A lower bound of Ruzsa's number related to the Erdős-Turán conjecture

Csaba Sándor[1*] Quan-Hui Yang[2†]

1. Department of Stochastics, Budapest University of Technology and Economics, H-1529 B.O. Box, Hungary

2. School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Abstract

For a set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_A(n)$ denote the number of ordered pairs $(a, a') \in A \times A$ such that $a + a' = n$. The celebrated Erdős-Turán conjecture says that, if $R_A(n) \geq 1$ for all sufficiently large integers n, then the representation function $R_A(n)$ cannot be bounded. For any positive integer m, Ruzsa's number R_m is defined to be the least positive integer r such that there exists a set $A \subseteq \mathbb{Z}_m$ with $1 \leq R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. In 2008, Chen proved that $R_m \leq 288$ for all positive integers m. In this paper, we prove that $R_m \geq 6$ for all integers $m \geq 36$. We also determine all values of R_m when $m \leq 35$.

2010 Mathematics Subject Classification: Primary 11B34,11B13.

Keywords and phrases: Representation function, Ruzsa's number, Erdős-Turán conjecture

1 Introduction

Let $\mathbb{N}$ be all nonnegative integers. For any set $A, B \subseteq \mathbb{N}$, let

$$R_{A,B}(n) = \sharp\{(a, b) : \ a \in A, \ b \in B, \ a + b = n\}.$$

Let $R_A(n) = R_{A,A}(n)$. If $R_A(n) \geq 1$ for all sufficiently large integers n, then we say that A is a basis of $\mathbb{N}$. The celebrated Erdős-Turán conjecture [7] states that if A is a basis of

*Email: csandor@math.bme.hu. This author was supported by the OTKA Grant No. K109789. This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

†Email: yangquanhui01@163.com. This author was supported by the National Natural Science Foundation for Youth of China, Grant No. 11501299, the Natural Science Foundation of Jiangsu Province, Grant Nos. BK20150889, 15KJB110014 and the Startup Foundation for Introducing Talent of NUIST, Grant No. 2014r029.

$\mathbb{N}$, then $R_A(n)$ cannot be bounded. Erdős [6] proved that there exists a basis A and two constants $c_1, c_2 > 0$ such that $c_1 \log n \leq R_A(n) \leq c_2 \log n$ for all sufficiently large integers n. Recently, Dubickas [5] gave the explicit values of c_1 and c_2. In 2003, Nathanson [14] proved that the Erdős-Turán conjecture does not hold on $\mathbb{Z}$. In fact, he proved that there exists a set $A \subseteq \mathbb{Z}$ such that $1 \leq R_A(n) \leq 2$ for all integers n. In the same year, Grekos et al. [8] proved that if $R_A(n) \geq 1$ for all n, then $\limsup_{n\to\infty} R_A(n) \geq 6$. Later, Borwein et al. [2] improved 6 to 8. In 2013, Konstantoulas [11] proved that if the upper density $\overline{d}(\mathbb{N} \setminus (A + A))$ of the set of numbers not represented as sums of two numbers of A is less than $1/10$, then $R_A(n) > 5$ for infinitely many natural numbers n. Chen [4] proved that there exists a basis A of $\mathbb{N}$ such that the set of n with $R_A(n) = 2$ has density one. Later, the second author [17] and Tang [16] generalized Chen's result. For the analogue of Erdős-Turán conjecture in groups, one can refer to [9], [10] and [12].

For a positive integer m, let $\mathbb{Z}_m$ be the set of residue classes mod m. For $A, B \subseteq \mathbb{Z}_m$, let $R_{A,B}(n)$ be the number of solutions of equation $a + b = n,\ a \in A,\ b \in B$. Let $R_A(n) = R_{A,A}(n)$. If $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, then A is called an additive basis of $\mathbb{Z}_m$.

In 1990, Ruzsa [15] found a basis A of $\mathbb{N}$ for which $R_A(n)$ is bounded in the square mean. Ruzsa's method implies that there exists a constant C such that for any positive integer m, there exists an additive basis A of $\mathbb{Z}_m$ with $R_A(n) \leq C$ for all $n \in \mathbb{Z}_m$. For each positive integer m, Chen [3] defined Ruzsa's number R_m to be the least positive integer r such that there exists an additive basis A of $\mathbb{Z}_m$ with $R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. In this paper, Chen also proved that $R_m \leq 288$ for all positive integers m and $R_{2p^2} \leq 48$ for all prime numbers p. Until now, this is the best upper bound about Ruzsa's number and there is no nontrivial lower bound. In fact, in the same paper, Chen says "We have $R_m \geq 3$ for $m \neq 1, 2, 3$. Now we cannot improve this trivial lower bound".

In this paper, we give a nontrivial lower bound of Ruzsa's number.

Theorem 1. *$R_m = 2$ if and only if $m = 2, 3$; $R_m = 3$ if and only if $m = 4, 5, 7$.*

Remark 1. *If $m > 1$ and $A \subseteq \mathbb{Z}_m$ is an additive basis, then $|A| \geq 2$. It follows that there exist two distinct elements $a, a' \in A$, and so $R_A(a + a') \geq 2$. Hence $R_m = 1$ if and only if $m = 1$.*

Theorem 2. *$R_m = 4$ if and only if $m = 6, 8, 9, 10, 11, 12, 13, 14, 15, 19$; $R_m = 5$ if and only if $m = 16, 17, 18, 20, 21, 22, 23, 24, 25, 27, 28, 35$.*

By Theorems 1 and 2, we have the following Corollary.

Corollary 1. *If $m \geq 36$, then $R_m \geq 6$.*

Remark 2. *Furthermore, if $m \leq 35$, then $R_m \leq 6$. We list all the values of R_m ($2 \leq m \leq 35$) and a set $A \subseteq \mathbb{Z}_m$ such that $1 \leq R_A(n) \leq R_m$ for all $n \in \mathbb{Z}_m$ in the Appendix.*

2 Proofs

In order to prove Theorems 1 and 2, we need some lemmas in the following. The first lemma due to Lev and Sárközy [13] is the main tool of our proofs.

Lemma 1. *(Lev and Sárközy's lower bound) If A is a subset of a finite non-trivial abelian group G, then for any real number c we have*

$$\sum_{g\in G}(R_A(g)-c)^2 \geq \frac{1}{|G|-1}\left(\frac{|A|^4}{|G|}-2|A|^3+|A|^2|G|\right).$$

Lemma 2. *Let $A\subseteq\mathbb{Z}_m$. If $R_A(n)\geq 1$ for all $n\in\mathbb{Z}_m$, then $|A|>\sqrt{2m}-1/2$.*

Proof. Since $R_A(n)\geq 1$ for all $n\in\mathbb{Z}_m$, we have

$$\begin{aligned}|A|^2=\sum_{n=0}^{m-1}R_A(n) &\geq |\{n:n\in\mathbb{Z}_m,R_A(n)=1\}|+2|\{n:n\in\mathbb{Z}_m,R_A(n)\geq 2\}|\\ &= 2|\{n:n\in\mathbb{Z}_m\}|-|\{n:n\in\mathbb{Z}_m,R_A(n)=1\}|\\ &= 2m-|\{n:n\in\mathbb{Z}_m,R_A(n)=1\}|\geq 2m-|A|.\end{aligned}$$

Hence $(|A|+1/2)^2>2m$, that is, $|A|>\sqrt{2m}-1/2$. □

Lemma 3. *Let $A\subseteq\mathbb{Z}_m$ and c be a positive integer. If $R_A(n)\leq c$ for all $n\in\mathbb{Z}_m$, then $|A|\leq\sqrt{cm}$.*

This lemma follows from $|A|^2=\sum_{n=0}^{m-1}R_A(n)\leq cm$ immediately.

Lemma 4. *(See [1, P. 827, Test C].) Suppose that v,λ,k $(v\geq k\geq\lambda)$ are positive integers. Let p be a prime divisor of $k-\lambda$ and let $w\geq 1$, $(w,p)=1$, be a divisor of v for which there exists an integer $f>0$ such that $p^f\equiv -1\ (mod\ w)$. If p^e exactly divides $k-\lambda$ and p^l $(l\geq 0)$ exactly divides v, then there exists a set $A\subseteq\mathbb{Z}_v$ with $|A|=k$ such that the congruence $a-a'\equiv b\ (mod\ v)$, $a,a'\in A$ has exactly λ distinct solutions for all $b\not\equiv 0\ (mod\ v)$ if and only if*

$$p^{\lfloor e/2\rfloor}<(v/w)p^{-l},$$

where $\lfloor x\rfloor$ denotes the largest integer $\leq x$.

Lemma 5. *Let A be an additive basis of $\mathbb{Z}_m$ and k,l be positive integers with $(l,m)=1$. Then $A+k$, lA is also an additive basis and*

$$\max_{n\in\mathbb{Z}_m}R_A(n)=\max_{n\in\mathbb{Z}_m}R_{A+k}(n)=\max_{n\in\mathbb{Z}_m}R_{lA}(n).$$

This lemma follows from $R_A(n)=R_{A+k}(n+2k)=R_{lA}(ln)$ for all $n\in\mathbb{Z}_m$ immediately.

Proof of Theorem 1. If $m \leq 11$, by the computer-based calculation, then we obtain that $R_m = 2$ if and only if $m = 2, 3$ and $R_m = 3$ if and only if $m = 4, 5, 7$. Now it suffices to prove that $R_m \leq 3$ implies $m \leq 11$. Suppose that $m \geq 12$ and there exists a subset $A \subseteq \mathbb{Z}_m$ such that $1 \leq R_A(n) \leq 3$ for all $n \in \mathbb{Z}_m$.

Putting $G = \mathbb{Z}_m$ and $c = 2$, by Lemma 1, we obtain that for any subset $A \subseteq \mathbb{Z}_m$,

$$\sum_{n=0}^{m-1} (R_A(n) - 2)^2 \geq \frac{|A|^2(m - |A|)^2}{m(m-1)}. \tag{1}$$

Since $1 \leq R_A(n) \leq 3$, it follows that

$$(R_A(n) - 2)^2 = \begin{cases} 1, & \text{if } R_A(n) \text{ is odd;} \\ 0, & \text{if } R_A(n) \text{ is even.} \end{cases}$$

Furthermore, if $R_A(n)$ is odd, then there exists $a \in A$ such that $n = 2a$, and so

$$\sum_{n=0}^{m-1} (R_A(n) - 2)^2 = \sum_{\substack{n=0 \\ 2 \nmid R_A(n)}}^{m-1} 1 \leq \sum_{a \in A} 1 = |A|. \tag{2}$$

By (1) and (2), we have

$$|A|(m - |A|)^2 \leq m(m-1) < m^2.$$

On the other hand, by Lemmas 2 and 3, we have $\sqrt{2m} - 1/2 < |A| \leq \sqrt{3m}$. Hence

$$|A|(m - |A|)^2 > (\sqrt{2m} - 1/2)(m - \sqrt{3m})^2 > m^2,$$

because $\sqrt{2m} - 1/2 > 4$ and $\sqrt{3m} \leq m/2$ for $m \geq 12$. This is a contradiction. □

Proof of Theorem 2. We first prove that $R_m \leq 5$ implies that $m \leq 500$. Suppose that $m > 500$ and there exists $A \subseteq \mathbb{Z}_m$ such that $1 \leq R_A(n) \leq 5$ for all $n \in \mathbb{Z}_m$. By Lemma 1, taking $G = \mathbb{Z}_m$ and $c = 3$, we get

$$\sum_{n=0}^{m-1} (R_A(n) - 3)^2 \geq \frac{|A|^2(m - |A|)^2}{m(m-1)}. \tag{3}$$

If $R_A(n)$ is odd, then $(R_A(n) - 3)^2 \leq 4$. If $R_A(n)$ is even, then $(R_A(n) - 3)^2 = 1$. Hence

$$\begin{aligned} & \sum_{n=0}^{m-1} (R_A(n) - 3)^2 \\ \leq\ & 4|\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is odd}\}| + |\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is even}\}| \\ =\ & m + 3|\{n : n \in \mathbb{Z}_m, R_A(n) \text{ is odd}\}| \leq m + 3|A|. \end{aligned} \tag{4}$$

By (3) and (4), we have

$$|A|^2(m - |A|)^2 \leq (m + 3|A|)m(m-1). \tag{5}$$

On the other hand, by Lemmas 2 and 3, we have $\sqrt{2m} - 1/2 < |A| \le \sqrt{5m}$. Hence

$$\begin{aligned} |A|^2(m-|A|)^2 &> (\sqrt{2m}-1/2)^2(m-\sqrt{5m})^2 > (1.9\cdot 0.9^2)m^3 \\ &> 1.3m^3 > (m+3\sqrt{5m})m^2 > (m+3|A|)m(m-1), \end{aligned}$$

because $\sqrt{2m} - 1/2 > \sqrt{1.9m}$, $m - \sqrt{5m} > 0.9m$ and $m + 3\sqrt{5m} < 1.3m$ for $m > 500$. This contradicts with the inequality (5). Thus, if $m > 500$, then $R_m \ge 6$.

Now we only need to consider cases $m \le 500$.

If $m \le 20$, then the computer-based calculation can run over all the sets $A \subseteq \mathbb{Z}_m$ with $\sqrt{2m} - 1/2 \le |A| \le \sqrt{5m}$ and we can determine these values of R_m. We obtain that $R_m = 4$ for $m \in \{6, 8, 9, 10, 11, 12, 13, 14, 15, 19\}$ and $R_{16} = R_{17} = R_{18} = R_{20} = 5$.

Next we assume that $21 \le m \le 500$. A routine computer-based calculation gives that the maximal pair of (m, k) satisfying that

$$21 \le m \le 500, \quad \sqrt{2m} - 1/2 \le |A| = k \le \sqrt{5m} \tag{6}$$

and the inequality (5) holds is $(m, k) = (91, 13)$. The value for such (m, k) is too large for the computer-based calculation to run over all the sets $A \subseteq \mathbb{Z}_{91}$ with $|A| = 13$.

In the following, we need three steps to reduce these values.

Our task is to find all exact pairs of (m, k) with the following property: There exists $A \subseteq \mathbb{Z}_m$ with $|A| = k$ such that $1 \le R_A(n) \le 5$ for all $n \in \mathbb{Z}_m$. In the first step, for $i \in \{1, 2, 3, 4, 5\}$, let

$$k_i = |\{n : n \in \mathbb{Z}_m, R_A(n) = i\}|.$$

Then

$$k_1 + k_2 + k_3 + k_4 + k_5 = k, \quad k_i \in \mathbb{N}\ (1 \le i \le 5), \tag{7}$$

$$k^2 = |A|^2 = \sum_{n=0}^{m-1} R_A(n) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5, \tag{8}$$

and

$$k_1 + k_3 + k_5 \le |A| = k, \quad \text{and the equality holds when } m \text{ is odd.} \tag{9}$$

By Lemma 1, taking $c = k^2/m$, we have

$$\sum_{n=0}^{m-1} \left(R_A(n) - \frac{k^2}{m}\right)^2 = \sum_{i=1}^{5} \left(i - \frac{k^2}{m}\right)^2 k_i \ge \frac{|A|^2(m-|A|)^2}{m(m-1)} = \frac{k^2(m-k)^2}{m(m-1)}. \tag{10}$$

By the computer-based calculation, the maximal values of (m, k) such that there exists nonnegative integers k_1, k_2, k_3, k_4, k_5 satisfying (6)-(10) is $(50, 12)$. This value is

also too large for the computer-based calculation to run over all subsets $A \subseteq \mathbb{Z}_{50}$ with $|A| = 12$.

In the second reduction step, we shall delete all pairs (m,k) for which $42 \le m \le 50$. Here we need to improve the Lev-Sárközy's bound. Clearly,

$$\begin{aligned}(11)\quad \sum_{n=0}^{m-1}\left(R_A(n)-\frac{k^2}{m}\right)^2 &= \sum_{n=0}^{m-1}R_A^2(n)-\frac{2k^2}{m}\sum_{n=0}^{m-1}R_A(n)+\frac{k^4}{m}\\ &= \sum_{n=0}^{m-1}R_A^2(n)-\frac{2k^2}{m}\cdot k^2+\frac{k^4}{m}=\sum_{n=0}^{m-1}R_A^2(n)-\frac{k^4}{m}.\end{aligned}$$

Next we use Lev-Sárközy's arguments to obtain a better lower bound for $\sum_{n=0}^{m-1}\left(R_A(n)-\frac{k^2}{m}\right)^2$. Clearly, the sum $\sum_{n=0}^{m-1}R_A^2(n)$ counts the number of solutions of the equation

$$a_1+a_2=a_3+a_4,\quad a_1,a_2,a_3,a_4\in A.$$

Rearranging these terms, one can rewrite this equation as $a_1-a_3=a_4-a_2$. Hence

$$\sum_{n=0}^{m-1}R_A^2(n)=\sum_{n=0}^{m-1}R_{A,-A}^2(n)=k^2+\sum_{n=1}^{m-1}R_{A,-A}^2(n).$$

Clearly, $\sum_{n=1}^{m-1}R_{A,-A}^2(n)=k^2-k$. Let $k^2-k=q(m-1)+r$, where q,r are nonnegative integers and $0\le r<m-1$. Then

$$q=\left\lfloor\frac{k^2-k}{m-1}\right\rfloor\quad\text{and}\quad r=k^2-k-\left\lfloor\frac{k^2-k}{m-1}\right\rfloor(m-1).$$

Hence

$$\begin{aligned}(12)\quad &\sum_{n=0}^{m-1}R_A^2(n)=k^2+\sum_{n=1}^{m-1}R_{A,-A}^2(n)\\ &\ge k^2+(q+1)^2r+q^2(m-1-r)\\ &= k^2+(2q+1)r+q^2(m-1)\\ &= k^2+\left(2\left\lfloor\frac{k^2-k}{m-1}\right\rfloor+1\right)\left(k^2-k-\left\lfloor\frac{k^2-k}{m-1}\right\rfloor(m-1)\right)+\left\lfloor\frac{k^2-k}{m-1}\right\rfloor^2(m-1).\end{aligned}$$

By (10), (11) and (12), we get the following better lower bound instead of (10).

$$\begin{aligned}(13)\quad \sum_{i=1}^{5}\left(i-\frac{k^2}{m}\right)^2k_i \ge\ & k^2+\left\lfloor\frac{k^2-k}{m-1}\right\rfloor^2(m-1)-\frac{k^4}{m}\\ &+\left(2\left\lfloor\frac{k^2-k}{m-1}\right\rfloor+1\right)\left(k^2-k-\left\lfloor\frac{k^2-k}{m-1}\right\rfloor(m-1)\right).\end{aligned}$$

By the computer-based calculation, we list all pairs of (m,k) such that there exist nonnegative integers k_1,k_2,k_3,k_4,k_5 satisfying (6)-(9) and (13) in the following.

$(m,k)\in\{(21,7),(21,8),(21,9),(22,7),(22,8),(22,9),(23,7),(23,8),\ (23,9),(24,8),(24,9),$ $(25,8),(25,9),\ (26,8),(26,9),(27,8),(27,9),(28,8),\ (28,9),(28,10),(29,8),(29,9),(29,10),$

$(30,9)$,$(30,10)$,$(31,9)$, $(31,10)$, $(32,9)$,$(32,10)$,$(33,9)$,$(33,10)$,$(34,10)$,$(35,10)$,$(36,10)$, $(36,11)$,$(37,11)$, $(38,11)$, $(39,11)$,$(40,11)$,$(41,11)$,$(45,12)\}$.

In the last step, we deal with cases $(m,k) = (40,11),(41,11),(45,12)$, since such values are also too large for the computer-based calculation.

Now we first deal with the largest case $(m,k) = (45,12)$. Take $v = 45$, $\lambda = 3$, $k = 12$, $p = 3$, $w = 5$, $f = 2$, $e = 2$, $l = 2$. By Lemma 4, it follows that there is no subset $A \subseteq \mathbb{Z}_{45}$ with $|A| = 12$ such that $R_{A,-A}(n) = 3$ for all $n \not\equiv 0 \pmod{45}$. In other words, for any set $A \subseteq \mathbb{Z}_{45}$, there exists $n \not\equiv 0 \pmod{45}$ such that $R_{A,-A}(n) \neq 3$. Noting that $\sum_{n=1}^{44} R_{A,-A}(n) = k^2 - k = 132$, we have

$$\sum_{n=1}^{44} R_{A,-A}^2(n) \geq 3^2 \times 42 + 2^2 + 4^2 = 398.$$

Hence, by (11) and (12), we have

$$\sum_{n=0}^{44} \left(R_A(n) - \frac{12^2}{45} \right)^2 = 12^2 + \sum_{n=1}^{44} R_{A,-A}^2(n) - \frac{12^4}{45} \geq 81.2.$$

On the other hand, we list all values of (k_1,k_2,k_3,k_4,k_5) when $(m,k) = (45,12)$ in the following.

k_1	k_2	k_3	k_4	k_5
0	24	0	9	12
1	22	0	11	11
2	20	0	13	10
4	16	0	17	8

k_1	k_2	k_3	k_4	k_5
5	14	0	19	7
6	12	0	21	6
7	10	0	23	5
8	8	0	25	4

k_1	k_2	k_3	k_4	k_5
9	6	0	27	3
10	4	0	29	2
11	2	0	31	1
12	0	0	33	0

For all the values list above, we have

$$\sum_{n=0}^{44} \left(R_A(n) - \frac{12^2}{45} \right)^2 = \sum_{i=1}^{5} \left(i - \frac{12^2}{45} \right)^2 k_i = 79.2.$$

This is a contradiction.

Finally, we deal with the cases $(m,k) = (41,11)$ and $(40,11)$, since the number of sets A for which the computer-based calculation can run over is about $\binom{39}{9}$. If $m = 41$, by Lemma 5, then we can assume that $0, 40 \in A$. Hence the number of such A is $\binom{39}{9}$, and the computer-based calculation can run over all such sets A. Now we consider the case $m = 40$. If there is an element in A coprime with 40, by Lemma 5, then we can assume that $0, 39 \in A$, and so the computer-based calculation can also deal with the case. If there is no element in A coprime with 40, then we can assume that $0 \in A$ and

$$A \subseteq \{0,2,4,5,6,8,10,12,14,15,16,18,20,22,24,25,26,28,30,32,34,35,36,38\}.$$

In this case, there are only $\binom{23}{10}$ sets A and we can deal with it with the computer-based calculation.

Using these idea, by the computer-based calculation, we obtain

$$R_m = 4 \text{ if and only if } m = 6, 8, 9, 10, 11, 12, 13, 14, 15, 19;$$

$$R_m = 5 \text{ if and only if } m = 16, 17, 18, 20, 21, 22, 23, 24, 25, 27, 28, 35.$$

□

3 Appendix

m	R_m	the set A	m	R_m	the set A
2	2	$\{0,1\}$	19	4	$\{0,1,5,7,8,15,18\}$
3	2	$\{0,1\}$	20	5	$\{0,1,2,5,6,13,16\}$
4	3	$\{0,1,2\}$	21	5	$\{0,1,2,3,4,6,13,16\}$
5	3	$\{0,1,2\}$	22	5	$\{0,1,2,4,5,9,15,17\}$
6	4	$\{0,3,4,5\}$	23	5	$\{0,1,2,3,5,11,14,18\}$
7	3	$\{0,1,2,4\}$	24	5	$\{0,1,2,6,9,10,12,17\}$
8	4	$\{0,3,5,6,7\}$	25	5	$\{0,1,2,4,9,12,20,22\}$
9	4	$\{0,4,6,7,8\}$	26	6	$\{0,1,2,5,15,19,20,22\}$
10	4	$\{0,1,2,3,6\}$	27	5	$\{0,1,2,3,5,11,15,18,23\}$
11	4	$\{0,4,6,8,9\}$	28	5	$\{0,1,2,4,5,8,10,17,22\}$
12	4	$\{0,1,6,8,9,11\}$	29	6	$\{0,1,2,3,4,6,10,17,22\}$
13	4	$\{0,5,7,8,11,12\}$	30	6	$\{0,1,2,3,4,5,7,11,17,22\}$
14	4	$\{0,4,8,9,11,12\}$	31	6	$\{0,1,2,3,4,5,9,13,20,25\}$
15	4	$\{0,6,8,11,12,14\}$	32	6	$\{0,1,2,3,4,5,8,15,20,26\}$
16	5	$\{0,1,2,3,4,7,11\}$	33	6	$\{0,1,2,3,4,6,10,14,21,26\}$
17	5	$\{0,1,2,3,4,7,12\}$	34	6	$\{0,1,2,3,4,6,13,19,26,29\}$
18	5	$\{0,1,2,3,5,8,12\}$	35	5	$\{0,1,4,5,10,12,16,19,26,34\}$

4 Acknowledgement

This work was done during the second author visiting to Budapest University of Technology and Economics. He would like to thank Dr. Sándor Kiss and Dr. Csaba Sándor for their warm hospitality. He also would like to thank Dr. Wenjun Cai for submitting his Matlab Program to a cluster of computers with 64G memory.

References

[1] L. D. Baumert, *Difference sets,* SIAM J. Appl. Math. 17(4), 826–833 (1969).

[2] P. Borwein, S. Choi and F. Chu, *An old conjecure of Erdős-Turán on additive bases,* Math. Comp. 75 (2005), 475-484.

[3] Y.-G. Chen, *The analogue of Erdős-Turán conjecture in $\mathbb{Z}_m$,* J. Number Theory 128 (2008), 2573-2581.

[4] Y.-G. Chen, *On the Erdős-Turán conjecture,* C. R. Math. Acad. Sci. Paris 350 (2012), 933-935.

[5] A. Dubickas, *A basis of finite and infinite sets with small representation,* The Electronic J. Combin. 19 (2012), R6.

[6] P. Erdős, *On a problem of Sidon in additive number theory,* Acta Sci. Math. (Szeged) 15 (1954), 255-259.

[7] P. Erdős and P. Turán, *On a problem of Sidon in additive number theory, and on some related problems,* J. Lond. Math. Soc. 16 (1941), 212-215.

[8] G. Grekos, L. Haddad, C. Helou and J. Pihko, *On the Erdős-Turán conjecture*, J. Number Theory 102 (2003), 339-352.

[9] L. Haddad and C. Helou, *Bases in some additive groups and the Erdős-Turán conjecture*, J. Combin. Theory Ser. A 108 (2004), 147-153.

[10] L. Haddad and C. Helou, *Additive bases representations in groups*, Integers 8 (2008), A5, 9 pp.

[11] I. Konstantoulas, *Lower bounds for a conjecture of Erdős and Turán,* Acta Arith. 159 (2013), 301-313.

[12] S. V. Konyagin and V. F. Lev, *The Erdős-Turán problem in infinite groups,* Additive number theory, 195-202, Springer, New York, 2010.

[13] V. F. Lev and A. Sárközy, *An Erdős-Fuchs type theorem for finite groups,* Integers 11(4) (2012), 487–494 .

[14] M. B. Nathanson, *Unique representation bases for integers,* Acta Arith. 108 (2003), 1-8.

[15] I. Z. Ruzsa, *A just basis,* Monatsh. Math. 109 (1990), 145-151.

[16] M. Tang, *On the Erdős-Turán conjecture,* J. Number Theory 150 (2015), 74-80.

[17] Q.-H. Yang, *A generalization of Chen's theorem on the Erdős-Turán conjecture,* Int. J. Number Theory 9 (2013), 1683-1686.