] \tag{6.14}$$

for some constant $C = C(d)$.

The main ingredient in the proof of (4.2) is the following proposition.

Proposition 6.5. *For every $d \geq 5$, there exist $L_{(0)} \geq 2$ and $\mathcal{C} = \mathcal{C}(d) < +\infty$ such that for any $L \geq L_{(0)}$, any $\alpha \leq L^{2-d+1/4}$ and any $N \geq 1$ we have*

$$\max_{\substack{\mathcal{T} \in \Lambda_N \\ (\mathcal{X},\mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}}} \mathbb{P}\left[\left(\bigcap_{\substack{x,z \in \mathcal{X} \\ |x-z|=1}} F_{x,z} \right) \cap \left(\bigcap_{y \in \mathcal{Y}} E_y \right) \right] \leq \left(\mathcal{C} L^{4-2d+1/2} \right)^{2^N}. \tag{6.15}$$

Together with (6.14) and the assumption $d \geq 5$, this proposition immediately yields the desired result (4.2) if we choose L large enough. We will explain why our method fails to prove (4.2) if $d = 3, 4$ and $R = 1$ in Remark 7.3.

The rest of this section is devoted to the proof of Proposition 6.5.

6.1 Reduction to coalescing walks with initial period of no coalescence

From now on, we fix not only $\mathcal{T} \in \Lambda_N$ (see Definition 4.2), but also $(\mathcal{X}, \mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}$ (see Definition 6.3). Recalling the definition of E_x in (6.16), let us define

$$\beta = \beta(L, d) = \mathbb{P}[E_x] = \mathbb{P}[E_0]. \tag{6.16}$$

Lemma 6.6. *We have*

$$\mathbb{P}\left[\left(\bigcap_{\substack{x,z \in \mathcal{X} \\ |x-z|=1}} F_{x,z} \right) \cap \left(\bigcap_{y \in \mathcal{Y}} E_y \right) \right] \leq \beta^{|\mathcal{Y}|} \cdot \mathbb{E}\left[\alpha^{\mathcal{N}_{\infty}(\mathcal{X})} \cdot \mathbb{1}_{\{\mathcal{N}_T(\mathcal{X}) = |\mathcal{X}|\}} \right]. \tag{6.17}$$

Remark 6.7. *Recall the definition of $\mathcal{N}_{\infty}(\cdot)$ in (3.5). The event $\{\mathcal{N}_T(\mathcal{X}) = |\mathcal{X}|\}$ on the right-hand side of (6.17) is simply the event that the walks started from the vertices of $\mathcal{X}$ do not coalesce with each other before time T.*

Proof. We will use the joint graphical construction of the system of coalescing walks and the configuration $\xi = \xi^{(\alpha)}$ described by equation (3.10). Since our set $\mathcal{X}$ is fixed, we can and will assume that, in the enumeration of $\mathbb{Z}^d$ that was needed for (3.9), the vertices in $\mathcal{X}$ come before all other vertices of $\mathbb{Z}^d$. We can thus write

$$\mathcal{X} = \{x_1, x_2, \ldots, x_{|\mathcal{X}|}\}. \tag{6.18}$$

The occurrence of each event E_y, for $y \in \mathcal{Y}$, can be decided from the Poisson processes in the graphical construction in the space-time box

$$\{z \in \mathbb{Z}^d : |z - y| \leq 1 + L/4\} \times [0, T], \qquad y \in \mathcal{Y}.$$

The occurrence of $\cap_{x,z\in\mathcal{X}, |x-z|=1} F_{x,z}$ can be decided from the random variables $\mathcal{U}_n : 1 \leq n \leq |\mathcal{X}|$ and the Poisson processes in the graphical construction in the space-time set

$$\left(\{w \in \mathbb{Z}^d : \mathrm{dist}(\{w\}, \mathcal{X}) \leq 1 + L/4\} \times [0, T]\right) \cup \left(\mathbb{Z}^d \times (T, \infty)\right).$$

Using (4.10), we see that these space-time sets are all disjoint, and thus

$$\mathbb{P}\left[\left(\bigcap_{\substack{x,z\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right) \cap \left(\bigcap_{y\in\mathcal{Y}} E_y\right)\right] = \mathbb{P}\left[\bigcap_{\substack{x,z\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right] \cdot \prod_{y\in\mathcal{Y}} \mathbb{P}\left[E_y\right] \overset{(6.16)}{=} \beta^{|\mathcal{Y}|} \cdot \mathbb{P}\left[\bigcap_{\substack{x,z\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right].$$

We now define $\mathcal{M}_\mathcal{X} = \{\eta(x_k) : 1 \leq k \leq |\mathcal{X}|\}$, where η is defined in (3.9). For every non-empty $A \subseteq \{1, \ldots, |\mathcal{X}|\}$ we have

$$\mathbb{P}\left[\{\mathcal{M}_\mathcal{X} = A\} \cap \bigcap_{\substack{x,z\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right] \overset{(3.10),(6.8)}{=} \alpha^{|A|} \cdot \mathbb{P}\left[\{\mathcal{M}_\mathcal{X} = A\} \cap \bigcap_{\substack{x,z\in\mathcal{X}:\\|x-z|=1}} E_{x,z}\right]. \tag{6.19}$$

Note that, by (3.5), (3.9) and (6.18), we have $|\mathcal{M}_\mathcal{X}| = \mathcal{N}_\infty(\mathcal{X})$. Therefore

$$\mathbb{P}\left[\bigcap_{\substack{\{x,z\}\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right] \overset{(6.19)}{=} \sum_{k=1}^{|\mathcal{X}|} \alpha^k \cdot \mathbb{P}\left[\{|\mathcal{M}_\mathcal{X}| = k\} \cap \bigcap_{\substack{x,z\in\mathcal{X}:\\|x-z|=1}} E_{x,z}\right]$$

$$= \mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X})} \cdot \mathbb{1}\left\{\bigcap_{\substack{x,z\in\mathcal{X}:\\|x-z|=1}} E_{x,z}\right\}\right] \overset{(4.10),(6.6),(6.7)}{\leq} \mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X})} \cdot \mathbb{1}_{\{\mathcal{N}_T(\mathcal{X})=|\mathcal{X}|\}}\right].$$

The proof of Lemma 6.6 is complete. $\qquad\square$

6.2 Reduction to independent random walks

Our next goal is to bound the expectation on the right-hand side of (6.17). We will need to take a close look at the coalescing walks $\{(Y_t^x)_{t\geq 0} : x \in \mathcal{X}\}$. For this, it will no longer be convenient to work with the graphical construction of the coalescing walks using the Poisson processes $(D_t^{(x,y)})$ that we described in Section 3. Rather, we will switch to a new probability space, in which we will give a different representation of the system of coalescing walks.

The following construction will depend on the set $\mathcal{X}$ which has been fixed at the beginning of Section 6.1 and also on the enumeration of $\mathcal{X}$ that was fixed in (6.18). Let P denote a probability measure under which one defines a collection of processes $\{(X_t^x)_{t\geq 0} : x \in \mathcal{X}\}$ satisfying:

- for each $x \in \mathcal{X}$, $(X_t^x)_{t \geq 0}$ is a continuous-time, nearest neighbor random walk on $\mathbb{Z}^d$ with jump rate 1 and $X_0^x = x$;

- these walks are all independent.

(We emphasize that this is *not* a system of coalescing walks). The expectation operator associated to P is denoted by E. We then define the processes:

- $\{(W_t^x)_{t \geq 0} : x \in \mathcal{X}\}$. They are defined by induction. Put $W_t^{x_1} = X_t^{x_1}$ for all t. Assume $W^{x_1}, \dots, W^{x_n}$ are defined and let

$$\sigma = \inf\{t : X_t^{x_{n+1}} = W_t^{x_k} \text{ for some } k \leq n\}.$$

On $\{\sigma = \infty\}$, let $W_t^{x_{n+1}} = X_t^{x_{n+1}}$ for all t. On $\{\sigma < \infty\}$, let K be the smallest index such that $X_\sigma^{x_{n+1}} = W_\sigma^{x_K}$. Put

$$W_t^{x_{n+1}} = \begin{cases} X_t^{x_{n+1}} & \text{if } t \leq \sigma; \\ W_t^{x_K} & \text{if } t > \sigma. \end{cases}$$

- $\{(Z_t^x)_{t \geq 0} : x \in \mathcal{X}\}$. These are defined exactly as above, with the only difference that in the induction step, σ is defined by

$$\sigma = \inf\{t \geq T : X_t^{x_{n+1}} = Z_t^{x_k} \text{ for some } k \leq n\}.$$

Claim 6.8. *(i)* $\{(W_t^x)_{t \geq 0} : x \in \mathcal{X}\}$ *is a system of coalescing walks started from* $\mathcal{X}$*; in particular, its law under* P *is the same as that of* $\{(Y_t^x)_{t \geq 0} : x \in \mathcal{X}\}$ *under* $\mathbb{P}$.

(ii) $\{(Z_t^x)_{t \geq 0} : x \in \mathcal{X}\}$ *is a system of random walks that move independently (with no coalescence) up to time* T *and after time* T*, behave as a system of coalescing walks.*

The proof of this claim is straightforward and we omit it.
Similarly to (3.5) we also define

$$\mathcal{N}_t^W = |\{W_t^x : x \in \mathcal{X}\}|, \qquad \mathcal{N}_\infty^W = \lim_{t \to \infty} \mathcal{N}_t^W$$
$$\mathcal{N}_t^Z = |\{Z_t^x : x \in \mathcal{X}\}|, \qquad \mathcal{N}_\infty^Z = \lim_{t \to \infty} \mathcal{N}_t^Z.$$

We now have

$$\mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X})} \cdot \mathbb{1}_{\{\mathcal{N}_T(\mathcal{X}) = |\mathcal{X}|\}}\right] = E\left[\alpha^{\mathcal{N}_\infty^W} \cdot \mathbb{1}_{\{X_t^x \neq X_t^y \text{ for all } x,y \in \mathcal{X}, \, x \neq y \text{ and } t \leq T\}}\right]$$
$$= E\left[\alpha^{\mathcal{N}_\infty^Z} \cdot \mathbb{1}_{\{X_t^x \neq X_t^y \text{ for all } x,y \in \mathcal{X}, \, x \neq y \text{ and } t \leq T\}}\right] \leq E\left[\alpha^{\mathcal{N}_\infty^Z}\right]. \qquad (6.20)$$

At this point one might be tempted to apply Lemma 5.3, i.e., to switch from coalescing to annihilating walks. In Remark 7.4 we explain why this method cannot be used to prove Theorem 1.1.

6.3 A stochastic domination result

In this subsection we give definitions and state preliminary results (Lemma 6.11, Lemma 6.12, Proposition 6.9 and Proposition 6.13) which will put us in position to prove Proposition 6.5 in Section 6.4. The following details the interdependence of these results.

	proved in Section	needed for the proof of
• Lemma 6.11	6.5	Proposition 6.5, Proposition 6.13
• Lemma 6.12	6.5	Proposition 6.13
• Proposition 6.13	6.6	Proposition 6.5
• Proposition 6.9	6.7	Lemma 6.11

Recall the notion of the enumeration $\mathcal{X} = \{x_1, x_2, \ldots, x_{|\mathcal{X}|}\}$ from (6.18). Let us define

$$U_n := \mathbb{1}\{\exists\, k < n,\, t \geq T \,:\, X_t^{x_n} = X_t^{x_k}\}, \quad 1 \leq n \leq |\mathcal{X}|, \qquad U = \sum_{n=1}^{|\mathcal{X}|} U_n. \qquad (6.21)$$

In words: U_n is the indicator of the event that the n'th walker hits any of the previous walkers after T. Recalling the construction of Section 6.2 we have

$$\mathcal{N}_\infty^Z = |\mathcal{X}| - \sum_{n=1}^{|\mathcal{X}|} \mathbb{1}\{\exists\, k < n,\, t \geq T \,:\, Z_t^{x_n} = Z_t^{x_k}\} \geq |\mathcal{X}| - U$$

and we can thus bound

$$E\left[\alpha^{\mathcal{N}_\infty^Z}\right] \leq \alpha^{|\mathcal{X}|} \cdot E\left[\alpha^{-U}\right]. \qquad (6.22)$$

Let us now describe the main ideas of this subsection. The indicator variables U_n, $1 \leq n \leq |\mathcal{X}|$ are not independent; however, in Proposition 6.13 we will argue that their sum can be dominated by a sum of independent variables. Let us explain now the heuristics for this domination. Suppose we reveal the paths $(X_t^{x_n})_{t \geq 0}, 1 \leq n \leq |\mathcal{X}|$ one by one, starting with $(X_t^{x_1})_{t \geq 0}$. We think of each path n as a trial: a *success* if it avoids all the previously revealed paths after time T (that is, if $U_n = 0$), and a *failure* otherwise. At the time of revealing path n, it should have a high probability of being a success (since the set $\{X_T^x \,:\, x \in \mathcal{X}\}$ is very sparse), unless some path of index $k < n$ behaved in an atypical manner that makes it exceptionally likely that $(X_t^{x_n})_{t \geq T}$ meets $(X_t^{x_k})_{t \geq T}$. In (6.27) below we will introduce the variable $V_{k,n}$ as the indicator of this event that path k *endangers* trial n. We then rely on two fundamental observations.

- First (see Lemma 6.11): since the random set $\{X_T^x \,:\, x \in \mathcal{X}\}$ is very sparse (as suggested by (6.13)), it is very unlikely that a path endangers a trial, so that the random variables $V_k = \sum_{n > k} V_{k,n}$, which represent the number of trials endangered by each path k, are equal to zero with high probability.

- Second (see Lemma 6.12): if trial n is not endangered by any path of index $k < n$, then it is very likely to be successful.

For any $x, y \in \mathbb{Z}^d$ let us define the random variable

$$M_\infty^{x,y,T} = \mathbb{P}[\exists\, s \geq T \,:\, X_s^y = X_s^x \mid X_u^x \,:\, 0 \leq u < \infty]. \qquad (6.23)$$

(the reason for the ∞ symbol in $M_\infty^{x,y,T}$ will become clear in Section 6.7).

Recall the definition of $\varepsilon = \varepsilon(d)$ and $\delta = \delta(d)$ from (6.1).

Proposition 6.9. *There exists $T_0 = T_0(d) < +\infty$ and $D_0 = D_0(d) < +\infty$ such that*

$$P[M_\infty^{x,y,T} > T^{1-\frac{d}{2}+\varepsilon}] \leq e^{-T^\delta}, \qquad\qquad x, y \in \mathbb{Z}^d, \qquad\qquad\qquad T \geq T_0, \quad (6.24)$$

$$P[M_\infty^{x,y,T} > |x-y|^{2-d+\varepsilon}] \leq e^{-|x-y|^\delta}, \qquad x, y \in \mathbb{Z}^d, |x-y| \geq D_0, \qquad T \geq 0. \quad (6.25)$$

Remark 6.10. *By (2.7) we have $E\left[M_\infty^{x,y,T}\right] \asymp T^{1-d/2} \wedge |x-y|^{2-d}$, thus (6.24) and (6.25) are bounds on the probability that the random variable $M_\infty^{x,y,T}$ deviates too much from its expectation. The reason for the choice of δ in (6.1) as ε/d will become apparent in the proof of Proposition 6.9. The bounds (6.24) are (6.25) are sufficient for our purposes, but we do not claim that they are optimal.*

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The proof of Proposition 6.9 is postponed to Section 6.7.

We now fix T_0 and D_0 as in Proposition 6.9. Given these choices, we may then assume that the renormalization constant L satisfies

$$T \stackrel{(6.5)}{=} L^{2-\varepsilon} \geq T_0, \qquad L \geq D_0. \tag{6.26}$$

We define for $1 \leq k < n \leq |\mathcal{X}|$ the random variables

$$V_{k,n} = \begin{cases} \mathbb{1}\left\{ M_\infty^{x_k,x_n,T} > T^{1-\frac{d}{2}+\varepsilon} \right\}, & \text{if} \quad |x_k - x_n| = 1, \\ \mathbb{1}\left\{ M_\infty^{x_k,x_n,T} > |x_n - x_k|^{2-d+\varepsilon} \right\}, & \text{otherwise.} \end{cases} \tag{6.27}$$

In words: $V_{k,n}$ is the indicator of the event that $(X_t^{x_k})_{t\geq 0}$ endangers $(X_t^{x_n})_{t\geq 0}$. We also define

$$\underline{V}_k = \left(V_{k,k+1}, \dots, V_{k,|\mathcal{X}|} \right), \qquad V_k = \sum_{n=k+1}^{|\mathcal{X}|} V_{k,n}. \tag{6.28}$$

Now by (6.23) and (6.27), for any $1 \leq k < n \leq |\mathcal{X}|$

$$V_{k,n} \quad \text{is measurable with respect to} \quad \sigma(X_t^{x_k} : t \geq 0), \tag{6.29}$$

therefore

$$\underline{V}_1, \dots, \underline{V}_{|\mathcal{X}|} \quad \text{are independent}, \tag{6.30}$$

$$V_1, \dots, V_{|\mathcal{X}|} \quad \text{are independent}. \tag{6.31}$$

Now by (6.28) for any $1 \leq n \leq |\mathcal{X}|$ the random variable V_n is the number of trajectories that the trajectory $(X_t^{x_n})$ endangers. Our stochastic domination result, Proposition 6.13, will involve the total number of paths that either endanger others or are endangered by others; hence, as an intermediate step, in the next lemma we stochastically dominate the random variable $V_n + \mathbb{1}\{V_n > 0\}$, which collects the $\sigma(X_t^{x_n} : t \geq 0)$-measurable terms in the sum that counts the total number of paths that either endanger others or are endangered by others.

Lemma 6.11. *If $e^{-T^\delta} + \sum_{k=1}^\infty 2^k \cdot e^{-(\frac{1}{2}\ell^k L)^\delta} \leq 1$ then for any $n \in \{1, \dots, |\mathcal{X}|\}$ the random variable $V_n + \mathbb{1}\{V_n > 0\}$ is stochastically dominated by a random variable V_0^* with probability mass function $\mathsf{p}_{V_0^*}$ supported on the set of integers $\{0\} \cup \{2^k + 1 : k \geq 0\}$ and given by*

$$\mathsf{p}_{V_0^*}(2) = e^{-T^\delta}, \quad \mathsf{p}_{V_0^*}(2^k + 1) = 2^k \cdot e^{-(\frac{1}{2}\ell^k L)^\delta}, \ k \geq 1, \quad \mathsf{p}_{V_0^*}(0) = 1 - \sum_{k>0} \mathsf{p}_{V_0^*}(k). \tag{6.32}$$

In particular,

$$P[V_n > 0] \leq P[V_0^* > 0] = e^{-T^\delta} + \sum_{k=1}^\infty 2^k \cdot e^{-(\frac{1}{2}\ell^k L)^\delta}. \tag{6.33}$$

The proof of Lemma 6.11 is postponed until Section 6.5.

Recall the definition of U_n from (6.21). In words, the next lemma states that if a path is not endangered by any of the previous paths, then it is very likely to avoid all of them.

Lemma 6.12. *For any $n \in \{1, \dots, |\mathcal{X}|\}$,*

$$P\left[U_n = 1 \mid X_t^{x_k} : 1 \leq k < n, \ t \geq 0\right] \cdot \mathbb{1}\left\{ \sum_{k=1}^{n-1} V_{k,n} = 0 \right\}$$

$$\leq T^{1-\frac{d}{2}+\varepsilon} + \sum_{k=1}^\infty 2^k \left(\frac{1}{2}\ell^k L \right)^{2-d+\varepsilon}. \tag{6.34}$$

The proof of Lemma 6.12 is postponed until Section 6.5.

In order to state the following proposition, and for the sake of clarity, we recapitulate some relevant definitions:

- U_n (for $1 \leq n \leq |\mathcal{X}|$) and U in (6.21);

- $V_{k,n}$ (for $1 \leq k < n \leq |\mathcal{X}|$), $\underline{V}_n$ and V_n (for $1 \leq n \leq |\mathcal{X}|$) in (6.27) and (6.28);

- V_0^* in Lemma 6.11.

We add to this list one more definition; let

$$p := p(L, d) = \frac{T^{1-\frac{d}{2}+\varepsilon} + \sum_{k=1}^{\infty} 2^k \left(\frac{1}{2}\ell^k L\right)^{2-d+\varepsilon}}{P[V_0^* = 0]}. \tag{6.35}$$

Proposition 6.13. *Let $U^* \sim \text{Bin}(|\mathcal{X}|, p)$ and let V^* be independent from U^*, where V^* is the sum of $|\mathcal{X}|$ i.i.d. copies of V_0^*. Then*

$$U \text{ is stochastically dominated by } U^* + V^*. \tag{6.36}$$

Remark 6.14. *If the n'th path is not endangered by previous paths then the parameter of the Bernoulli variable U_n is bounded by the right-hand side of (6.34). The indicators U_n, $1 \leq n \leq |\mathcal{X}|$ are not independent, but we can "hide" their correlations by slightly increasing the parameters of these indicators (c.f. (6.34) and (6.35)) and by adding V^*. Hence, it can happen that the term of index n contributes to the dominating random variable in (6.36) even if it ends up being a success, that is, if $U_n = 0$. This justifies the terminology used in the Introduction: we "throw away" some paths in order to guarantee independence. Similarly, since we add V^* to the dominating random variable in (6.36), we "throw away" paths endangered by others and paths which endanger others.*

This method resembles the "sprinkling technique" which has been successfully applied in the context of random interlacements (e.g., in [Sz12, Section 2]) and Gaussian free field (e.g., in [RS13, Proposition 2.2]).

The proof of Proposition 6.13 will be carried out in Section 6.6 using a coupling argument.

6.4   Proof of Proposition 6.5

By (3.10) the left-hand side of (6.15) is a non-decreasing function of α, so it is enough to prove (6.15) for

$$\alpha = L^{2-d+\frac{1}{4}}. \tag{6.37}$$

Remark 6.15. *Let us comment about the choice of α. For the sake of this heuristic argument let us assume that $\mathcal{Y} = \emptyset$ in (6.39) below, so that $|\mathcal{X}| = 2 \cdot 2^N$, see (6.10). Comparing the combinatorial term $\left(CL^d\right)^{2^N}$ of (6.14) with the terms*

$$\alpha^{|\mathcal{X}|} \cdot e^{|\mathcal{X}|p/\alpha} = \left(\alpha^2 e^{2p/\alpha}\right)^{2^N}$$

in (6.43) below, we see that if we want $\mathbb{P}[B(L_N) \xleftrightarrow{*\xi} B(2L_N)^c] \ll 1$ then it is a good idea to choose α so that

$$\alpha^2 L^d \ll 1, \qquad p/\alpha = \mathcal{O}(1). \tag{6.38}$$

Now p is not much bigger than L^{2-d} (see (6.47) below), so if $d \geq 5$, then (6.37) is a good choice if we want α to satisfy the bounds (6.38).

Let us fix $\mathcal{T} \in \Lambda_N$ (see Definition 4.2) and $(\mathcal{X}, \mathcal{Y}) \in \mathcal{P}_{\mathcal{T}}$ (see Definition 6.3). We have

$$\mathbb{P}\left[\left(\bigcap_{\substack{\{x,z\}\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right) \cap \left(\bigcap_{y\in\mathcal{Y}} E_y\right)\right] \overset{(6.17),(6.20)}{\leq} \beta^{|\mathcal{Y}|} E\left[\alpha^{\mathcal{N}_\infty^Z}\right] \overset{(6.22),(6.36)}{\leq}$$

$$\beta^{|\mathcal{Y}|} \cdot \alpha^{|\mathcal{X}|} \cdot E\left[\left(\frac{1}{\alpha}\right)^{U^*}\right] \cdot E\left[\left(\frac{1}{\alpha}\right)^{V^*}\right]. \quad (6.39)$$

Now we bound the terms on the right-hand side of (6.39).

$$\beta \overset{(6.6),(6.16)}{=} \mathbb{P}\left[\max_{0\leq t\leq T} |Y_t^0| > \frac{1}{4}L\right] \overset{(2.3),(6.5)}{\leq} 2d\exp\left(-\frac{1}{8}L\ln\left(1 + \frac{d}{4}L^{-1+\varepsilon}\right)\right). \quad (6.40)$$

Recall from Proposition 6.13 that $U^* \sim \text{Bin}(|\mathcal{X}|, p)$, where $p = p(L,d)$ was defined in (6.35). For a random variable $Z \sim \text{Bin}(m,r)$ and $\theta \geq 0$, we have $E[\theta^Z] \leq e^{mr\theta}$, thus

$$E\left[\left(\frac{1}{\alpha}\right)^{U^*}\right] \leq e^{|\mathcal{X}|p/\alpha}. \quad (6.41)$$

Recall from Proposition 6.13 that V^* is the sum of $|\mathcal{X}|$ independent copies of V_0^*.

$$E\left[\left(\frac{1}{\alpha}\right)^{V_0^*}\right] \overset{(6.32)}{=} \mathsf{p}_{V_0^*}(0) + \frac{e^{-T^\delta}}{\alpha^2} + \sum_{k=1}^\infty \left(\frac{1}{\alpha}\right)^{2^k+1} 2^k e^{-(\frac{1}{2}\ell^k L)^\delta} \overset{(6.5),(6.37)}{=}$$

$$\mathsf{p}_{V_0^*}(0) + L^{2d-\frac{9}{2}} \cdot e^{-L^{(2-\varepsilon)\delta}} + \sum_{k=1}^\infty \exp\left((2^k+1)(d-\frac{9}{4})\ln(L) + k\ln(2) - \frac{1}{2^\delta}(\ell^\delta)^k L^\delta\right)$$

$$=: q \overset{(*)}{=} q(L,d), \quad (6.42)$$

where in $(*)$ the parameter q is indeed only a function of L and d, because of the definition of ε and δ in (6.1) and ℓ in (6.2). We can thus bound

$$\mathbb{P}\left[\left(\bigcap_{\substack{\{x,z\}\in\mathcal{X}\\|x-z|=1}} F_{x,z}\right) \cap \left(\bigcap_{y\in\mathcal{Y}} E_y\right)\right] \overset{(6.39),(6.41),(6.42)}{\leq} \beta^{|\mathcal{Y}|} \cdot \alpha^{|\mathcal{X}|} \cdot e^{|\mathcal{X}|p/\alpha} \cdot q^{|\mathcal{X}|} =$$

$$\exp\left\{|\mathcal{Y}|\ln\beta + \frac{1}{2}|\mathcal{X}|(2\ln q + 2\ln\alpha + 2p/\alpha)\right\}. \quad (6.43)$$

Recall our definition of ℓ from (6.2). We will choose L big enough so that it satisfies multiple criteria, as we now discuss. By (6.26) we need

$$L > L_{(1)} := T_0^{\frac{1}{2-\varepsilon}} \vee D_0.$$

Having already fixed ε, δ and ℓ, we assume that L satisfies

$$L \geq L_{(2)}, \quad \text{so that} \quad \exp\left(-L^{(2-\varepsilon)\delta}\right) + \sum_{k=1}^\infty 2^k \cdot \exp\left(-(\frac{1}{2}\ell^k L)^\delta\right) \leq \frac{1}{2}, \quad (6.44)$$

so that the condition of Lemma 6.11 is satisfied for L. We will also assume

$$L \geq L_{(3)}, \quad \text{so that} \quad q(L,d) \overset{(**)}{\leq} 2. \tag{6.45}$$

The inequality $(**)$ can be achieved because $p_{V_0^*}(0) \leq 1$ (see (6.32)) and by our choice of ℓ in (6.2) we have $(\ell^\delta)^k = 3^k$, thus the sum of the other terms in the definition (6.42) of q can be made arbitrarily small by making L large. Next we will show that

$$p(L,d) \leq 4L^{2-d+\frac{1}{4}} \overset{(6.37)}{=} 4\alpha \quad \text{if} \quad L \geq L_{(2)}. \tag{6.46}$$

To show that this inequality indeed holds, we estimate

$$p = p(L,d) \overset{(6.5),(6.35),}{=} \frac{L^{(2-\varepsilon)(1-\frac{d}{2}+\varepsilon)} + \sum_{k=1}^\infty 2^k \left(\frac{1}{2}\ell^k L\right)^{2-d+\varepsilon}}{P[V_0^* = 0]}$$

$$\overset{(6.1)}{\leq} \frac{L^{2-d+\frac{1}{4}} + L^{2-d+\varepsilon}\sum_{k=1}^\infty 2^k \left(\frac{1}{2}\ell^k\right)^{2-d+\varepsilon}}{P[V_0^* = 0]} \overset{(6.2),(6.33),(6.44)}{\leq}$$

$$2\left(L^{2-d+\frac{1}{4}} + L^{2-d+\varepsilon}\right) \overset{(6.1)}{\leq} 4L^{2-d+\frac{1}{4}}. \tag{6.47}$$

We can now bound the expression in the exponential in the right-hand side of (6.43):

$$|\mathcal{Y}|\ln(\beta) + \frac{|\mathcal{X}|}{2}\left(2\ln q + \frac{p}{\alpha} + 2\ln\alpha\right) \overset{(6.45),(6.46),}{\leq} |\mathcal{Y}|\ln(\beta) + \frac{|\mathcal{X}|}{2}\left(2\ln 2 + 4 + 2\ln\alpha\right) \overset{(6.37),(6.40)}{\leq}$$

$$|\mathcal{Y}|\left(\ln(2d) - \frac{1}{8}L\ln\left(1 + \frac{d}{4}L^{-1+\varepsilon}\right)\right) + \frac{|\mathcal{X}|}{2}\left(\widehat{\mathcal{C}} + \left(4 - 2d + \frac{1}{2}\right)\ln L\right) \overset{(*)}{\leq}$$

$$\left(|\mathcal{Y}| + \frac{|\mathcal{X}|}{2}\right)\left(\widehat{\mathcal{C}} + \left(4 - 2d + \frac{1}{2}\right)\ln L\right) \overset{(6.10)}{=} 2^N \cdot \left(\widehat{\mathcal{C}} + \left(4 - 2d + \frac{1}{2}\right)\ln L\right), \tag{6.48}$$

where $(*)$ holds for $L \geq L_{(4)}$. Plugging (6.48) back in (6.43), we obtain that the statement of Proposition 6.5 holds with ℓ as in (6.2) and $L_{(0)} := L_{(1)} \vee L_{(2)} \vee L_{(3)} \vee L_{(4)}$.

6.5 Proof of Lemmas 6.11 and 6.12

We now prove the two lemmas of Section 6.3 bounding the probability that random walk paths endanger (Lemma 6.11) and intersect (Lemma 6.12) each other. These proofs simply put together results that have already been established. For Lemma 6.11, we combine Proposition 6.9 – which bounds the probability that a path endangers another path that starts at a given distance from it – with (6.13) – which bounds the number of points of $\mathcal{X}$ that are within a given distance from a fixed point $x \in \mathcal{X}$. Lemma 6.12 is even simpler and follows from a combination of (6.13) with the definition of "endangering" in (6.27).

Proof of Lemma 6.11. Fix $n \in \{1, \ldots, |\mathcal{X}|\}$. We take a bijection

$$\theta : \{0, 1, \ldots, |\mathcal{X}| - n\} \to \{n, n+1, \ldots, |\mathcal{X}|\}$$

with the property that

$$0 = |x_{\theta(0)} - x_n| \leq |x_{\theta(1)} - x_n| \leq \cdots \leq |x_{\theta(|\mathcal{X}|-n)} - x_n|.$$

We have

$$|\{i \geq n : |x_i - x_n| \leq \ell^k L/2\}| \leq |\mathcal{X} \cap B(x_n, \ell^k L/2)| \overset{(6.13)}{\leq} 2^k, \qquad k \geq 1,$$

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so that
$$|x_{\theta(i)} - x_n| > \ell^k L/2 \qquad \text{for all } i \geq 2^k, \ k \geq 1.$$

By our definition of ℓ (see (6.1),(6.2)) and L (see (6.26)) we have $\ell^k L/2 \geq D_0$, for any $k \geq 1$, moreover $T \geq T_0$ (see (6.26)), therefore we can use Proposition 6.9 to bound the probability of the event in the indicator $V_{n,\theta(i)}$ (see (6.27)) that trajectory n endangers trajectory $\theta(i)$:

$$P\left[V_{n,\theta(i)} = 1\right] \leq \begin{cases} e^{-T^\delta} & \text{if } i = 1; \\ e^{-\left(\ell^k L/2\right)^\delta} & \text{if } i \geq 2^k, \ k \geq 1. \end{cases} \tag{6.49}$$

Now, if $i \geq 2$, we have

$$P[V_n \geq i] \leq P[V_n \geq 2^{\lfloor \log_2 i \rfloor}] \leq \sum_{j \geq 2^{\lfloor \log_2 i \rfloor}} P[V_{n,\theta(j)} = 1] \overset{(6.49)}{\leq} \sum_{k=\lfloor \log_2 i \rfloor}^{\infty} 2^k \cdot e^{-\left(\ell^k L/2\right)^\delta}$$

and similarly,

$$P[V_n \geq 1] \leq e^{-T^\delta} + \sum_{k=1}^{\infty} 2^k \cdot e^{-\left(\ell^k L/2\right)^\delta}.$$

We then obtain

$$P[V_n + \mathbb{1}\{V_n > 0\} \geq 1] = P[V_n + \mathbb{1}\{V_n > 0\} \geq 2] \leq e^{-T^\delta} + \sum_{k=1}^{\infty} 2^k \cdot e^{-\left(\ell^k L/2\right)^\delta}$$

and, for $i > 2$,

$$P[V_n + \mathbb{1}\{V_n > 0\} \geq i] \leq P[V_n \geq i - 1] \leq \sum_{k=\lfloor \log_2(i-1) \rfloor}^{\infty} 2^k \cdot e^{-\left(\ell^k L/2\right)^\delta}.$$

The statement of the lemma now follows from comparing these inequalities with the definition of the law of V_0^* in (6.32). $\qquad \square$

Proof of Lemma 6.12. We have

$$P\left[U_n = 1 \,\middle|\, X_t^{x_k} : 1 \leq k < n, \ t \geq 0\right]$$

$$\overset{(6.21)}{\leq} \sum_{m=1}^{n-1} P\left[\exists s \geq T \ : \ X_s^{x_n} = X_s^{x_m} \,\middle|\, X_t^{x_k} : 1 \leq k < n, \ t \geq 0\right]$$

$$= \sum_{m=1}^{n-1} P\left[\exists s \geq T \ : \ X_s^{x_n} = X_s^{x_m} \,\middle|\, X_t^{x_m} : \ t \geq 0\right] \overset{(6.23)}{=} \sum_{m=1}^{n-1} M_\infty^{x_m, x_n, T},$$

so that

$$P\left[U_n = 1 \mid X_t^{x_k} : 1 \leq k < n, \ t \geq 0\right] \cdot \mathbb{1}\left\{\sum_{k=1}^{n-1} V_{k,n} = 0\right\} \leq \sum_{m=1}^{n-1} M_\infty^{x_m, x_n, T} \cdot \mathbb{1}\left\{V_{m,n} = 0\right\}.$$

Now, by (6.27),

$$M_\infty^{x_m, x_n, T} \cdot \mathbb{1}\left\{V_{m,n} = 0\right\} \leq \begin{cases} T^{1-\frac{d}{2}+\varepsilon} & \text{if } \quad |x_m - x_n| = 1; \\ |x_m - x_n|^{2-d+\varepsilon} & \text{otherwise.} \end{cases}$$

The proof of (6.34) can now be completed by applying Definition 6.3 and (6.13) as we did in the proof of Lemma 6.11; we omit the details.

$\qquad \square$

6.6 Proof of Proposition 6.13

In this section we will prove our stochastic domination result using a coupling argument. The key idea lies in the definition of some auxiliary random variables U_n^*, $1 \le n \le |\mathcal{X}|$, so let us start by explaining this informally (the precise definition is given in (6.65)). Define the events

$$A_n = \left\{ \sum_{k=1}^{n-1} V_{k,n} = 0, \ V_n = 0 \right\}, \quad 1 \le n \le |\mathcal{X}|. \tag{6.50}$$

In words: A_n is the event that the n'th random walk path is not endangered by previous paths and does not endanger upcoming paths. We will specify the key properties of $U_n^*, 1 \le n \le |\mathcal{X}|$ using the events A_n in (6.51) and (6.52) below. Suppose we fix n and we reveal all the paths $\{X_t^{x_i} : i < n, \ t \ge 0\}$, and moreover we reveal the vector $\underline{V}_n$ (defined in (6.28)). Given all this information, we are able to determine whether or not A_n has occurred. Now,

(a) assume A_n has occurred. At this point, we have full knowledge of all the paths with index smaller than n, and also some partial knowledge of the n'th path: we know $\underline{V}_n$, in fact we know that A_n occurred, which implies $V_n = \sum_{k=n+1}^{|\mathcal{X}|} V_{n,k} = 0$. In Lemma 6.17, we argue that the conditional probability of $\{U_n = 1\}$ given all this information is at most p (defined in (6.35)), a number that is not much larger than the bound we had given in (6.34) (which did not include the conditioning on $\{V_n = 0\}$). We are thus able to define U_n^* so that $U_n \le U_n^*$ and $U_n^* \sim \text{Bernoulli}(p)$.

(b) if A_n has not occurred, we simply prescribe (using extra, auxiliary randomness) that U_n^* is Bernoulli(p).

The sum $\sum_{n=1}^{|\mathcal{X}|} U_n \cdot \mathbb{1}_{A_n}$ is then dominated by $\sum_{n=1}^{|\mathcal{X}|} U_n^* \cdot \mathbb{1}_{A_n} \le \sum_{n=1}^{|\mathcal{X}|} U_n^*$, and the sum $\sum_{n=1}^{|\mathcal{X}|} U_n \cdot \mathbb{1}_{A_n^c}$ is dominated by $\sum_{n=1}^{|\mathcal{X}|} (V_n + \mathbb{1}\{V_n > 0\})$ (see (6.53) below), which in turn is dominated by $\sum_{n=1}^{|\mathcal{X}|} V_n^*$, a sum of i.i.d. random variables distributed as V_0^* from Lemma 6.11. Finally, the desired independence properties of our construction follow from the fact that the distribution of U_n^* is the same regardless of the conditioning; this is formalized in Lemma 6.18.

Proof of Proposition 6.13. In a series of lemmas we will construct, by extending the probability space of the random walks $(X_t^{x_n}), 1 \le n \le |\mathcal{X}|$, random variables $U_1^*, \ldots, U_{|\mathcal{X}|}^*$ satisfying

$$U_n \cdot \mathbb{1}_{A_n} \le U_n^* \cdot \mathbb{1}_{A_n}; \tag{6.51}$$
$$U_n^* \sim \text{Ber}(p) \text{ and is independent of } \left((U_k^*)_{1 \le k \le n-1}, (\underline{V}_k)_{1 \le k \le |\mathcal{X}|} \right). \tag{6.52}$$

Here we show how this construction implies (6.36). We let $U^* = \sum_{n=1}^{|\mathcal{X}|} U_n^*$. We have

$$U \overset{(6.21)}{=} \sum_{n=1}^{|\mathcal{X}|} U_n \overset{(6.51)}{\le} \sum_{n=1}^{|\mathcal{X}|} (U_n^* + \mathbb{1}_{A_n^c}) \overset{(6.50)}{\le} U^* + \sum_{n=1}^{|\mathcal{X}|} \left(\mathbb{1}_{\{V_n > 0\}} + \sum_{k=1}^{n-1} V_{k,n} \right)$$

$$= U^* + \sum_{n=1}^{|\mathcal{X}|} \mathbb{1}_{\{V_n > 0\}} + \sum_{k=1}^{|\mathcal{X}|} \sum_{n=k+1}^{|\mathcal{X}|} V_{k,n} \overset{(6.28)}{=} U^* + \sum_{k=1}^{|\mathcal{X}|} (V_k + \mathbb{1}_{\{V_k > 0\}}). \tag{6.53}$$

Now (6.52) implies that $U^* \sim \text{Bin}(|\mathcal{X}|, p)$ and is independent of $V_1, \ldots, V_{|\mathcal{X}|}$, which are also independent by (6.31). Putting this together with Lemma 6.11 we obtain that

$$U^* + \sum_{n=1}^{|\mathcal{X}|} (\mathbb{1}_{\{V_n > 0\}} + V_n) \text{ is stochastically dominated by } U^* + V^*,$$

where V^* is a sum of $|\mathcal{X}|$ independent copies of V_0^*. This completes the proof of Proposition 6.13 given (6.51) and (6.52). $\qquad\square$

The rest of this subsection is devoted to the construction of random variables $U_1^*, \ldots, U_{|\mathcal{X}|}^*$ satisfying (6.51) and (6.52). We start recalling a few standard facts about conditional expectations.

Lemma 6.16. *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space.*

1. *[Wi91, Section 9.7, Property (k)] If X is an $\mathcal{F}$-measurable and bounded random variable (r.v.), $\mathcal{G}, \mathcal{G}' \subset \mathcal{F}$ are sigma-algebras and $\mathcal{G}'$ is independent of $\sigma(\mathcal{G} \cup \sigma(X))$, then*

$$\mathsf{E}[X \,|\, \mathcal{G}, \mathcal{G}'] = \mathsf{E}[X \,|\, \mathcal{G}]. \tag{6.54}$$

2. *[Kl05, Theorem 2.24] Let $\mathcal{H} \subseteq \mathcal{F}$ be a sigma-algebra, Z be an $\mathcal{F}$-measurable r.v. independent of $\mathcal{H}$, Y be an $\mathcal{H}$-measurable r.v., and $f : \mathbb{R}^2 \to \mathbb{R}$ be Borel-measurable and bounded. If we define $g(y) := \mathsf{E}[f(y, Z)]$ for $y \in \mathbb{R}$ then*

$$\mathsf{E}\left[f(Y, Z) \,|\, \mathcal{H}\right] = g(Y). \tag{6.55}$$

We now extend the probability space of the walks $X_t^x, t \geq 0, x \in \mathcal{X}$ with an independent collection of auxiliary random variables

$$\zeta_k, \ 1 \leq k \leq |\mathcal{X}|, \qquad \text{i.i.d. and unifomly distributed on } [0, 1]. \tag{6.56}$$

For $1 \leq n \leq |\mathcal{X}|$, we introduce the sigma-field

$$\sigma_n := \sigma\left((\zeta_k : 1 \leq k \leq n), \ (\underline{V}_k : 1 \leq k \leq |\mathcal{X}|), \ (X_t^{x_k} : t \geq 0, \ 1 \leq k \leq n)\right)$$
$$\stackrel{(6.28),(6.29)}{=} \sigma\left((\zeta_k : 1 \leq k \leq n), \ (\underline{V}_k : n < k \leq |\mathcal{X}|), \ (X_t^{x_k} : t \geq 0, \ 1 \leq k \leq n)\right). \tag{6.57}$$

Recalling (6.21) we also define the random variable

$$p_n := P\left[U_n = 1 \,|\, \sigma_{n-1}\right] = P\left[U_n = 1 \,\left|\, \begin{array}{c} (\zeta_k)_{1 \leq k \leq n-1}, \ (\underline{V}_k)_{n \leq k \leq |\mathcal{X}|}, \\ X_t^{x_k}, t \geq 0, 1 \leq k \leq n-1 \end{array} \right.\right]. \tag{6.58}$$

Lemma 6.17. *The number p defined in (6.35), the event A_n, $1 \leq n \leq |\mathcal{X}|$ defined in (6.50) and the random variable p_n defined in (6.58) satisfy*

$$p_n \cdot \mathbb{1}_{A_n} \leq p. \tag{6.59}$$

Proof. By (6.21), (6.29), (6.54) and the fact that $(X_t^{x_k}), 1 \leq k \leq |\mathcal{X}|$ are independent random walks, we have

$$p_n = P\left[U_n = 1 \,|\, \underline{V}_n, \ X_t^{x_k}, t \geq 0, \ k \leq n-1\right].$$

Recall from (6.28) that V_n is the indicator of the event that the path $(X_t^{x_n})$ does not endanger any upcoming paths. We now claim that

$$p_n \cdot \mathbb{1}\{V_n = 0\} = \frac{P\left[U_n = 1, \ V_n = 0 \,|\, X_t^{x_k}, t \geq 0, \ k \leq n-1\right]}{P\left[V_n = 0 \,|\, X_t^{x_k}, t \geq 0, \ k \leq n-1\right]} \cdot \mathbb{1}\{V_n = 0\}. \tag{6.60}$$

Before we prove this, let us see how it allows us to conclude. Noting that

$$P\left[V_n = 0 \,|\, X_t^{x_k}, t \geq 0, \ k \leq n-1\right] \stackrel{(6.29),(6.54)}{=} P\left[V_n = 0\right], \tag{6.61}$$

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we have

$$p_n \cdot \mathbb{1}_{A_n} \overset{(6.50)}{=} p_n \cdot \mathbb{1}\{V_n = 0\} \mathbb{1}\left\{\sum_{k=1}^{n-1} V_{k,n} = 0\right\}$$

$$\overset{(6.60),(6.61)}{\leq} \frac{P\left[U_n = 1 \mid X_t^{x_k}, t \geq 0,\ k \leq n-1\right] \cdot \mathbb{1}\left\{\sum_{k=1}^{n-1} V_{k,n} = 0\right\}}{P[V_n = 0]} \cdot \mathbb{1}\{V_n = 0\}.$$

By applying Lemma 6.12 to the numerator and Lemma 6.11 to the denominator, we conclude that the right-hand side is smaller than p (see (6.35)), thus (6.59) holds.

It remains to prove (6.60). To this end, we abbreviate

$$\mathcal{V} = \sigma(\underline{V}_n), \qquad \mathcal{G} = \sigma(X_t^{x_k} : t \geq 0,\ 1 \leq k \leq n-1),$$

thus $p_n = P[U_n = 1 | \mathcal{G}, \mathcal{V}]$, so we must then prove that

$$\mathbb{1}_{\{V_n = 0\}} \cdot P[V_n = 0|\mathcal{G}] \cdot P[U_n = 1|\mathcal{G}, \mathcal{V}] = \mathbb{1}_{\{V_n=0\}} \cdot P[U_n = 1,\ V_n = 0|\mathcal{G}]. \tag{6.62}$$

Since V_n is $\mathcal{V}$-measurable and $P[V_n = 0|\mathcal{G}]$ is $\mathcal{G}$-measurable, (6.62) is the same as

$$E\left[\mathbb{1}_{\{U_n = 1,\ V_n = 0\}} \cdot P[V_n = 0|\mathcal{G}] \mid \mathcal{G}, \mathcal{V}\right] = \mathbb{1}_{\{V_n = 0\}} \cdot P[U_n = 1,\ V_n = 0|\mathcal{G}]. \tag{6.63}$$

We now check that the right-hand side of (6.63) satisfies the definition of the left-hand side. First, note that $\mathbb{1}_{\{V_n=0\}} \cdot P[U_n = 1,\ V_n = 0|\mathcal{G}]$ is measurable with respect to $\sigma(\mathcal{G}, \mathcal{V})$. Second, for any event $C \in \sigma(\mathcal{G}, \mathcal{V})$, we must check that

$$E\left[\mathbb{1}_C \cdot \mathbb{1}_{\{U_n=1,\ V_n=0\}} \cdot P[V_n = 0|\mathcal{G}]\right] = E[\mathbb{1}_C \cdot \mathbb{1}_{\{V_n=0\}} \cdot P[U_n = 1,\ V_n = 0|\mathcal{G}]]. \tag{6.64}$$

Now $\mathcal{V}$ is an atomic sigma-algebra (since it is generated by finitely many events, see (6.28)) and $\{V_n = 0\}$ is an atom of $\mathcal{V}$, therefore the event $C \cap \{V_n = 0\}$ is equal to $G \cap \{V_n = 0\}$ for some $G \in \mathcal{G}$. Using this, (6.64) is equivalent to

$$E\left[\mathbb{1}_G \cdot \mathbb{1}_{\{U_n=1,\ V_n=0\}} \cdot P[V_n = 0|\mathcal{G}]\right] = E[\mathbb{1}_{\{V_n=0\}} \cdot E[\mathbb{1}_G \cdot \mathbb{1}_{\{U_n=1,\ V_n=0\}}|\mathcal{G}]].$$

By taking $E[\cdot|\mathcal{G}]$ inside the expectation, we see that both sides are equal to

$$E\left[E[\mathbb{1}_G \cdot \mathbb{1}_{\{U_n=1,\ V_n=0\}}|\mathcal{G}] \cdot P[V_n = 0|\mathcal{G}]\right].$$

The proof of Lemma 6.17 is complete. $\qquad\square$

We are now ready to define

$$U_n^* := \mathbb{1}_{A_n} \cdot \left(U_n + (1 - U_n) \cdot \mathbb{1}\left\{\zeta_n \leq \frac{p - p_n}{1 - p_n}\right\}\right) + \mathbb{1}_{A_n^c} \cdot \mathbb{1}\{\zeta_n \leq p\}. \tag{6.65}$$

Lemma 6.18. U_n^* *satisfies* (6.51) *and* (6.52).

Proof. That (6.51) is satisfied is obvious, so we turn to (6.52).

Recalling the definitions of U_n from (6.21), $\underline{V}_k$ from (6.28), A_n from (6.50), ζ_k from (6.56), σ_n from (6.57) and p_n from (6.58) we note that

$$A_n,\ U_n,\ \zeta_n \text{ and } p_n \text{ are all } \sigma_n\text{-measurable}, 1 \leq n \leq |\mathcal{X}|.$$

Consequently, $U_1^*, \ldots, U_{n-1}^*$ are all σ_{n-1}-measurable. Since $(\underline{V}_k)_{n \leq k \leq |\mathcal{X}|}$ are also σ_{n-1}-measurable, we see that (6.52) will follow once we show that

$$\mathbb{E}[U_n^* \mid \sigma_{n-1}] = p. \tag{6.66}$$

We start with

$$E\left[U_n^* \mid \sigma_{n-1}\right] \overset{(6.65)}{=} E\left[U_n \cdot \mathbb{1}_{A_n} \mid \sigma_{n-1}\right] + E\left[(1-U_n) \cdot \mathbb{1}_{A_n} \cdot \mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}\right]$$
$$+ P\left[A_n^c \cap \{\zeta_n \leq p\} \mid \sigma_{n-1}\right]$$

$$\overset{(*)}{=} \mathbb{1}_{A_n} \cdot E\left[U_n \mid \sigma_{n-1}\right] + \mathbb{1}_{A_n} \cdot E\left[(1-U_n) \cdot \mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}\right] + \mathbb{1}_{A_n^c} \cdot P\left[\zeta_n \leq p \mid \sigma_{n-1}\right]$$

$$\overset{(6.58)}{=} \mathbb{1}_{A_n} \cdot p_n + \mathbb{1}_{A_n} \cdot E\left[(1-U_n) \cdot \mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}\right] + \mathbb{1}_{A_n^c} \cdot p,$$

where in $(*)$ we used that $A_n \in \sigma_{n-1}$. The proof of (6.66) will be complete once we show

$$\mathbb{1}_{A_n} \cdot E\left[(1-U_n) \cdot \mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}\right] = \mathbb{1}_{A_n} \cdot (p - p_n). \tag{6.67}$$

To this end, we first calculate

$$\mathbb{1}_{A_n} \cdot E\left[(1-U_n) \cdot \mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}, \, (X_t^{x_n})_{t \geq 0}\right] \tag{6.68}$$

$$= \mathbb{1}_{A_n} \cdot (1-U_n) \cdot E\left[\mathbb{1}_{\left\{\zeta_n \leq \frac{p-p_n}{1-p_n}\right\}} \,\middle|\, \sigma_{n-1}, \, (X_t^{x_n})_{t \geq 0}\right]$$

$$\overset{(**)}{=} \mathbb{1}_{A_n} \cdot (1-U_n) \cdot \frac{p-p_n}{1-p_n}, \tag{6.69}$$

where $(**)$ follows from (6.59) and (6.55), which can be applied because p_n is σ_{n-1}-measurable and ζ_n is independent of $\sigma(\sigma_{n-1}, (X_t^{x_n})_{t \geq 0})$.

To conclude the proof of (6.67), note that taking $E[\,\cdot\,|\sigma_{n-1}]$ on (6.68) (and again using the fact that $A_n \in \sigma_{n-1}$) gives the left-hand side of (6.67), whereas taking $E[\,\cdot\,|\sigma_{n-1}]$ on (6.69) (and using (6.58)) gives the right-hand side of (6.67). The proof of (6.66) and Lemma 6.18 is complete. $\qquad\square$

6.7 Proof of Proposition 6.9

The goal of this section is to prove Proposition 6.9. Recall the definition of $M_\infty^{x,y,T}$ from (6.23). We generalize this definition by setting, for any $t \in [T, \infty)$,

$$M_t^{x,y,T} = P[\exists\, u \geq T \,:\, X_u^y = X_u^x \mid \mathcal{F}_t^x], \qquad \mathcal{F}_t^x = \sigma\left(X_u^x : 0 \leq u \leq t\right). \tag{6.70}$$

This defines a martingale indexed by $t \in [T, \infty]$. In order to simplify notation, we will omit the superscripts that indicate dependence on x, y and T.

Let us now outline the strategy of proof of (6.24) (the proof of (6.25) will follow as a corollary). As suggested in Remark 6.10, we have $E[M_\infty] \leq CT^{1-d/2}$. In fact we have $M_T \leq C_0 T^{1-d/2}$ for some deterministic constant C_0 (see (6.72)), because X^y walks independently of X^x, so the conditional probability that they meet after T given any possible outcome of X_T^x is bounded by $C_0 T^{1-d/2}$. Given this bound on M_T, the event $\{M_\infty > T^{1-\frac{d}{2}+\varepsilon}\}$ can only occur if the terminal value M_∞ of the martingale deviates too much from M_T. This is where Theorem 2.4 comes into play. In order to apply this theorem, we will obtain estimates on the size of the jumps of (M_t) for $t \geq T$ and on its predictable quadratic variation $\langle M \rangle_\infty - \langle M \rangle_T$; these estimates are given in (6.73) and (6.74). We derive these estimates by first giving a useful equivalent definition of M_t in Claim 6.20 and then comparing M_t with $M_t^{(e)}$, which arises from M_t by artificially forcing the walk $(X_s^x)_{s \geq 0}$ to jump at time t in the direction of the unit vector $e \sim 0$, see Definition 6.21. Specifically, in Lemma 6.24 we show that the jumps of M can be bounded in terms of $|M_t - M_t^{(e)}|$ and the predictable quadratic variation $\langle M \rangle_\infty - \langle M \rangle_T$ can be expressed as

an integral of $(M_t - M_t^{(e)})^2$. The difference $|M_t - M_t^{(e)}|$ is bounded in Lemma 6.25 using the random walk facts of Section 2.3.

Recall that (M_t) is càdlàg. Denote by

$$\Delta M_T = \sup_{t \geq T} |M_t - M_{t-}| \tag{6.71}$$

the maximal jump size of M_t after time T. Recall the notion of $\langle M \rangle_t$ from Definition 2.3.

Lemma 6.19. *There exist dimension-dependent constants C_0, C_1, C_2 such that the following bounds almost surely hold:*

$$M_T \leq C_0 T^{1-\frac{d}{2}}, \tag{6.72}$$

$$\Delta M_T \leq C_1 T^{\frac{1}{2}-\frac{d}{2}}, \tag{6.73}$$

$$\langle M \rangle_\infty - \langle M \rangle_T \leq C_2 T^{2-d}. \tag{6.74}$$

Before we prove Lemma 6.19 we use it to prove Proposition 6.9.

Proof of Proposition 6.9. We first prove (6.24):

$$P\left[M_\infty > T^{1-\frac{d}{2}+\varepsilon}\right] \overset{(6.72)}{\leq} P\left[M_\infty - M_T > T^{1-\frac{d}{2}+\varepsilon} - C_0 T^{1-\frac{d}{2}}\right] \overset{(*)}{\leq}$$

$$P\left[M_\infty - M_T > \frac{1}{2}T^{1-\frac{d}{2}+\varepsilon}\right] \overset{(2.2),(6.73),(6.74)}{\leq}$$

$$\exp\left(-\frac{1}{2}\frac{\frac{1}{2}T^{1-\frac{d}{2}+\varepsilon}}{C_1 T^{\frac{1}{2}-\frac{d}{2}}}\ln\left(1 + \frac{\frac{1}{2}T^{1-\frac{d}{2}+\varepsilon}C_1 T^{\frac{1}{2}-\frac{d}{2}}}{C_2 T^{2-d}}\right)\right) =$$

$$\exp\left(-\frac{1}{4C_1}T^{\frac{1}{2}+\varepsilon}\ln\left(1 + \frac{C_1}{2C_2}T^{-\frac{1}{2}+\varepsilon}\right)\right) \overset{(*)}{\leq} \exp\left(-T^\varepsilon\right), \quad (6.75)$$

where the inequalities marked by $(*)$ hold if T is large enough. We have proved that (6.24) would hold even if we defined δ to be equal to ε, so it also holds if $\delta = \varepsilon/d$ as in (6.1).

We now turn to (6.25). We fix a small constant $\sigma \in (0,1)$ (to be chosen later in (6.83) as $\sigma = \varepsilon/4$). We keep the notation σ with the hope that it makes the proof more transparent. Given this σ we define

$$\widehat{T} = 2^{\frac{1}{d/2-1-\sigma}} \cdot |x-y|^{2-\frac{2(\varepsilon-2\sigma)}{d-2-2\sigma}}, \tag{6.76}$$

so that

$$\widehat{T}^{1-\frac{d}{2}+\sigma} = \frac{1}{2}|x-y|^{2-d+\varepsilon}. \tag{6.77}$$

Note that, since $\frac{2(\varepsilon-2\sigma)}{d-2-2\sigma} < \frac{2\varepsilon}{d-4} \overset{(6.1)}{<} 1$, we have $2 - \frac{2(\varepsilon-2\sigma)}{d-2-2\sigma} > 1$, so (6.76) implies that

$$\widehat{T} > |x-y| \quad \text{if} \quad x \neq y \in \mathbb{Z}^d. \tag{6.78}$$

Having fixed some $x \neq y \in \mathbb{Z}^d$, we now start to bound the left-hand side of (6.25).

$$P[M_\infty > |x-y|^{2-d+\varepsilon}] \overset{(6.70)}{\leq} P\left[P[\exists\, u \geq \widehat{T} \ : \ X_u^y = X_u^x \mid \mathcal{F}_\infty^x] > \frac{1}{2}|x-y|^{2-d+\varepsilon}\right] +$$

$$P\left[P[\exists\, u \leq \widehat{T} \ : \ X_u^y = X_u^x \mid \mathcal{F}_\infty^x] > \frac{1}{2}|x-y|^{2-d+\varepsilon}\right]. \quad (6.79)$$

Assuming that $|x - y|$ is large enough (and hence $\widehat{T}$ is large enough), we bound the first term on the right-hand side of (6.79) analogously to (6.75), with $\widehat{T}$ in place of T and σ in place of ε:

$$P\left[P[\exists\, u \geq \widehat{T} \; : \; X_u^y = X_u^x \mid \mathcal{F}_\infty^x] > \frac{1}{2}|x - y|^{2-d+\varepsilon}\right] \overset{(6.77)}{\leq} \exp\left(-\widehat{T}^\sigma\right) \overset{(6.78)}{\leq} e^{-|x-y|^\sigma}.$$

(6.80)

Now we bound the second term on the right-hand side of (6.79) using Markov's inequality:

$$P\left[P[\exists\, u \leq \widehat{T} \; : \; X_u^y = X_u^x \mid \mathcal{F}_\infty^x] > \frac{1}{2}|x - y|^{2-d+\varepsilon}\right] \leq \frac{P[\exists\, u \leq \widehat{T} \; : \; X_u^y = X_u^x]}{\frac{1}{2}|x - y|^{2-d+\varepsilon}}, \quad (6.81)$$

and

$$P[\exists\, u \leq \widehat{T} \; : \; X_u^y = X_u^x] \overset{(2.9)}{=} P[\exists\, u \leq 2\widehat{T} \; : \; X_u^{x-y} = 0] \leq P\left[\max_{u \leq 2\widehat{T}} |X_u^0| \geq |x - y|\right]$$

$$\overset{(2.3)}{\leq} 2d\exp\left(-\frac{1}{2}|x - y|\ln\left(1 + \frac{d \cdot |x - y|}{2\widehat{T}}\right)\right). \quad (6.82)$$

The expression on the right-hand side of (6.82) suggests that $\widehat{T}$ should be much smaller than $|x - y|^2$. With this in mind, and inspecting (6.76), we set

$$\sigma = \varepsilon/4. \tag{6.83}$$

If $|x - y|$ is large enough, (6.76) then implies that

$$2d\exp\left(-\frac{1}{2}|x - y|\ln\left(1 + \frac{d \cdot |x - y|}{2\widehat{T}}\right)\right) \leq \exp\left(-|x - y|^{\frac{\varepsilon}{d-2}}\right). \tag{6.84}$$

Putting the above bounds together we obtain

$$P[M_\infty > |x - y|^{2-d+\varepsilon}] \overset{(6.79),(6.80),(6.81),(6.82),(6.84)}{\leq}$$

$$e^{-|x-y|^{\varepsilon/4}} + \frac{\exp\left(-|x - y|^{\frac{\varepsilon}{d-2}}\right)}{\frac{1}{2}|x - y|^{2-d+\varepsilon}} \overset{(*)}{\leq} \exp\left(-|x - y|^{\varepsilon/d}\right),$$

where $(*)$ holds if $|x - y|$ is large enough. This completes the proof of (6.25) with $\delta = \varepsilon/d$, as required by (6.1). The proof of Proposition 6.9 is complete, given Lemma 6.19. $\square$

Now we prepare the ground for the proof of Lemma 6.19. We begin with stating a useful equivalent formula for the martingale M_t, $t \in [T, \infty)$.

Claim 6.20. *For any $t \geq T$,*

$$M_t = P[\,\exists\, u \geq T : X_u^y = X_{u \wedge t}^x \mid \mathcal{F}_t^x]. \tag{6.85}$$

Proof. Given $t \geq T$ let us define the event

$$A = \{\,\exists\, s \in [T, t) \; : \; X_s^y = X_s^x\,\}. \tag{6.86}$$

The statement follows from

$$M_t \overset{(6.70)}{=} P[A \mid \mathcal{F}_t^x]$$
$$+ \sum_{v, w \in \mathbb{Z}^d} P[A^c \cap \{X_t^y = w\} \mid \mathcal{F}_t^x] \cdot \mathbb{1}[X_t^x = v] \cdot P[\exists\, s \geq 0 : X_s^w = X_s^v]$$
$$\overset{(2.9)}{=} P[A \mid \mathcal{F}_t^x] + \sum_{v, w \in \mathbb{Z}^d} P[A^c \cap \{X_t^y = w\} \mid \mathcal{F}_t^x] \cdot \mathbb{1}[X_t^x = v] \cdot P[\exists\, s \geq 0 : X_s^w = v]$$
$$= P[A \mid \mathcal{F}_t^x] + P[A^c \cap \{\exists\, s \geq t : X_s^y = X_t^x\} \mid \mathcal{F}_t^x] = P[\exists\, s \geq T : X_s^y = X_{s \wedge t}^x \mid \mathcal{F}_t^x].$$

$\square$

Definition 6.21. *For any $t \in [T, +\infty)$ let us define for $e \in \mathbb{Z}^d$, $e \sim 0$, the random variable*

$$M_t^{(e)} = P[\exists u \geq T \ : \ X_u^y = X_{u \wedge t}^x + e \cdot \mathbb{1}_{\{u \geq t\}} \mid \mathcal{F}_t^x]. \tag{6.87}$$

The $\mathcal{F}_t^x$-measurable random variable $M_t^{(e)}$ is a perturbed version of M_t where we artificially force the walk $(X_s^x)_{s \geq 0}$ to jump at time t in the direction of the unit vector $e \sim 0$. Recall that we assume that our random walks and martingales are càdlàg.

Definition 6.22. *Denote by $\tau_1 < \tau_2 < \ldots$ the jump times of the random walk (X_t^x) and let $\tau_0 = 0$. For any $n \geq 1$ let $e_n = X_{\tau_n}^x - X_{\tau_n-}^x$ denote the direction of the jump of (X_t^x) at time τ_n.*

Note that

$$(\tau_n - \tau_{n-1})_{n \geq 1} \text{ are i.i.d. with Exp(1) distribution.} \tag{6.88}$$

The next claim states that M_t only jumps when X_t^x jumps and in between jumps M_t is constant.

Claim 6.23. *For any $n = 1, 2, \ldots$ we have*

$$M_{\tau_n} = M_{\tau_n-}^{(e_n)}, \tag{6.89}$$

$$M_t = M_{\tau_{n-1}}, \quad \tau_{n-1} \leq t < \tau_n. \tag{6.90}$$

Proof. Let $\gamma : [0, \infty) \to \mathbb{Z}^d$ be a càdlàg function with $\gamma(0) = x$. This γ will play the role of a possible realization of of $(X_u^x)_{u \geq 0}$. Assume that for some $T \leq s < t$ and $e \in \mathbb{Z}^d$, $e \sim 0$ the trajectory γ satisfies

$$\gamma(r) = \gamma(s) \text{ for all } r \in [s, t] \quad \text{and} \quad \gamma(t) = \gamma(t_-) + e. \tag{6.91}$$

The two statements of the claim are immediate consequences of (6.85), (6.87) and

$$P[\exists u \geq T \ : \ X_u^y = \gamma(u \wedge t)] = \lim_{r \nearrow t} P[\exists u \geq T \ : \ X_u^y = \gamma(u \wedge r) + e \cdot \mathbb{1}_{\{u \geq r\}}]; \tag{6.92}$$

$$P[\exists u \geq T \ : \ X_u^y = \gamma(u \wedge r)] = P[\exists u \geq T \ : \ X_u^y = \gamma(u \wedge s)] \text{ for all } r \in [s, t). \tag{6.93}$$

(6.93) holds because, by (6.91), $\gamma(u \wedge s) = \gamma(u \wedge r)$ for all u. To establish (6.92) we note that, again by (6.91), for fixed $r \in (s, t)$ the symmetric difference of the events

$$\{ \exists u \geq T \ : \ X_u^y = \gamma(u \wedge t) \} \quad \text{and} \quad \{ \exists u \geq T \ : \ X_u^y = \gamma(u \wedge r) + e \cdot \mathbb{1}_{\{u \geq r\}} \}$$

is contained in the event that X^y has a jump between times r and t. $\qquad \square$

Lemma 6.24. *We have*

$$\Delta M_T \leq \sup_{t \geq T} \max_{e \sim 0} |M_t^{(e)} - M_t|, \tag{6.94}$$

$$\langle M \rangle_t - \langle M \rangle_T = \frac{1}{2d} \sum_{e \sim 0} \int_T^t (M_s^{(e)} - M_s)^2 \, \mathrm{d}s. \tag{6.95}$$

Proof. The inequality (6.94) immediately follows from (6.71) and Claim 6.23.

Now we prove (6.95). Recall Definition 2.3. The right-hand side of (6.95) is adapted to $(\mathcal{F}_t^x)$ and continuous in t, hence it is predictable (see Definition 2.2), thus we only need to check that for any $T \leq s \leq t$ we have

$$E[M_t^2 - M_s^2 \mid \mathcal{F}_s^x] = E\left[\frac{1}{2d} \sum_{e \sim 0} \int_s^t (M_u^{(e)} - M_u)^2 \, \mathrm{d}u \mid \mathcal{F}_s^x \right]. \tag{6.96}$$

Let us define for $\delta > 0$ and $u \geq T$ the random variable

$$\psi_u^\delta := \frac{1}{\delta} \, E[(M_{u+\delta} - M_u)^2 \,|\, \mathcal{F}_u^x] \overset{(*)}{=} \frac{1}{\delta} \, E[M_{u+\delta}^2 - M_u^2 \,|\, \mathcal{F}_u^x], \qquad (6.97)$$

where $(*)$ follows from the fact that M_t is a bounded martingale. Using (6.89), (6.90) and that (X_t) is a continuous-time simple random walk on $\mathbb{Z}^d$ we obtain

$$\lim_{\delta \to 0_+} \psi_u^\delta = \frac{1}{2d} \sum_{e \sim 0} (M_u^{(e)} - M_u)^2, \qquad \mathbb{P} - \text{a.s.} \qquad (6.98)$$

It follows from the definition (6.97) that for any $\delta > 0$ we have

$$E\left[\int_s^t \psi_u^\delta \mathrm{d}u \,|\, \mathcal{F}_s^x\right] = E\left[\frac{1}{\delta} \int_t^{t+\delta} M_u^2 \, \mathrm{d}u - \frac{1}{\delta} \int_s^{s+\delta} M_u^2 \, \mathrm{d}u \,|\, \mathcal{F}_s^x\right].$$

From this, (6.88) and Claim 6.23 it follows that

$$\lim_{\delta \to 0_+} E\left[\int_s^t \psi_u^\delta \mathrm{d}u \,|\, \mathcal{F}_s^x\right] = E\left[M_t^2 - M_s^2 \,|\, \mathcal{F}_s^x\right], \qquad \mathbb{P} - \text{a.s.} \qquad (6.99)$$

Now (6.96) will follow from (6.98) and (6.99) by dominated convergence as soon as we prove that for any $u \geq T$ and $0 < \delta \leq 1$ we have $\psi_u^\delta \leq 1$. This bound follows from (6.88) and Claim 6.23. $\qquad \square$

Lemma 6.25. *There exists $C > 0$ such that for any $t \geq T \geq 1$ and $e \sim 0$,*

$$|M_t^{(e)} - M_t| \leq Ct^{\frac{1}{2} - \frac{d}{2}}. \qquad (6.100)$$

Before we prove Lemma 6.25, let us deduce Lemma 6.19 from it.

Proof of Lemma 6.19. We begin with (6.72). We first observe that, for any $y, z \in \mathbb{Z}^d$,

$$\sum_{w \in \mathbb{Z}^d} p_T(y, w) \cdot P\left[\exists t \geq 0 : X_t^w = X_t^z\right] \overset{(2.11),(2.6)}{\leq} \int_T^\infty p_t(y, z) \mathrm{d}t \overset{(2.7)}{\leq} C_0 T^{1 - \frac{d}{2}}. \qquad (6.101)$$

With this at hand, we derive (6.72):

$$M_T \overset{(6.70)}{=} \sum_{z,w} \mathbb{1}\{X_T^x = z\} \cdot p_T(y, w) \cdot P\left[\exists t \geq T : X_t^y = X_t^x \,|\, X_T^x = z, \, X_T^y = w\right]$$

$$= \sum_z \mathbb{1}\{X_T^x = z\} \cdot \sum_w p_T(y, w) \cdot P\left[\exists t \geq 0 : X_t^w = X_t^z\right] \overset{(6.101)}{\leq} C_0 T^{1 - \frac{d}{2}}.$$

The bound (6.73) follows from (6.94) and (6.100). Now we prove (6.74):

$$\langle M \rangle_\infty - \langle M \rangle_T \overset{(6.95)}{=} \frac{1}{2d} \sum_{e \sim 0} \int_T^\infty (M_s^{(e)} - M_s)^2 \, \mathrm{d}s \overset{(6.100)}{\leq} \int_T^\infty C s^{1-d} \, \mathrm{d}s = CT^{2-d}.$$

$$\square$$

Proof of Lemma 6.25. Given $t \geq T$ we define the event A by (6.86). We have

$$M_t \overset{(6.85),(2.8)}{=} P[A \,|\, \mathcal{F}_t^x] + \sum_{v,w \in \mathbb{Z}^d} P[A^c \cap \{X_t^y = w\} \,|\, \mathcal{F}_t^x] \cdot \mathbb{1}[X_t^x = v] \cdot \frac{g(v, w)}{g(0, 0)},$$

$$M_t^{(e)} \overset{(6.87),(2.8)}{=} P[A \,|\, \mathcal{F}_t^x] + \sum_{v,w \in \mathbb{Z}^d} P[A^c \cap \{X_t^y = w\} \,|\, \mathcal{F}_t^x] \cdot \mathbb{1}[X_t^x = v] \cdot \frac{g(v + e, w)}{g(0, 0)},$$

thus we obtain (6.100):

$$|M_t^{(e)} - M_t| \leq \sum_{v,w \in \mathbb{Z}^d} P[A^c \cap \{X_t^y = w\} \mid \mathcal{F}_t^x] \cdot \mathbb{1}[X_t^x = v] \cdot \frac{|g(v+e,w) - g(v,w)|}{g(0,0)} \overset{(2.5)}{\leq}$$

$$\sum_{v,w \in \mathbb{Z}^d} P[X_t^y = w] \cdot \mathbb{1}[X_t^x = v] \cdot |g(v+e,w) - g(v,w)| \overset{(2.15)}{\leq} C t^{\frac{1}{2} - \frac{d}{2}}.$$

$\square$

7 Concluding remarks

Remark 7.1. *In order to informally explain why $d \geq 5$ is easier than $d = 4$ and especially $d = 3$ when it comes to proving $\alpha_c > 0$ for the nearest-neighbour voter model on $\mathbb{Z}^d$, let us introduce a toy model. Recall the graphical construction (3.2) of the coalescing random walks $(Y_t^x)_{t \geq 0, x \in \mathbb{Z}^d}$ and assume that $R = 1$. Denote by μ^* the law of the random element $(\xi(x))_{x \in \mathbb{Z}^d}$ of $\{0,1\}^{\mathbb{Z}^d}$ that we obtain by defining*

$$\xi(x) = \begin{cases} 1 & \text{if } Y_t^x = Y_t^0 \text{ for some } t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In words, the coalescence class of the origin is occupied, and every other vertex of $\mathbb{Z}^d$ is vacant.

As a first step in the direction of (4.2), one might first want to show

$$\lim_{L \to \infty} \mu^* \left[B(0,L) \overset{*\xi}{\longleftrightarrow} B(0,2L)^c \right] = 0. \tag{7.1}$$

When $d \geq 5$, this follows from (4.2) and the fact that

$$\mu_{\alpha_0} \left[B(0,L) \overset{*\xi}{\longleftrightarrow} B(0,2L)^c \right] \geq \alpha_0 \cdot \mu^* \left[B(0,L) \overset{*\xi}{\longleftrightarrow} B(0,2L)^c \right].$$

We believe that (7.1) can be proved in the $d = 4$ case using a careful implementation of similar ideas. However, the question of (7.1) is to the best of our knowledge open in the $d = 3$ case and we think new ideas are needed for the proof.

We also note that if $d \geq 3$ and ξ has law μ^*, then by [BClG01, Theorem 3] the sequence of rescaled random measures $\frac{1}{N} \sum_{x \in \mathbb{Z}^d} \xi(x) \delta_{x/\sqrt{N}}$ converge in law with respect to the topology of vague convergence on the space of Radon measures on $\mathbb{R}^d$, and the limit object is a variant of super-Brownian motion. It is also known (see [P95, Section 4] and [P02, Theorem III.6.3]) that for $d \geq 4$ the closed support of super-Brownian motion is totally disconnected, but the open problem stated on [P02, page 119] is still not solved, i.e. the closed support of super-Brownian motion in $d = 3$ may or may not contain non-trivial connected subsets. The combination of these facts also indicate that (7.1) may be easier to verify for $d = 4$ than for $d = 3$.

Remark 7.2. *One reason why the proof of Theorem 1.2 in Section 5 is so short is that we chose $L = 1$ so that crossing an annulus on the bottom level of our renormalization scheme just means that a single site is of type 1. Let us explain why this choice is insufficient when it comes to proving Theorem 1.1. In this heuristic argument we will also keep track of the dependence on ℓ of the combinatorial terms and probabilities in order to make sure that making ℓ large will not be helpful either.*

If $R = 1$ and $L = 1$, then (similarly to (5.3)) we obtain

$$\mu_\alpha[B(L_N - 2) \overset{*\xi}{\longleftrightarrow} B(2L_N)^c] \overset{(3.6)}{\leq} \widehat{C}(\ell)^{2^N} \max_{\mathcal{T} \in \Lambda_N} \mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X}_\mathcal{T})}\right], \qquad (7.2)$$

where $\widehat{C}(\ell) \asymp \ell^{2d-2}$. For any $\mathcal{T} \in \Lambda_N$, we have $\mathbb{E}\left[\alpha^{\mathcal{N}_\infty(\mathcal{X}_\mathcal{T})}\right] \geq \alpha \mathbb{P}[\mathcal{N}_\infty(\mathcal{X}_\mathcal{T}) = 1]$. Recalling (5.2) we can construct a scenario where $\mathcal{N}_\infty(\mathcal{X}) = 1$ (i.e., all walkers coalesce) by first coalescing the walkers starting from $\mathcal{T}(m_1)$ and $\mathcal{T}(m_2)$ (see (4.6)) for every $m \in T_{(N-1)}$, and then coalescing the resulting walkers with their respective "sibling", etc. Recursively repeating this procedure from the leaves to the root of the binary tree we obtain that

$$\mathbb{P}[\mathcal{N}_\infty(\mathcal{X}_\mathcal{T}) = 1] \gtrsim \left(C\ell^{2-d}\right)^{2^{N-1}} \cdot \left(C\ell^{2(2-d)}\right)^{2^{N-2}} \ldots \left(C\ell^{N(2-d)}\right)^{2^0} \asymp \left(C\ell^{(2-d)\cdot 2}\right)^{2^N}.$$

If we use this to (heuristically) lower bound the right-hand side of (7.2), we obtain

$$\widehat{C}(\ell)^{2^N} \cdot \alpha \cdot \left(C\ell^{(2-d)\cdot 2}\right)^{2^N} \asymp \left(\widetilde{C}\ell^2\right)^{2^N},$$

which may go to infinity as $N \to \infty$ if the constant $\widetilde{C} = \widetilde{C}(d)$ happens to be too big.

Remark 7.3. Let us explain why the method of Section 6 fails to prove (4.2) if $d = 3, 4$ and $R = 1$ by arguing that the right-hand side of (6.14) does not go to zero. Rather than fixing the value of ℓ as in (6.2), in this heuristic argument we will keep track of the dependence on ℓ as well as on L of the terms on the right-hand side of (6.14). If we assume $\mathcal{Y} = \emptyset$, then by (6.10) we have $|\mathcal{X}| = 2 \cdot 2^N$. Similarly to Remark 7.2, we will bound the probability of the event on the right-hand side of (6.14) from below. For any fixed $\alpha > 0$ we can bound

$$\mathbb{P}\left[\bigcap_{\substack{\{x,z\} \in \mathcal{X} \\ |x-z|=1}} F_{x,z}\right] \overset{(3.5),(6.8)}{\geq} \alpha \mathbb{P}\left[\mathcal{N}_\infty(\mathcal{X}) = 1, \bigcap_{\substack{\{x,z\} \in \mathcal{X} \\ |x-z|=1}} E_{x,z}\right]. \qquad (7.3)$$

Now the probability that $E_{x,z}$ occurs and yet $Y_t^x = Y_t^z$ for some $t > T$ is roughly $\sqrt{T}^{2-d} = L^{(1-\varepsilon/2)(2-d)}$ by (2.7) and (6.5), moreover we can use the binary tree structure of $\mathcal{X}$ to construct a scenario where $\mathcal{N}_\infty(\mathcal{X}) = 1$ and give a (heuristic) lower bound on the probability on the right-hand side of (7.3) by

$$\left(L^{(1-\varepsilon/2)(2-d)}\right)^{2^N} \cdot \prod_{k=1}^N \left(L\ell^k\right)^{(2-d)2^{N-k}} \asymp \left(L^{(2-d)(2-\varepsilon/2)} \cdot \ell^{(2-d)\cdot 2}\right)^{2^N}.$$

If we multiply this with the combinatorial term $\left(L^d\ell^{2d-2}\right)^{2^N}$ that appears on the right-hand side of (6.14) then the resulting product goes to infinity as $N \to \infty$.

Remark 7.4. Let us explain why the "decorrelation via annihilation" method developed in Section 5 cannot be used to prove Theorem 1.1. Let us assume $\mathcal{Y} = \emptyset$ (so that by (6.10) we have $|\mathcal{X}| = 2 \cdot 2^N$) and bound the probability of the event of the right-hand side of (6.14):

$$\mathbb{P}\left[\bigcap_{\substack{\{x,z\} \in \mathcal{X} \\ |x-z|=1}} F_{x,z}\right] \overset{(6.17),(6.20)}{\leq} E\left[\alpha^{\mathcal{N}_\infty^Z}\right].$$

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Now we try to bound this using the idea of Lemma 5.3, i.e., we let random walks starting from the vertices of $\mathcal{X}$ run independently until time T and then we let them annihilate each other. Let us denote by $\mathcal{N}_\infty^{Z'}$ the number of walkers that do not get annihilated. Similarly to Lemma 5.3, we have $E\left[\alpha^{\mathcal{N}_\infty^{Z}}\right] \leq E\left[\alpha^{\mathcal{N}_\infty^{Z'}}\right]$, but using an argument similar to the one used in Remark 7.3 we can (non-rigorously) bound

$$E\left[\alpha^{\mathcal{N}_\infty^{Z'}}\right] \geq P[\mathcal{N}_\infty^{Z'} = 0] \gtrsim \left(L^{(1-\varepsilon/2)(2-d)}\right)^{2^N},$$

and this term is not small enough to beat the combinatorial term $(CL^d)^{2^N}$ on the right-hand side of (6.14).

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