CONVERGENCE OF THE MATRIX TRANSFORMATION METHOD FOR THE FINITE DIFFERENCE APPROXIMATION OF FRACTIONAL ORDER DIFFUSION PROBLEMS

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(Received November 3, 2015)

Abstract. Numerical solution of fractional order diffusion problems with homogeneous Dirichlet boundary conditions is investigated on a square domain. An appropriate extension is applied to have a well-posed problem on $\mathbb{R}^2$ and the solution on the square is regarded as a localization. For the numerical approximation a finite difference method is applied combined with the matrix transformation method. Here the discrete fractional Laplacian is approximated with a matrix power instead of computing the complicated approximations of fractional order derivatives. The spatial convergence of this method is proved and demonstrated in some numerical experiments.

Keywords: fractional diffusion problem, finite differences, matrix transformation method

MSC 2010: 35R11, 65M06, 65M12

1. Introduction

Numerical solution techniques for fractional order diffusion problems have been intensively studied in the last decade. The corresponding mathematical models describe superdiffusion or subdiffusion, which were observed in several phenomena [3], [8] due to the increasing accuracy of the measurement techniques. Since in these models fractional order differential operators are used they are tied closely with the theory of the fractional calculus [15], [23], [24], which has a long history. Moreover, a novel framework has been recently elaborated to generalize the Fick's law and the fractional calculus, called the non-local calculus [6], [7].

The research has been supported by the Hungarian Research Fund OTKA (grant K112157). F. Izsák acknowledges the support of the Hungarian Research Fund OTKA (grant K104666) and the support of the Bolyai research fellowship of the Hungarian Academy of Sciences.

The majority of the numerical solution techniques is based on finite difference discretizations. A stable method based on shifted finite differences was first developed in [19]. Based of this work, higher-order methods [27], [30] were constructed and analyzed and the results were extended to some related non-linear problems [16]. The analysis has been extended also for the finite volume discretization, see [10] and [29].

A non-trivial aspect of the modeling and the precise error analysis is the handling of boundary conditions. The non-local nature of the fractional order diffusion operators [6] and the reduced regularity implies that the classical Dirichlet type boundary condition may not make sense. An approach to solve this problem has been developed in [25] in the one-dimensional situation dealing both with the homogenous Neumann and the Dirichlet type boundary conditions.

A difficulty in the practice of the numerical approximations is to compute the involved finite differences in two (or three) space dimension [28], [30]. To alleviate this procedure, the so-called matrix transform (or matrix transfer) method (MTM) has been proposed in [12], [13] and [17] and generalized in [14] for time and space-fractional diffusion problems. This approach makes possible to deal with the sparse matrix $\widehat{A}$ corresponding to the standard Laplacian operator $-\Delta$: for the discretization of $-\Delta^{\alpha}$ we have to use $\widehat{A}^{\alpha}$. The computational experiments confirmed the favor of this method. A corresponding error analysis was carried out only for the finite element methods with respect to the L_2-norm, see [26].

The aim of the present work is twofold.

- The first objective is to define a well-posed problem which corresponds to the space-fractional diffusion equation and involves homogeneous Dirichlet type boundary conditions.
- Second, we intend to develop a convergence theory for the matrix transformation method corresponding to the finite difference approximation and establish the order of convergence in the L_2-norm.

In the rest of the paper, after some preliminaries, an extension operator is introduced corresponding to homogeneous Dirichlet type boundary conditions. It is pointed out that in this way we arrive at a well-posed problem. We verify then the approximation property of the matrix transform approach. Based on this, a corresponding general semidiscrete numerical scheme is defined and the spatial convergence of this method is proved. Whenever our final result concerns finite difference approximation, the analysis is mainly based on spectral arguments and we use some recent results of the numerical aspects of the semigroup theory. The work is closed with some numerical experiments, which confirm the presented convergence theory.

2. Mathematical preliminaries

We examine the fractional diffusion equation in $\mathbb{R}^2$ using its divergence form.

2.1. **Fractional Laplacian and its eigenspace.** We define first the fractional Laplacian operator on the domain $\Omega = (0,1) \times (0,1)$ and the fractional Hilbert spaces following [13].

Definition 1. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\{\lambda_j\}_{j\in\mathbb{N}}$ denote the eigenfunctions and the corresponding eigenvalues of the Laplace operator $(-\Delta_D) : L_2(\Omega) \to L_2(\Omega)$, which is defined on a bounded Lipschitz domain Ω with homogeneous Dirichlet boundary conditions. These functions form a complete orthonormal set in $L_2(\Omega)$. For $\alpha \in \mathbb{R}^+$ we introduce*

$$\mathcal{F}_\alpha = \left\{ f = \sum_{n=1}^{\infty} c_n \varphi_n, \quad c_n = \langle f, \varphi_n \rangle : \quad \sum_{n=1}^{\infty} |c_n|^2 |\lambda_n|^\alpha < \infty \right\}$$

such that the fractional Laplacian $(-\Delta_D)^{\alpha/2} : \mathcal{F}_\alpha \to L_2(\Omega)$ with homogeneous Dirichlet boundary conditions is defined with

$$(-\Delta_D)^{\alpha/2} f := \sum_j \lambda_j^{\alpha/2} c_j \varphi_j. \tag{2.1}$$

While both the operator $-\Delta_D$ and the linear space $\mathcal{F}_\alpha$ depend on Ω, this is not shown for the sake of simplicity. Note that alternative definitions of the fractional Laplacian are available. Corresponding to the pointwise approximation of the Laplacian, in [5] for the case $\Omega = \mathbb{R}^2$ its fractional power is defined as:

$$-\Big(- \Delta \Big)^{\alpha/2} u(\underline{x}) := \frac{C_\alpha}{2} \int_{\mathbb{R}^2} \frac{u(\underline{x}+\underline{y}) + u(\underline{x}-\underline{y}) - 2u(\underline{x})}{|\underline{y}|^{2+\alpha}} \,\mathrm{d}\underline{y}\,, \tag{2.2}$$

$$\text{where} \quad C_\alpha = \left(\int_{\mathbb{R}^2} \frac{1-\cos\zeta_1}{|\zeta|^{2+\alpha}} d\zeta \right)^{-1}.$$

According to [20], the right hand side of (2.2) can be given as

$$\operatorname{div} J(\underline{x}), \text{ where } \quad J(\underline{x}) = \frac{C_\alpha}{\alpha^2} \operatorname{grad} \int_{\mathbb{R}^2} \frac{u(\underline{y})}{|\underline{x}-\underline{y}|^\alpha} \,\mathrm{d}\underline{y}\,,$$

which is a divergence form corresponding to a non-local Fick's law [7]. Accordingly, we will use the following definition of the fractional Laplacian on $\mathbb{R}^2$:

$$-\Big(- \Delta \Big)^{\alpha/2} u(\underline{x}) := \frac{C_\alpha}{\alpha^2} \Delta \left(\int_{\mathbb{R}^2} \frac{u(\underline{y})}{|\underline{x}-\underline{y}|^\alpha} \,\mathrm{d}\underline{y} \right)(\underline{x}). \tag{2.3}$$

Definition 2. *For $u \in \mathcal{F}_{\frac{\alpha}{2}}$ with $\alpha \in \mathbb{R}^+$ let*

$$\|u\|_{\frac{\alpha}{2}} = \left(|u|_0^2 + |u|_{\frac{\alpha}{2}}^2\right)^{\frac{1}{2}},$$

where

$$|u|_{\frac{\alpha}{2}}^2 = \sum_{k,l=1}^{\infty} \Big((k\pi)^2 + (l\pi)^2\Big)^{\frac{\alpha}{2}} |u_{k,l}|^2$$

with the Fourier coefficients $u_{k,l}$ of u. Then $\mathbb{H}^{\frac{\alpha}{2}}(\Omega) := (\mathcal{F}_{\frac{\alpha}{2}}, \|\cdot\|_{\frac{\alpha}{2}})$, see [4], [21].

dsds *Remarks:* Usually, $\mathbb{H}^{\frac{\alpha}{2}}(\Omega)$ is only defined for $\alpha \in (0,2)$ as this can be related with the classical Sobolev spaces, see [21] and [22].
According to Definition 2, we frequently use $\|\cdot\|_0$ for the $L_2(\Omega)$-norm.

Definition 3. *For each $N \in \mathbb{N}^+$ the linear space $S_N \subset L_2(\Omega)$ is defined with*

$$S_N = \operatorname{span}\Big\{2\sin(k\pi x)\sin(l\pi y) : x, y \in [0,1], 1 \leq k, l \leq N-1, \quad k, l \in \mathbb{N}\Big\}$$

and the corresponding projection operator $P_N : L_2(\Omega) \to S_N$ with

$$P_N f(x,y) = \sum_{k,l=1}^{N-1} f_{k,l} 2\sin(k\pi x)\sin(l\pi y),$$

where

$$f(x,y) = \sum_{k,l=1}^{\infty} f_{k,l} 2\sin(k\pi x)\sin(l\pi y).$$

Remarks: Since P_N is a projection, we obviously have $\|P_N\| \leq 1$ and $P_N\Big|_{S_N}$ is the identity.
Also, since P_N projects to the eigenfunctions of the operator $(-\Delta_D)^{\alpha/2}$, we have

$$(-\Delta)^{\alpha/2} P_N f = P_N(-\Delta)^{\alpha/2} f \quad \forall f \in \mathcal{F}_{\frac{\alpha}{2}}. \tag{2.4}$$

2.2. **Discretization and Fourier interpolation.** We define a uniform grid with the gridsize $h = \frac{1}{N}$ and the corresponding interior gridpoints

$$\Omega_h := \{(x_k, y_l) = (kh, lh) : 1 \leq k, l \leq N-1\}.$$

In the error estimates we always assume that $h < 1$, because we are interested in the fine-grid limit. The finite difference method results in a nodal approximation. We can relate it to a continuous analytic solution by using the following interpolation.

Definition 4. *Let $\mathcal{I}_N : \mathbb{R}^{(N-1)\times(N-1)} \to S_N$ denote the sine Fourier interpolation given by*

$$(\mathcal{I}_N \mathbf{f})(x,y) = \sum_{k,l=1}^{N-1} \overline{f}_{k,l} 2\sin(k\pi x)\sin(l\pi y),$$

where

$$\overline{f}_{k,l} = h^2 \sum_{m,n=1}^{N-1} f(m,n) 2\sin(k\pi x_m)\sin(l\pi y_n).$$

Here the entries of $\mathbf{f}$ *in the interior gridpoints are denoted with* $f(m,n)$ *for* $1 \leq m,n \leq N-1$.

Remarks: One can easily verify that for any $u \in C(\bar{\Omega})$ we have

$$\left(\mathcal{I}_N \left(u\Big|_{\Omega_h}\right)\right)(x) = u(x) \quad \forall x \in \Omega_h, \tag{2.5}$$

moreover, for any $g \in C_0(\bar{\Omega})$ and $\Theta \in S_N$ we have

$$(g,\Theta) = (\mathcal{I}_N(g\Big|_{\Omega_h}),\Theta) \tag{2.6}$$

where $(\cdot,\cdot)$ denotes the $L_2(\Omega)$-inner product and $C_0(\bar{\Omega})$ denotes the continuous functions on Ω with vanishing boundary values.

3. Results

Our objective is to find $u : [0,T] \to C(\overline{\Omega})$ such that

$$\begin{cases} \frac{\partial u}{\partial t}(t,\underline{x}) = -\mu\left(-\Delta_D\right)^{\alpha/2} u(t,\underline{x}), & \underline{x} \in \Omega,\ t \in (0,T) \\ u(t,\underline{x}) = 0, \quad \underline{x} \in \partial\Omega,\ t \in [0,T), \\ \lim_{t\to 0} u(t,\underline{x}) = u(0,\underline{x}) = u_0(\underline{x}), & \underline{x} \in \Omega. \end{cases} \tag{3.1}$$

Here $\Omega = (0,1)\times(0,1)$, $u_0 \in \mathcal{F}_{\alpha+1}$ are given and we assume that $\alpha \in (1,2]$. In the applications, this assumption is not restrictive, at the same time, it implies sufficient smoothness.

Lemma 1. *If* $f \in \mathcal{F}_2$ *then* $f \in C_0(\Omega)$.

Proof. Taking the eigenfunction expansion

$$f(x,y) = \sum_{k,l=1}^{\infty} f_{k,l} 2\sin(k\pi x)\sin(l\pi y) \tag{3.2}$$

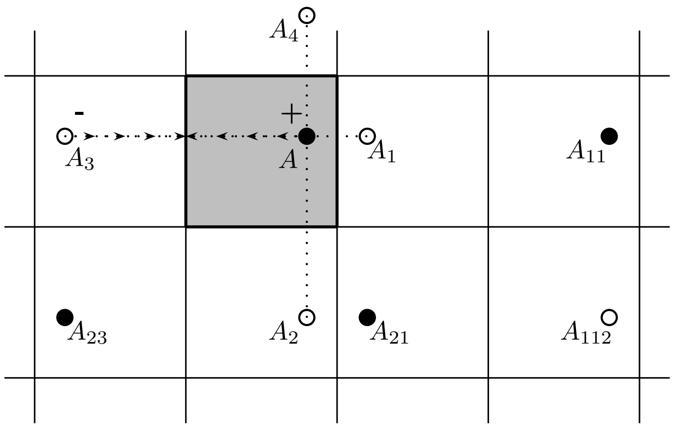

FIGURE 1. The extension procedure $\bar{\cdot}$ and annihilation between A_3 and A. Values equal to $u(A)$ are denoted with $\bullet$, while their negative with $\circ$. Ω is the shaded domain.

the relation $f \in \mathcal{F}_2$ implies that $\sum_{k,l=1}^{\infty} f_{k,l}^2 (k^2+l^2)^2$ is convergent. Then using the Cauchy–Schwartz inequality we have for all $N \in \mathbb{N}$ that

$$\sqrt{\sum_{k,l=1}^{N} (f_{k,l}(k^2+l^2))^2}\sqrt{\sum_{k,l=1}^{N} \frac{1}{(k^2+l^2)^{\frac{4}{3}}}} \geq \sum_{k,l=1}^{N} \frac{|f_{k,l}|(k^2+l^2)}{(k^2+l^2)^{\frac{2}{3}}} \geq \sum_{k,l=1}^{N} |f_{k,l}|,$$

where the left hand side and hence the right hand side is finite. Therefore, the series in (3.2) converges uniformly, which results in a continuous sum with vanishing boundary values as stated. □

3.1. **Extension corresponding to homogeneous Dirichlet boundary conditions.** The extension $\bar{\cdot} : C_0(\Omega) \to C(\mathbb{R}^2)$ is defined as follows: The reflection of $A \in \Omega$ across the faces of Ω is denoted with A_1, A_2, A_3 and A_4 in a fixed order. With these

$$\overline{u}(A_1) = \overline{u}(A_2) = \overline{u}(A_3) = \overline{u}(A_4) := -u(A).$$

Following this procedure repeatedly on the boundary of the new unit squares and using the equality $u(A_{i,j}) = -u(A_i) = -u(A_j) = u(A)$ we have that the extension $\bar{\cdot}$ is well-defined on $\mathbb{R}^2$. See also Figure 1.

Remarks: Note that this is an odd extension in the sense that the following identities are valid for all $x, y \in \mathbb{R}$:

$$\begin{aligned} \overline{u}(1+x,y) &= -\overline{u}(1-x,y); \quad \overline{u}(x,y) = -\overline{u}(-x,y) \\ \overline{u}(x,1+y) &= -\overline{u}(x,1-y); \quad \overline{u}(x,y) = -\overline{u}(x,-y). \end{aligned} \tag{3.3}$$

A physical motivation of the extension procedure is that taking particles in Ω with positive weight, their mirror images should be equipped with the negative weights since in this way, after a collision on $\partial\Omega$ they will be annihilated making the boundary an absorbing wall. This is also depicted in Figure 1 between A and A_3.

Note that for any integers k and l the extension of the function $\sin k\pi x \sin l\pi y$ from Ω to $\mathbb{R}^2$ is given with the same formula, which is used without further reference.

3.1.1. *The extended problem and its solution.* Using the extension procedure we pose the following extended problem for $\overline{u}$:

$$\begin{cases} \frac{\partial \overline{u}}{\partial t}(t,\underline{x}) = -\tilde{\mu}\,(-\Delta)^{\alpha/2}\,\overline{u}(t,\underline{x}), \quad \underline{x} \in \mathbb{R}^2,\ t \in (0,T) \\ \lim_{t\to 0}\overline{u}(t,\underline{x}) = \overline{u}(0,\underline{x}) = \overline{u}_0(\underline{x}), \quad \underline{x} \in \mathbb{R}^2, \end{cases} \tag{3.4}$$

where $u_0 \in C(\overline{\Omega})$ and $\tilde{\mu} > 0$ are given.

To highlight the relation between (3.1) and (3.4) our main tool is the fact that the definitions in (2.1) and (2.2) are equivalent in a sense. Using also the formulation in (2.3) we state the following.

Theorem 1. *Using the assumptions for* (3.1)*, the following equality holds true for all* $u \in \mathcal{F}_\alpha(\Omega)$*:*

$$(-\Delta_D)^{\alpha/2}u(\underline{x}) = \frac{1}{2\tilde{C}_\alpha}\frac{\alpha^2}{C_\alpha}(-\Delta)^{\alpha/2}\overline{u}(\underline{x}), \quad \underline{x} \in \Omega$$

with a constant $\tilde{C}_\alpha \in \mathbb{R}^+$*.*

The technical proof is postponed to the appendix.

Lemma 2. *Using the assumptions for* (3.1)*, the solution of* (3.4) *is smooth in the sense that for* $t > 0$ $\overline{u}(t,\cdot) \in C^\infty(\mathbb{R}^2)$ *and it satisfies homogeneous "boundary" condition:* $\overline{u}(t,x,y) = 0$ *for* $(x,y) \in \partial\Omega$*.*

Proof. We first note that (3.4) is well-posed and its solution can be given as

$$\overline{u}(t,x,y) = \big(\Phi_t * \overline{u}_0\big)(x,y),$$

where Φ_t denotes the fundamental solution of (3.4), see [18]. With a straightforward generalization of Lemma 2.3 in [25] we obtain that $\Phi_t \in C^\infty(\mathbb{R}^2)$ which implies also the required smoothness of $\overline{u}(t,\cdot)$.

Concerning the boundary conditions we only show that $u(t,x_0,1) = 0$ for $x_0 \in (0,1)$, the proofs for the remaining cases can be obtained similarly. Using the fact that Φ_t is even in both of its variables and the equalities in (3.3) we obtain

$$\begin{aligned}
&\overline{u}(t,x_0,1) = \lim_{\epsilon_n\to 0-} \overline{u}(t,x_0,1-\epsilon_n) = \lim_{\epsilon_n\to 0-} \overline{u}_0 * \Phi_t(x_0,1-\epsilon_n) \\
&= \lim_{\epsilon_n\to 0-} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{u}_0(x_0-x,1-\epsilon_n-y)\Phi_t(x,y)\,\mathrm{d}y\ \mathrm{d}x \\
&= -\lim_{\epsilon_n\to 0-} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{u}_0(x_0-x,1+\epsilon_n+y)\Phi_t(x,y)\,\mathrm{d}y\ \mathrm{d}x \\
&= -\lim_{\epsilon_n\to 0-} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{u}_0(x_0-x,1+\epsilon_n+y)\Phi_t(x,-y)\,\mathrm{d}y\ \mathrm{d}x \\
&= -\lim_{\epsilon_n\to 0-} \big(\overline{u}_0 * \Phi_t\big)(x_0,1+\epsilon_n) = -\lim_{\epsilon_n\to 0-} \overline{u}(t,x_0,1+\epsilon_n) = -\overline{u}(t,x_0,1),
\end{aligned}$$

which gives that $\overline{u}(t,x_0,1) = 0$. □

3.2. **Analytic solution with sine Fourier expansion.** Using Theorem 1 the solution of (3.1) is nothing but the restriction of the solution of (3.4) to Ω. We also need its Fourier expansion, which is given in the following.

Theorem 2. *Using the assumptions for (3.1), for all $t > 0$ there exists a unique solution $u(t,\cdot)$ of (3.1) such that $u(t,\cdot) = \overline{u}(t,\cdot)\Big|_\Omega \in \mathcal{F}_\alpha$, where $\tilde{\mu} = \mu\frac{\alpha^2}{2\tilde{C}_\alpha C_\alpha}$. Moreover, $\|\overline{u}(t,\cdot)\Big|_\Omega\|_{\frac{\alpha}{2}} \le \|\overline{u}_0\Big|_\Omega\|_{\frac{\alpha}{2}}$ and $u(t,\cdot)$ satisfies the homogeneous Dirichlet boundary conditions.*

Proof. We seek the solution of (3.1) in the following form:

$$u(t,x,y) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} u_{k,l}(t) 2\sin(k\pi x)\sin(l\pi y).$$

Taking the scalar product of (3.1) with the function $2\sin(k\pi x)\sin(l\pi y)$ on Ω, we get the following system of differential equations:

$$\begin{cases} u'_{k,l}(t) = -\mu\Big((k\pi)^2 + (l\pi)^2\Big)^{\alpha/2} u_{k,l}(t) \\ u_{k,l}(0) = u_{0,k,l}, \end{cases}$$

where $u_{0,k,l}$ are the coefficients of the Fourier series of u_0 and $k,l \in \mathbb{N}$. Therefore, we have that

$$u(t,x,y) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} u_{0,k,l} \exp\Big\{ -t\mu\Big((k\pi)^2 + (l\pi)^2\Big)^{\alpha/2}\Big\} 2\sin(k\pi x)\sin(l\pi y).$$

such that $u(t,\cdot) \in \mathcal{F}_\alpha$ for any $\alpha > 0$.

To see uniqueness, we note that using Theorem 1 for the extension of any solution of (3.1) we have

$$\partial_t \overline{u}(t,x,y) = \partial_t u(t,x,y) = -\mu(-\Delta_D)^{\frac{\alpha}{2}} u(t,x,y) = -\mu \frac{\alpha^2}{2\tilde{C}_\alpha C_\alpha}(-\Delta)^{\frac{\alpha}{2}} \overline{u}(t,x,y)$$

for all $(x,y) \in \Omega$ and by the extension procedure,

$$\partial_t \overline{u}(t,x,y) = -\tilde{\mu}(-\Delta)^{\frac{\alpha}{2}} \overline{u}(t,x,y)$$

for all $(x,y) \in \mathbb{R}^2$. On the other hand, this solution (as mentioned in Lemma 2) is unique, which also implies the uniqueness of the solution of (3.1).

Since $-t\mu\Big((k\pi)^2 + (l\pi)^2\Big)^{\alpha/2} \le 0$, we have that

$$\begin{aligned}
&\|(\overline{u}(t,\cdot)\Big|_\Omega)\|^2_{\frac{\alpha}{2}} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[u_{0,k,l}\exp\Big\{-t\mu\Big((k\pi)^2+(l\pi)^2\Big)^{\alpha/2}\Big\}]^2 \\
&+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\Big((k\pi)^2+(l\pi)^2\Big)^{\alpha/2}[u_{0,k,l}\exp\Big\{-t\mu\Big((k\pi)^2+(l\pi)^2\Big)^{\alpha/2}\Big\}]^2 \\
&\le \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[u_{0,k,l}]^2 + \Big((k\pi)^2+(l\pi)^2\Big)^{\alpha/2}[u_{0,k,l}]^2 = \|(\overline{u}_0\Big|_\Omega)\|^2_{\frac{\alpha}{2}}
\end{aligned}$$

as stated. □

3.3. **Numerical solution of** (3.1)**.** We analyze here the numerical solution of (3.1) using the matrix transformation method (MTM). First we investigate an approximation with a spectral method using Fourier projection for the initial value. Using this result, we will show that the MTM also gives a good approximation.

3.3.1. *The spectral method.* We will use the following approximation results, which are stated in [4] for periodic boundary conditions, but can be adopted (with a minimal change) to the case of Dirichlet boundary conditions.

Proposition 1. *For all $0 \le \eta \le s$ there is a constant C such that for all $u \in \mathbb{H}^s(\Omega)$ we have*

$$\|u - P_N u\|_\eta \le C N^{\eta - s}|u|_s, \tag{3.5}$$

moreover, if $s > 1$ then we also have

$$\|u - \mathcal{I}_N(u|_{\Omega_h})\|_\eta \le C N^{\eta - s}|u|_s. \tag{3.6}$$

First we define an approximation $u_N(t,\cdot) \in S_N$ as the solution of the following problem:

$$\begin{cases} \frac{\partial u_N}{\partial t}(t,\underline{x}) = -\mu\,(-\Delta_D)^{\alpha/2}\,u_N(t,\underline{x}), & \underline{x} \in \Omega,\ t \in (0,T), \\ \lim_{t\to 0} u_N(t,\underline{x}) = u_N(0,\underline{x}) = \mathcal{I}_N(u_0\Big|_{\Omega_h})(\underline{x}), & \underline{x} \in \Omega. \end{cases} \tag{3.7}$$

A corresponding error estimate is given as follows:

Theorem 3. *Let u_N be the solution of the problem in* (3.7) *and u is the solution of* (3.1)*, using again the assumptions here. Then there exists a constant C independent of u_0 such that for all $t \in (0,T)$ the following error estimation is valid:*

$$\|u(t) - u_N(t)\|_0 \le Ch\|u_0\|_{\alpha+1}, \quad t \in [0,T_0].$$

Proof. Let $Lf := \mu(-\Delta)^{\alpha/2}f$ and $L_N f = P_N Lf$. Taking the scalar product of (3.1) and (3.7) with a function $\Theta \in S_N$ we get the following equalities:

$$\big(\partial_t P_N u, \Theta\big) + \big(LP_N u, \Theta\big) = -\big(L(u - P_N u), \Theta\big), \tag{3.8}$$

$$\big(\partial_t u_N, \Theta\big) + \big(Lu_N, \Theta\big) = 0, \tag{3.9}$$

where, indeed, the notations $P_N(u(s))$, $LP_N(u(s))$ and $u_N(s)$ should be used, which for the brevity is simplified over the proof. The difference of (3.8) and (3.9) gives for $e = u_N - P_N u$ and $\Theta = e$ the following:

$$\big(\partial_t e, e\big) + \big(Le, e\big) = \big(L(u - P_N u), e\big).$$

Since L is positive definite, we obtain

$$\begin{aligned} &2\|e\|_0\partial_t\|e\|_0 = \partial_t\|e\|_0^2 = \int_\Omega \partial_t e^2 = \int_\Omega 2e\partial_t e \\ &= 2\big(L(u-P_N u), e\big) - 2\big(Le, e\big) \le 2\big(L(u-P_N u), e\big) \le 2\|e\|_0\|L(u - P_N u)\|_0, \end{aligned}$$

which implies

$$\partial_t\|e\|_0 \le \|L(u - P_N u)\|_0.$$

Integrating both sides on $[0,t]$ gives

$$\|e(t)\|_0 \le \|e_0\|_0 + t\sup_{(0,t)}\|L(u - P_N u)\|_0. \tag{3.10}$$

Using (3.7), (3.5) and (3.6) with $\eta = 0$, we get
(3.11)
$$\|e_0\|_0 = \|u_N(0) - P_N u(0)\|_0 \le \|\mathcal{I}_N(u_0\Big|_{\Omega_h}) - u_0\|_0 + \|u_0 - P_N u(0)\|_0 \le CN^{-\alpha}\|u_0\|_\alpha$$

and (3.5) implies that

$$\|L(u - P_N u)\|_0 \le \|u - P_N u\|_\alpha \le N^{-1}\|u\|_{\alpha+1}. \tag{3.12}$$

Therefore, inserting (3.11) and (3.12) into (3.10) with the inequality in Theorem 2 gives the following estimate:

$$\|e(t)\|_0 \le CN^{-\alpha}\|u_0\|_\alpha + tN^{-1}\|u\|_{\alpha+1}.$$

Combining this with the triangle inequality and using (3.5) with $\eta = 0$ we obtain

$$\|u(t) - u_N(t)\|_0 \le \|u(t) - P_N u(t)\|_0 + \|e\|_0 \le 2CN^{-\alpha}\|u_0\|_\alpha + tN^{-1}\|u\|_{\alpha+1}. \tag{3.13}$$

Finally, using that $\alpha > 1$, we obtain $2CN^{-\alpha}\|u_0\|_\alpha \le 2CN^{-1}\|u_0\|_{\alpha+1}$, which compared with (3.13) completes the proof of the theorem. □

3.3.2. *The matrix transformation method.* In this subsection, we establish the order of convergence for the MTM. $\widehat{A}_h \in \mathbb{R}^{(N-1)^2\times(N-1)^2}$ denotes the matrix corresponding to the standard five-point difference scheme of the Laplacian $(-\Delta)$ with homogeneous Dirichlet boundary conditions.

The eigenvectors of the matrix $\widehat{A}_h$ and the corresponding eigenvalues in the order of increasing value are given for $k, l \in \{1, 2, ..., N-1\}$ by

$$\Big(v_{k,l}\Big)_{i,j} = 2\sin(k\pi ih)\sin(l\pi jh) \quad \text{and} \quad \lambda_{k,l} = \Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 + \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2.$$

Since $\widehat{A}_h$ is positive definite, we can take its singular value decomposition $V^T\Lambda V$. Here the k-th column of V is the k-th eigenvector of $\widehat{A}_h$ belonging to the k-th eigenvalue and the diagonal matrix Λ contains the corresponding eigenvalues.

The basic idea of the MTM is to use the matrix $\widehat{A}_h^{\alpha/2}$ for the approximation of the operator $\big(-\Delta_D\big)^{\alpha/2}$. Accordingly, we define $\widehat{A}_h^{\alpha/2} := V^T\Lambda^{\alpha/2}V$ and with this we have to solve the semidiscretized problem

$$\begin{cases} \dfrac{\partial \widehat{U}_N}{\partial t}(t) = -\mu \widehat{A}_h^{\alpha/2}\widehat{U}_N(t), \quad \forall t \ge 0, \\ \widehat{U}_N(0) = u_0(\cdot)\Big|_{\Omega_h}, \end{cases} \tag{3.14}$$

where $\widehat{U}_N(t)$ is a vector in $\mathbb{R}^{(N-1)^2}$ and its components are given in the gridpoints of Ω_h.

3.4. **Convergence result.** We also introduce the operator $A_h^{\alpha/2}: S_N \to S_N$ with

$$A_h^{\alpha/2} u(x,y) = \sum_{k,l=1}^{N-1} u_{k,l} 2\sin(k\pi x)\sin(l\pi y)\Big[\Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 + \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2\Big]^{\alpha/2},$$

$$\text{where} \quad u(x,y) = \sum_{k,l=1}^{N-1} u_{k,l} 2\sin(k\pi x)\sin(l\pi y).$$

Remark: The operators A_h and $\widehat{A}_h$ are equivalent in the following sense

$$\Big[A_h u(x,y)\Big]\Big|_{\Omega_h} = \widehat{A}_h\Big[u(x,y)\Big|_{\Omega_h}\Big] \text{ and } \mathcal{I}_N\Big\{\widehat{A}_h\Big[u(x,y)\Big|_{\Omega_h}\Big]\Big\} = A_h u(x,y), \tag{3.15}$$

for all $u \in S_N$. To relate the operators $(-\Delta)$ and A_h we need the following estimate.

Proposition 2. *For arbitrary $\alpha \in (0,2]$ and integers k,l with $1 \le k,l \le N-1$ there is a mesh-independent constant $C_{\alpha,0}$ such that the following estimation is valid:*

$$\Big[(k\pi)^2 + (l\pi)^2\Big]^{\alpha/2} - \Big[\Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 + \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2\Big]^{\alpha/2} \le C_{\alpha,0} h^\alpha((k\pi)^{2\alpha} + (l\pi)^{2\alpha}).$$

Proof. We first note that using the Taylor expansion of the function $\sin^2$ around zero we have that

$$\Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 = (k\pi)^2 - h^2\frac{(k\pi)^4}{12}\cos\xi_k$$

is satisfied for all $1 \le k \le N-1$ with some $\xi_k \in [0,\pi]$.

Since $\sin x \le x$ is satisfied for $x \ge 0$, we obviously get

$$B = \Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 + \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2 \le (k\pi)^2 + (l\pi)^2 = A$$

such that $\cos\xi_k \ge 0$ should also be satisfied. Therefore, using the inequality $A^{\frac{\alpha}{2}} - B^{\frac{\alpha}{2}} \le (A-B)^{\frac{\alpha}{2}}$ for $\frac{\alpha}{2} \in (0,1]$, we finally obtain the following estimate for all $1 \le k,l \le N-1$:

$$\begin{aligned}
&\Big[(k\pi)^2 + (l\pi)^2\Big]^{\alpha/2} - \Big[\Big(\frac{2}{h}\sin\frac{k\pi h}{2}\Big)^2 + \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2\Big]^{\alpha/2} \\
&\le \Big(h^2\frac{(k\pi)^4}{12}\cos\xi_k\Big)^{\alpha/2} + \Big(h^2\frac{(l\pi)^4}{12}\cos\xi_l\Big)^{\alpha/2} \le \left[\left(\frac{(k\pi)^4}{12}\right)^{\alpha/2} + \left(\frac{(l\pi)^4}{12}\right)^{\alpha/2}\right] h^\alpha \\
&= \frac{1}{12^{\alpha/2}} h^\alpha((k\pi)^{2\alpha} + (l\pi)^{2\alpha}),
\end{aligned}$$

which proves the statement. □

We can now quantify the difference between $A_h^{\alpha/2}$ and $(-\Delta)^{\alpha/2}$.

Lemma 3. *If $u \in S_N$ and $\alpha \in (0,2]$, then we have*

$$\|\big(-\Delta_D\big)^{\alpha/2}u - A_h^{\alpha/2}u\|_0 \le C_N h^\alpha |u|_{\alpha+1}.$$

Proof. If $u \in S_N$ with $u(x,y) = \sum_{k,l=1}^{N-1} u_{k,l} 2\sin(k\pi x)\sin(l\pi y)$ then

$$\big(-\Delta_D\big)^{\alpha/2}u(x,y) = C_\alpha \sum_{k,l=1}^{N-1} u_{k,l}\big((k\pi)^2 + (l\pi)^2\big)^{\alpha/2} 2\sin(k\pi x)\sin(l\pi y)$$

$$A_h^{\alpha/2}u(x,y) = \sum_{k,l=1}^{N-1} u_{k,l}\left(\left(\frac{2}{h}\sin\frac{k\pi h}{2}\right)^2 + \left(\frac{2}{h}\sin\frac{l\pi h}{2}\right)^2\right)^{\alpha/2} 2\sin(k\pi x)\sin(l\pi y)$$

such that using Proposition 2 we obtain

$$\begin{aligned}
&\|\big(-\Delta_D\big)^{\alpha/2}u - A_h^{\alpha/2}u\|_0^2 \\
&= \sum_{k,l=1}^{N-1} u_{k,l}^2 \left(\big((k\pi)^2 + (l\pi)^2\big)^{\alpha/2} - \left(\left(\frac{2}{h}\sin\frac{k\pi h}{2}\right)^2 + \left(\frac{2}{h}\sin\frac{l\pi h}{2}\right)^2\right)^{\alpha/2}\right) \\
&\le C_{\alpha,0} h^\alpha \sum_{k,l=1}^{N-1} u_{k,l}^2 \left((k\pi)^{2\alpha} + (l\pi)^{2\alpha}\right) \le C_{\alpha,0} h^\alpha \sum_{k,l=1}^{N-1} u_{k,l}^2 \left((k\pi)^2 + (l\pi)^2\right)^{\alpha+1} \\
&= C_{\alpha,0} h^\alpha |u|_{\alpha+1}^2,
\end{aligned}$$

as stated in the lemma. □

To use a key approximation theorem, we pose some assumptions following the setting in [2].

Assumption 1: For the Banach spaces $\big(X_n\big)_{n\in\mathbb{N}}$ and X, the operators $P_n : X \to X_n$ and $J_n : X_n \to X$ satisfy the following:

- there exists a constant $K > 0$ such that $\|P_n\|, \|J_n\| \le K \;\; \forall n \in \mathbb{N}$,
- $P_n J_n = I_n$, where I_n is the identity operator on the space X_n,
- $J_n P_n f \to f$, $\forall f \in X$ for $n \to \infty$.

Assumption 2: For the generators $\big(A_n\big)_{n\in\mathbb{N}}$ and A of the strongly continuous semigroups $\big(T_n\big)_{n\in\mathbb{N}}$ and T on $\big\{X_n\big\}_{n\in\mathbb{N}}$ and X, respectively, we have

- $\|T_n(t)\| \le Me^{\omega t}$ $\forall n \in \mathbb{N}$ for some constants $M > 1$ and $\omega \in \mathbb{R}$

On the Banach space $(Y, \|\cdot\|_Y)$ with $Y \subset D(A)$ dense we have the following:

- $\|T(t)\|_Y \le Me^{\omega t}$ with the above constants $M > 1$ and ω,

- for all $g \in Y$ there exists a sequence (y_n) with $y_n \in D(A_n)$ which satisfies the following:

$$\|y_n - P_n g\|_{X_n} \to 0 \quad \text{and} \quad \|A_n y_n - P_n A g\|_{X_n} \to 0 \quad \text{for } n \to \infty. \tag{3.16}$$

Theorem 4. *Suppose that Assumption 1 and 2 hold true and there exist constants $C > 0$ and $p \in \mathbb{N}$ such that for all $f \in Y$ the following inequality holds:*

$$\|A_n P_n f - P_n A f\|_{X_n} \leq C \frac{\|f\|_Y}{n^p}.$$

Then for all $t > 0$ there exists a constant $C' > 0$ such that we have the error estimate

$$\|T_n(t) P_n f - P_n T(t) f\|_{X_n} \leq C' \frac{\|f\|_Y}{n^p}$$

and this convergence is uniform in t on compact intervals.

This statement is an easy consequence of Corollary 1.11 in [9], page 163 and the detailed proof can be found in [2].

To use the above results we investigate the following problem:

$$\begin{cases} \frac{\partial U_N}{\partial t}(t, \underline{x}) = -\mu A_h^{\alpha/2} U_N(t, \underline{x}), & \underline{x} \in \Omega,\ t \in (0, T) \\ U_N(t) \in S_N \quad t \in (0, T) \\ \lim_{t \to 0} U_N(t, \underline{x}) = \mathcal{I}_N(u_0\Big|_{\Omega_h})(\underline{x}), & \underline{x} \in \Omega,\ t \in (0, T), \end{cases} \tag{3.17}$$

which is related with (3.14) to obtain the main result. We intend to use also Proposition 2 and Theorem 3 and therefore, we need to assume some smoothness on the initial condition and restrict the exponent of the Dirichlet Laplacian to preserve the initial accuracy.

Theorem 5. *Using the assumptions for (3.1), we have that the numerical solution $\widehat{U}_N$ in (3.14) satisfies the following error estimate:*

$$\|u(t) - \mathcal{I}_N(\widehat{U}_N(t))\|_0 \leq hC\|u_0\|_{\alpha+1} \quad t \in [0, T].$$

Proof. Using (3.17) and the interpolation property of $\mathcal{I}_N$ in (2.5) we obviously have that

$$U_N(0, \underline{x}) = (\mathcal{I}_N(u_0\Big|_{\Omega_h}))(\underline{x}) = u_0(\underline{x}) \quad \text{for } \underline{x} \in \Omega_h. \tag{3.18}$$

Therefore, applying (3.17) in the gridpoints using (3.15) gives that

$$\frac{\partial U_N}{\partial t}(t, \underline{x}) = -\mu A_h^{\alpha/2} U_N(t, \underline{x}) = -\mu \widehat{A}_h^{\alpha/2} U_N(t, \underline{x})\Big|_{\Omega_h}. \tag{3.19}$$

The equalities in (3.18) and (3.19) imply that (3.17) in the gridpoints coincides with (3.14) such that

$$U_N(t)\Big|_{\Omega_h} = \widehat{U}_N(t) \quad \text{and vice versa} \quad \mathcal{I}_N(\widehat{U}_N(t)) = U_N(t). \tag{3.20}$$

If $u \in S_N$ then using Lemma 3 we get

$$\|\big(-\Delta\big)^{\alpha/2}u - A_h^{\alpha/2}u\|_0 \le C_\alpha h|u|_{\alpha+1}.$$

We can now apply the Theorem 4 with the following choice for the function spaces:

$$X_n = (S_N, \|\cdot\|_0),\ X = L_2(\Omega),\ Y = \mathcal{F}_{\alpha+1}(\Omega)$$

and for the corresponding operators:

$$P_n = P_N,\ J_n = S_N \hookrightarrow L_2(\Omega),\ A_n = A_h^{\alpha/2},\ \text{and } A = \big(-\Delta\big)^{\alpha/2},$$

where $\hookrightarrow$ denotes the identical embedding, i.e. $J_n(v) = v$. Finally, we define the semigroups $T_n(t) \in S_N$ and $T(t) \in L_2(\Omega)$ to be the solution operators of (3.17) and (3.1), respectively.

To verify **Assumption 1**, we first note that $\|P_N\| = 1$, see the remark after the definition of P_N. The subspace S_N is also equipped with the $\|\cdot\|_0$-norm, so that $\|J_n\| = 1$. Since P_N is a projection, we also have $P_n(J_n(u_k)) = P_N(u_k) = u_k$ for all $u_k \in S_N$. Finally, the orthogonal system $\Big\{2\sin(k\pi x)\sin(l\pi y) : x, y \in [0,1], k, l \in \mathbb{N}^+\Big\}$ is complete and therefore, $J_n(P_n(v)) \to v$ for all $v \in L_2(\Omega)$.

For the first estimate in **Assumption 2**, we note that

$$T_n(U_N(0,\cdot)) = U_N(t,\cdot) = \exp\{-\mu A_h^{\alpha/2}t\}U_N(0,\cdot)$$

and therefore, $\|T_n\| = \|\exp\{-\mu A_h^{\alpha/2}t\}\| \le 1$ since the matrix A_h is positive definite. Using Theorem 2 we have that $\|u(t,\cdot)\|_\alpha \le \|u(0,\cdot)\|_\alpha$ for the solution of (3.1) such that we obtain $\|T(t)\|_Y \le 1$.
Finally, with the choice $y_n = P_N g$ the first item in (3.16) is obviously satisfied and the second one is an easy consequence of (2.4) and Lemma 3:

$$\|A_n P_N g - P_N A g\|_0 = \|\big(-\Delta_D\big)^{\alpha/2}g - A_h^{\alpha/2}g\|_0 \le C_{\alpha,0}h^\alpha\|g\|_{\alpha+1}.$$

Finally, using $A_n P_n u = A_h^{\alpha/2}P_N u$ and (2.4) again with Lemma 3 gives that

$$\begin{aligned}\|A_n P_n u - P_n A u\|_0 &= \|A_h^{\alpha/2}P_N u - P_N(-\Delta)^{\alpha/2}u\|_0\\ &= \|A_h^{\alpha/2}P_N u - (-\Delta)^{\alpha/2}P_N u\|_0 \le C_{\alpha,0}h^\alpha\|u\|_{\alpha+1}\end{aligned}$$

such that the assumption in Theorem 4 is satisfied. Therefore, Theorem 4 implies the following inequality

$$\|u_N(t) - U_N(t)\|_0 \leq C_1 h^\alpha \|u_0\|_{\alpha+1}.$$

According to Theorem 3, we also have

$$\|u(t) - u_N(t)\|_0 \leq hC_2\|u_0\|_{\alpha+1},$$

such that using (3.20) the triangle inequality and $\alpha > 1$ implies

$$\begin{aligned}&\|u(t) - \mathcal{I}_N)(\widehat{U}_N(t))\|_0 = \|u(t) - U_N(t)\|_0 \leq \|u(t) - u_N(t)\|_0 + \|u_N(t) - U_N(t)\|_0 \\ &\leq \big(C_1 + C_2\big)h\|u_0\|_{\alpha+1}\end{aligned}$$

as stated in the theorem. □

Remarks: Theorem 5 gives the spatial accuracy of the MTM method, which is a consequence of the standard five-point stencil in the underlying finite difference discretization. The accuracy of the full discretization depends on the time integration to approximate $\widehat{U}_N(t)$.

The method presented here can also be applied for rectangular domains in any space dimensions. In such a case we know the eigenvalues of the matrix A_h corresponding to the Dirichlet Laplacian $-\Delta_D$ and the extension procedure in Section 3.1 can also be applied. By using sharp estimates for the eigenvalues of A_h one could mimic the presented analysis. At the same time, in this case additional strong smoothness assumptions would be necessary to ensure that applying the fractional Dirichlet Laplacian will lead to solutions with homogeneous boundary conditions since in general, the extension procedure can not be performed.
The treatment of inhomogeneous boundary conditions is still an open problem. Even the correct formulation of a corresponding continuous problem is still under discussion in the literature [1].

4. Numerical experiments

We investigate the following test problem:

$$(4.1)\qquad \begin{cases}\frac{\partial \overline{u}}{\partial t}(t,\underline{x}) = -0.01\,(-\Delta)^{1.4/2}\,\overline{u}(t,\underline{x}), \quad t\in(0,1),\\ u(0,x,y) = \Big[100\Big(x(1-x) + \big(x(1-x)\big)^2\Big)\Big(y(1-y) + \big(y(1-y)\big)^2\Big)\Big],\end{cases}$$

where $\underline{x} = (x,y) \in \Omega = (0,1)\times(0,1)$. Note that this is a restriction of the corresponding problem in $\mathbb{R}^2$ such that according to Lemma 2, the homogeneous Dirichlet boundary condition is also satisfied. The sine Fourier series of the analytic solution of (4.1) is

$$u(t,x,y) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} 230400 \sin(k\pi x)\sin(l\pi y)\frac{\big(1+(-1)^{k+1}\big)\big(1+(-1)^{l+1}\big)}{(kl)^5\pi^{10}}$$
$$\cdot \exp\big[-0.01t\big((k\pi)^2+(l\pi)^2\big)^{0.7}\big].$$

The semidiscretization of (4.1) with MTM for the point values in Ω_h is given by

$$\begin{cases} \dfrac{\partial \widehat{U}_N}{\partial t}(t) = -0.01\widehat{A}_h^{0.7}\widehat{U}_N(t), \quad t\in(0,1), \\ \widehat{U}_N(0) = 100\Big(x(1-x)+\big(x(1-x)\big)^2\Big)\Big(y(1-y)+\big(y(1-y)\big)^2\Big)\Big|_{\Omega_h}. \end{cases} \tag{4.2}$$

To solve this ODE we used the implicit Euler and Crank–Nicolson method. Based on these approximations, we can estimate the error $\|\mathcal{I}_N\Big[\widehat{U}_N(t)\Big](\cdot)-\overline{u}(t,\cdot)\|_0$.

To compare $(\mathcal{I}_N)\Big[\widehat{U}_N(t)\Big](\cdot)$ with $u(t,\cdot)$ we cut off the Fourier-series of $u(t,\cdot)$ at the first N terms in both variables, this results an extra error term of order $O(h^2)$, which does not harm the accuracy of the method since $u\in\mathcal{F}_{\alpha+2}$.

h	time step	IE $\|\cdot\|_0$	convergence order	CN $\|\cdot\|_0$	convergence order
0.2	0.2	0.0111		0.0083	
0.1	0.1	0.0036	1.6245	0.0021	1.9822
0.05	0.05	0.0013	1.4695	5.309610^{-4}	1.9837
0.025	0.025	5.015310^{-4}	1.3741	1.32810^{-4}	1.9993
0.0125	0.0125	2.178310^{-4}	1.2031	3.320310^{-5}	1.9999

Table 1. Error and convergence for the test problem in (4.1) using the MTM with implicit Euler (IE) and Crank–Nicolson (CN) method

In this case we obtain a second order convergence for the full discretization if the Crank–Nicolson method is used for the time steps. We can not expect this (full) convergence order, if the implicit Euler method is applied. Accordingly, we obtain an order near to *one*.

Appendix

4.1. **Equivalence of different forms of the fractional Laplacian.** For the proof of Theorem 1 we recall the Bessel functions $K_\nu(z)$, $I_\nu(z)$ and the modified Struve function $\mathbf{L}_\nu(z)$; see the definitions in the work [11] at the points of 8.55, 8.43 and 8.407. We summarize these properties in the following.

Proposition 3. *For all* $a \in \mathbb{R}^+$ *and* $\beta, \mu, \nu \in \mathbb{C}$ *with* $\operatorname{Re}\beta > 0, \operatorname{Re}\mu > -1/2$ *and* $\operatorname{Re}\nu < 1/2$ *with* $\nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$ *the following properties hold:*

1. $\int_0^\infty \left(\beta^2 + x^2\right)^{\nu-1/2} \cos(ax)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\Gamma\left(\nu + \frac{1}{2}\right)K_{-\nu}(a\beta)$,
2. $\int_0^\infty \left(\beta^2 + x^2\right)^{\nu-1/2} \sin(ax)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\left(\frac{2\beta}{a}\right)^\nu \cos(\pi\nu)\Gamma\left(\nu + \frac{1}{2}\right)\Big(I_{-\nu}(a\beta) - \mathbf{L}_\nu(a\beta)\Big)$,
3. $K_\nu(x) = K_{-\nu}(x)$,
4. $\int_0^\infty x^\mu K_\mu(ax)\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}(2a)^\mu \Gamma\left(\mu + \frac{1}{2}\right)\left(b^2 + a^2\right)^{-\mu-1/2}$.

In the following statement, we use the notation Δ for the differential operator $\partial_{xx} + \partial_{yy}$.

Proposition 4. *For each* $\alpha \in (1, 2]$ *there exists a constant* $\tilde{C}_\alpha > 0$ *such that for all* $k, l \in \mathbb{N}$ *and* $x, y \in \mathbb{R}^2$ *we have*

$$\Big(-\Delta\Big) \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(k\pi s_1)\sin(l\pi s_2)}{\left[(x - s_1)^2 + (y - s_2)^2\right]^{\alpha/2}} \mathrm{d}s_1 \mathrm{d}s_2 = \tilde{C}_\alpha \Big[(k\pi)^2 + (l\pi)^2\Big]^{\alpha/2} 2\sin(k\pi x)\sin(l\pi y).$$

Proof:

(4.3)

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sin(k\pi s_1)\sin(l\pi s_2)}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}}\,\mathrm{d}s_1\,\mathrm{d}s_2$$

$$= \int_{\mathbb{R}} \sin(l\pi s_2)\Big\{\int_{\mathbb{R}} \frac{\sin(k\pi s_1)}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}}\,\mathrm{d}s_1\Big\}\,\mathrm{d}s_2$$

$$= \int_{\mathbb{R}} \sin(l\pi s_2)\Big\{\int_{\mathbb{R}} \frac{\sin(k\pi(x+s_1))}{\big[s_1^2+(y-s_2)^2\big]^{\alpha/2}}\,\mathrm{d}s_1\Big\}\,\mathrm{d}s_2$$

$$= \int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\alpha}}\Big\{\int_{\mathbb{R}} \frac{\sin(k\pi(x+s_1))}{\big[\big(\frac{s_1}{y-s_2}\big)^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1\Big\}\,\mathrm{d}s_2$$

$$= \int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\alpha-1}}\Big\{\int_{\mathbb{R}} \frac{\sin(k\pi(x+s_1|y-s_2|))}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1\Big\}\,\mathrm{d}s_2 \qquad .$$

$$= \int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\alpha-1}}$$

$$\cdot\Big\{\sin(k\pi x)\int_{\mathbb{R}} \frac{\cos(k\pi s_1|y-s_2|)}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1 - \cos(k\pi x)\int_{\mathbb{R}} \frac{\sin(k\pi s_1|y-s_2|)}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1\Big\}\,\mathrm{d}s_2$$

$$= \sin(k\pi x)\int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\alpha-1}}\,\mathrm{d}s_2\int_{\mathbb{R}} \frac{\cos(k\pi s_1|y-s_2|)}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1$$

$$= 2\sin(k\pi x)\int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\alpha-1}}\,\mathrm{d}s_2\int_0^{\infty} \frac{\cos(k\pi s_1|y-s_2|)}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1.$$

Using the first formula of proposition 3 with parameters $\beta = 1, a = k\pi|y-v|, \nu = \frac{1-\alpha}{2}$ we obtain

$$\int_0^{\infty} \frac{\cos(k\pi s_1|y-s_2|)}{\big[s_1^2+1\big]^{\alpha/2}}\,\mathrm{d}s_1$$
$$= \frac{1}{\sqrt{\pi}}\Big(\frac{2}{k\pi}\Big)^{\frac{1-\alpha}{2}}\cos\big(\pi\frac{1-\alpha}{2}\big)\Gamma\big(1-\frac{\alpha}{2}\big)K_{\frac{\alpha-1}{2}}(k\pi|y-s_2|)\frac{1}{|y-s_2|^{\frac{1-\alpha}{2}}}.$$

Inserting this into (4.3) gives the following equality:

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sin(k\pi s_1)\sin(l\pi s_2)}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}}\,\mathrm{d}s_1\,\mathrm{d}s_2 = \frac{2}{\sqrt{\pi}}\sin(k\pi x)\Big(\frac{2}{k\pi}\Big)^{\frac{1-\alpha}{2}} \\ &\cdot\cos\big(\pi\frac{1-\alpha}{2}\big)\Gamma\big(1-\frac{\alpha}{2}\big)\int_{\mathbb{R}} \frac{\sin(l\pi s_2)}{|y-s_2|^{\frac{\alpha-1}{2}}}K_{\frac{\alpha-1}{2}}(k\pi|y-s_2|)\,\mathrm{d}s_2.\end{aligned} \tag{4.4}$$

We also have

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{\sin\left(l\pi s_2\right)}{|y-s_2|^{\frac{\alpha-1}{2}}} K_{\frac{\alpha-1}{2}}\left(k\pi|y-s_2|\right) \mathrm{d}s_2 \\
&= \int_{-\infty}^{y} \frac{\sin\left(l\pi s_2\right)}{|y-s_2|^{\frac{\alpha-1}{2}}} K_{\frac{\alpha-1}{2}}\left(k\pi|y-s_2|\right) \mathrm{d}s_2 + \int_{y}^{\infty} \frac{\sin\left(l\pi s_2\right)}{|y-s_2|^{\frac{\alpha-1}{2}}} K_{\frac{\alpha-1}{2}}\left(k\pi|y-s_2|\right) \mathrm{d}s_2 \\
&= \int_{0}^{\infty} \sin\big(l\pi(y-s_2)\big) s_2^{-\frac{\alpha-1}{2}} K_{\frac{1-\alpha}{2}}\left(k\pi s_2\right) \mathrm{d}s_2 \\
&+ \int_{0}^{\infty} \sin\big(l\pi(y+s_2)\big) s_2^{-\frac{\alpha-1}{2}} K_{\frac{1-\alpha}{2}}\left(k\pi s_2\right) \mathrm{d}s_2 \\
&= 2\sin\big(l\pi y\big) \int_{0}^{\infty} \cos\big(l\pi s_2\big) s_2^{-\frac{\alpha-1}{2}} K_{\frac{1-\alpha}{2}}\left(k\pi s_2\right) \mathrm{d}s_2.
\end{aligned}
$$

Using the fourth formula of proposition 3 with the parameters $\mu = \frac{1-\alpha}{2}, a = k\pi, b = l\pi$ gives

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{\sin\left(l\pi s_2\right)}{|y-s_2|^{\frac{\alpha-1}{2}}} K_{\frac{\alpha-1}{2}}\left(k\pi|y-s_2|\right) \mathrm{d}s_2 \\
&= 2\sin\big(l\pi y\big)\Gamma\big(1-\frac{\alpha}{2}\big)\frac{\sqrt{\pi}}{2}(2k\pi)^{\frac{1-\alpha}{2}}\Big((k\pi)^2+(l\pi)^2\Big)^{\frac{\alpha}{2}-1}
\end{aligned}
$$

and therefore, with the aid of (4.4) we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\sin\left(k\pi s_1\right)\sin\left(l\pi s_2\right)}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}} \mathrm{d}s_1\,\mathrm{d}s_2 \\
&= \sin\big(l\pi y\big)\sin\big(k\pi x\big)\Big[(k\pi)^2+(l\pi)^2\Big]^{\frac{\alpha}{2}-1} 2^{2-\alpha}\sin\big(\frac{\alpha\pi}{2}\big)\Big[\Gamma\big(1-\frac{\alpha}{2}\big)\Big]^2.
\end{aligned}
$$

Applying the operator $\big(-\Delta\big)$ for the last formula we get the statement with the constant

$$
\tilde{C}_\alpha = 2^{2-\alpha}\sin\big(\frac{\alpha\pi}{2}\big)\Big[\Gamma\big(1-\frac{\alpha}{2}\big)\Big]^2.
$$

Proof of Theorem 1: Let $u(x,y) = \sum_{k,l=1}^{\infty} u_{k,l} 2\sin k\pi x \sin l\pi y$ be the spectral expansion of $u \in \mathcal{F}_\alpha$. The extension $\overline{u}$ defined on $\mathbb{R}^2$ is automatically obtained just by extending the domain of x and y in the same formula. Using (2.1), Proposition 4 and finally (2.3) we obtain that

$$(-\Delta_D)^{\alpha/2}u(x,y) = \sum_{k,l=1}^{\infty} u_{k,l}\Big[(k\pi)^2+(l\pi)^2\Big]^{\alpha/2} 2\sin k\pi x \sin l\pi y$$

$$= \frac{1}{2\tilde{C}_\alpha}\sum_{k,l=1}^{\infty} u_{k,l}\Big(-\Delta\Big)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{2\sin(k\pi s_1)\sin(l\pi s_2)}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}}\,\mathrm{d}s_1\,\mathrm{d}s_2\right)(x,y)$$

$$= \frac{1}{2\tilde{C}_\alpha}\Big(-\Delta\Big)$$

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\big[(x-s_1)^2+(y-s_2)^2\big]^{\alpha/2}} \sum_{k,l=1}^{\infty} u_{k,l} 2\sin(k\pi s_1)\sin(l\pi s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\right)(x,y)$$

$$= \frac{1}{2\tilde{C}_\alpha}\Big(-\Delta\Big)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\overline{u}(s_1,s_2)}{|(x,y)-(s_1,s_2)|^{\alpha}}\,\mathrm{d}s_1\,\mathrm{d}s_2\right)(x,y) = \frac{1}{2\tilde{C}_\alpha}\frac{\alpha^2}{C_\alpha}(-\Delta)^{\alpha/2}\overline{u}(x,y)$$

as we stated in the theorem.

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