Intersecting P-free families

Dániel Gerbner [*]   Abhishek Methuku [†]   Casey Tompkins [‡§]

June 3, 2015

Abstract

We study the problem of determining the size of the largest intersecting P-free family for a given partially ordered set (poset) P. In particular, we find the exact size of the largest intersecting B-free family where B is the butterfly poset and classify the cases of equality. The proof uses a new generalization of the partition method of Griggs, Li and Lu. We also prove generalizations of two well-known inequalities of Bollobás and Greene, Katona and Kleitman in this case. Furthermore, we obtain a general bound on the size of the largest intersecting P-free family, which is sharp for an infinite class of posets originally considered by Burcsi and Nagy, when n is odd. Finally, we give a new proof of the bound on the maximum size of an intersecting k-Sperner family and determine the cases of equality.

1   Introduction

We denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the power set of $[n]$ by $2^{[n]}$. The family of all k-element subsets of $[n]$ is denoted by $\binom{[n]}{k}$. We refer to $\binom{[n]}{k}$ as the kth *level* in $2^{[n]}$. A collection $\mathcal{F} \subseteq 2^{[n]}$ is called an *antichain* if there do not exist $F, G \in \mathcal{F}$ with $F \subset G$. Let P and Q be partially ordered sets (posets). Then, P is said to be a *subposet* of Q if there exists an injection ϕ from P to Q such that $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q. Note, importantly, that the implication is only required in one direction.

The starting point for all forbidden poset problems is the well-known theorem of Sperner [17]:

Theorem 1 (Sperner [17]). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Moreover, equality occurs if and only if $\mathcal{F}$ is a level of maximum size in $2^{[n]}$.

Observe that every collection $\mathcal{F} \subseteq 2^{[n]}$ may itself be viewed as a poset under the containment relation. A k-chain, denoted by P_k, is defined to be the poset on the set $\{x_1, x_2, \ldots, x_k\}$ with the relations $x_1 \leq x_2 \leq \ldots \leq x_k$. Sperner's theorem is equivalent to the statement that the size of a collection $\mathcal{F} \subseteq 2^{[n]}$ containing no 2-chain as a subposet is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

[*]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences e-mail: gerbnerd@gmail.com

[†]Central European University, Budapest, Hungary. e-mail: abhishekmethuku@gmail.com

[‡]Central European University, Budapest, Hungary

[§]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences e-mail: ctompkins496@gmail.com

An important generalization of Sperner's theorem, due to Erdős [4], determines the size of the largest family containing no $(k+1)$-chain. Such a family is called *k-Sperner*. We use the notation $\Sigma(n, k)$ to denote the sum of the k largest binomial coefficients of the form $\binom{n}{i}$, $0 \le i \le n$.

Theorem 2 (Erdős [4])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be k-Sperner, then*

$$|\mathcal{F}| \le \Sigma(n, k).$$

Moreover, equality occurs if and only if $\mathcal{F}$ is the union of k of the largest levels in $2^{[n]}$.

The general study of forbidden poset problems was initiated in the paper of Katona and Tarján [13]. They determined the size of the largest family of sets containing neither a V (the poset on $\{x, y, z\}$ with relations $x \le y, z$) nor a Λ (the poset on $\{x, y, z\}$ with relations $x, y \le z$). They also gave an estimate on the maximum size of V-free families which we will make use of.

Theorem 3 (Katona, Tarján [13])**.** *Assume that $\mathcal{F} \subseteq 2^{[n]}$ contains no V as a subposet, then*

$$|\mathcal{F}| \le \left(1 + \frac{2}{n}\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The following function is the main object of study in forbidden poset problems:

$$\mathrm{La}(n, P) = \max_{\mathcal{F} \subseteq 2^{[n]}} \{|\mathcal{F}| : \mathcal{F} \text{ does not contain } P \text{ as a subposet}\}.$$

The value of $\mathrm{La}(n, P)$ has been determined or estimated for a variety of posets P. The butterfly poset, B, is defined on four elements w, x, y, z with relations $w, x \le y, z$. Of central importance to the present paper is a result of De Bonis, Katona and Swanepoel [3] which gave the exact result for $\mathrm{La}(n, B)$.

Theorem 4 (De Bonis, Katona, Swanepoel [3])**.**

$$\mathrm{La}(n, B) = \Sigma(n, 2).$$

Moreover, equality holds if and only if the family is the union of two of the largest levels in $2^{[n]}$.

Now we will mention some theorems where the family is also required to be intersecting. Milner [14] determined the size of the largest t-intersecting antichain. In the case $t = 1$, Milner's result yields

Theorem 5 (Milner [14])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting antichain, then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}.$$

This result follows from a more general inequality of Greene, Katona and Kleitman [8] (See also [12] and [16] for other simple proofs).

Theorem 6 (Greene, Katona, Kleitman [8])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting antichain, then*

$$\sum_{\substack{F \in \mathcal{F} \\ |F| \le \frac{n}{2}}} \frac{1}{\binom{n}{|F|-1}} + \sum_{\substack{F \in \mathcal{F} \\ |F| > \frac{n}{2}}} \frac{1}{\binom{n}{|F|}} \le 1.$$

2

In the case when $\mathcal{F}$ consists of only sets of size at most $\frac{n}{2}$, Bollobás [1] proved a stronger inequality generalizing the Erdős-Ko-Rado theorem [5].

Theorem 7 (Bollobás [1]). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting antichain and assume that for all $F \in \mathcal{F}$ we have $|F| \leq \frac{n}{2}$, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 1.$$

Note that Theorems 6 and 7 are implied by a more general result of Péter Erdős, Frankl and Katona [6] which determined the profile polytope for intersecting antichains.

In the course of determining the profile polytope for complement-free k-Sperner families, Gerbner [7] proved a generalization of Milner's theorem (the 1-intersecting case) to the k-Sperner setting.

Theorem 8 (Gerbner [7]). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting k-Sperner family, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i}, & \text{if } n \text{ is odd} \\ \binom{n-1}{\frac{n}{2}-1} + \sum_{i=\frac{n}{2}+1}^{\frac{n}{2}+k-1} \binom{n}{i} + \binom{n-1}{\frac{n}{2}+k}, & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

For simplicity we denote the right-hand side of (1) by $\sum_I(n,k)$. For any given P, we define

$$\mathrm{La}_I(n,P) = \max_{\mathcal{F} \subseteq 2^{[n]}} \{|\mathcal{F}| : \mathcal{F} \text{ does not contain } P \text{ as a subposet and } \mathcal{F} \text{ is intersecting}\}.$$

In this language, Theorem 8 states that $\mathrm{La}_I(n, P_{k+1}) = \sum_I(n,k)$, where P_{k+1} is the path poset of length $k+1$. Before we state our main results we need to introduce some notation. For all n and $k \leq \frac{n}{2}$, define

$$\mathcal{H}_{0,n,k} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 1} \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + 2} \cup \ldots \cup \binom{[n]}{\lfloor \frac{n}{2} \rfloor + k},$$

and in the case when n is even, for any $x \in [n]$, define

$$\mathcal{H}_{x,n,k} = \{F : F \in \binom{[n]}{\frac{n}{2}} : x \in F\} \cup \binom{[n]}{\frac{n}{2}+1} \cup \ldots \cup \binom{[n]}{\frac{n}{2}+k-1} \cup \{F : F \in \binom{[n]}{\frac{n}{2}+k} : x \notin F\}.$$

We determine the exact value of $\mathrm{La}_I(n,B)$, the maximum size of an intersecting butterfly-free family, for $n \geq 18$. In particular, we show that $\mathrm{La}_I(n,B) = \Sigma_I(n,2)$. The cases of equality are also obtained.

Theorem 9. *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family of subsets of $[n]$ where $n \geq 18$. Then,*

$$|\mathcal{F}| \leq \Sigma_I(n,2).$$

Equality holds if and only if:

- *For n odd, $\mathcal{F} = \mathcal{H}_{0,n,2}$;*

- *For n even, $\mathcal{F} = \mathcal{H}_{x,n,2}$ for some $x \in [n]$.*

3

The proof of this theorem can be seen as a generalization of the partition method of Griggs, Li and Lu [10, 9] to a weighted setting involving cyclic permutations. We also show that a variant of the LYM-type inequalities, Theorems 6 and 7, hold in this case.

Theorem 10. *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family with $2 \leq |F| \leq n - 2$ for all $F \in \mathcal{F}$, then*

$$\sum_{\substack{F \in \mathcal{F} \\ |F| \leq \frac{n}{2}}} \frac{1}{\binom{n}{|F|-1}} + \sum_{\substack{F \in \mathcal{F} \\ |F| > \frac{n}{2}}} \frac{1}{\binom{n}{|F|}} \leq 2.$$

Theorem 11. *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting B-free family with $2 \leq |F| \leq n/2$ for $F \in \mathcal{F}$, then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 2.$$

Next we obtain an upper bound on $\mathrm{La}_I(n, P)$ for an arbitrary poset P in the case when n is odd. Let $h(P)$ be the height of the poset P, that is, the size of the longest chain in P.

Theorem 12. *Assume n is odd and $\frac{|P|+h(P)}{2}$ is an integer. Let $\mathcal{F}$ be an intersecting P-free family of subsets of $[n]$, $n \geq 4$. Then,*

$$|\mathcal{F}| \leq \sum_{i=1}^{\frac{|P|+h(P)}{2}-1} \binom{n}{\lfloor \frac{n}{2} \rfloor + i}.$$

Note 1. *Let $e(P)$ denote the maximum number of consecutive levels in $2^{[n]}$ which do not contain a copy of P as a subposet for any n. Burcsi and Nagy determined the exact value of $\mathrm{La}(n, P)$ for infinitely many posets P for which $e(P) = \frac{|P|+h(P)}{2} - 1$. For all these posets we have equality in Theorem 12. In the cases where n is even or $\frac{|P|+h(P)}{2}$ is not an integer, a similar bound can be obtained, but it is not sharp in general.*

Finally, we give a new proof of Theorem 8 which avoids the usage of profile polytopes. We also classify the cases of equality.

Theorem 13. *Let $\mathcal{F}$ be an intersecting k-Sperner family of subsets of $[n]$. Then,*

$$|\mathcal{F}| \leq \Sigma_I(n, k).$$

If $k \leq \frac{n}{2}$, then equality holds in the following cases:

- *For n odd, $\mathcal{F} = \mathcal{H}_{0,n,k}$;*

- *For n even and $k = 1$, $\mathcal{F} = \mathcal{H}_{0,n,k}$ or $\mathcal{H}_{x,n,k}$ for some $x \in [n]$;*

- *For n even and $k > 1$, $\mathcal{F} = \mathcal{H}_{x,n,k}$ for some $x \in [n]$.*

Note 2. *If $k > \frac{n}{2}$ there can be many extremal families. For example, if n is even and $k = \frac{n}{2} + 1$, we may take any maximal intersecting family on level $\frac{n}{2}$ (of which there are many) in addition to all complete levels from $\frac{n}{2} + 1$ to n.*

The paper is organized as follows. In Section 2 we introduce Katona's cycle method [11] and prove some simple lemmas. In Section 3 we prove Theorem 9 determining the exact value of $La_I(n, B)$. In Section 4 we prove Theorem 10 and Theorem 11. In Section 5 we prove Theorem 12 about general posets P. Finally, in Section 6 we prove Theorem 13 about intersecting k-Sperner families.

2 Cycle method

A *cyclic permutation* of $[n]$ (in the sense of Katona [11]) is an arrangement of the numbers 1 through n along a circle. Sets of consecutive elements along the circle are called *intervals*. The collection of all intervals along σ of size r is denoted $\mathcal{L}_r^\sigma$. Most of our proofs will proceed by double counting pairs (F, σ) where $F \in \mathcal{F}$ and σ is a cyclic permutation. Moreover, we will always assume $\varnothing, [n] \notin \mathcal{F}$ (we handle the remaining cases separately). For any collection $\mathcal{H}$ of sets, let $\mathcal{H}^\sigma = \{F : F \in \mathcal{H} \text{ and } F \text{ is an interval along } \sigma\}$. In the double counting we will use the following weight function:

$$w(F, \sigma) = \begin{cases} \binom{n}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval along } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

Observe that, on the one hand, we have

$$\sum_{F \in \mathcal{F}} \sum_\sigma w(F, \sigma) = \sum_{F \in \mathcal{F}} |F|! \, (n - |F|)! \binom{n}{|F|} = n! \, |\mathcal{F}| \, .$$

On the other hand,

$$\sum_\sigma \sum_{F \in \mathcal{F}} w(F, \sigma) = \sum_\sigma \sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} .$$

For notational simplicity we will often work with the simplest case of a cyclic permutation where the numbers $1, 2, \ldots, n$ occur in that order. We call this cyclic permutation the *canonical cyclic permutation*. It is clear that when we are working with one fixed cyclic permutation we may assume it is canonical because renaming the elements will not change the intersection or containment structure of its intervals. Let A_i^j denote the interval $\{i, i+1, \ldots, i+j-1\}$ (addition involving the base set is always taken modulo n) where i is called the first element of A_i^j and $i+j-1$ is called the last element of A_i^j. We can partition all intervals along σ into chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$ where $\mathcal{C}_i = \{\{i\}, \{i, i+1\}, \ldots, \{i, i+1, \ldots, i+n-1\}\}$. We call this partition the *canonical chain decomposition*. It will be helpful in proving the following well-known result.

Lemma 1. *Let $\mathcal{G}$ be an antichain of intervals along a cyclic permutation σ, then $|\mathcal{G}| \leq n$, and equality holds if and only if $\mathcal{G} = \mathcal{L}_r^\sigma$ for some r.*

Proof. We may assume that σ is canonical. Let us consider the canonical chain decomposition. Since at most one interval from each chain may be in our collection, we have that either we take fewer than n intervals or every chain contains exactly one interval from $\mathcal{G}$. Suppose we are in the latter case and that some two intervals in $\mathcal{G}$ had different sizes. Then, there must exist chains $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ where the interval we take in $\mathcal{C}_i$ is larger than the one we take in $\mathcal{C}_{i+1}$. That is, we have $A_i^{j_1}, A_{i+1}^{j_2} \in \mathcal{G}$ with $j_1 > j_2$. However, this implies that we have $A_{i+1}^{j_2} \subseteq A_i^{j_1}$, a contradiction. $\qquad \square$

If we add the additional constraint that the intervals are intersecting and assume that they are of size at most $\frac{n}{2}$, then we have the following better bound (following Katona [11]).

Lemma 2. *Let $\mathcal{G}$ be an intersecting antichain of intervals along a cyclic permutation σ where all the intervals are of size at most $\frac{n}{2}$, then $|\mathcal{G}| \leq \frac{n}{2}$.*

5

Proof. Suppose without loss of generality that the interval $A_1^k = \{1, 2, \ldots, k\}$ is in $\mathcal{G}$. Since $\mathcal{G}$ is intersecting, every interval of $\mathcal{G}$ has either its first element or its last element in A_1^k. Also notice that if $i \in \{1, 2, \ldots, k\}$ is the last element of an interval of $\mathcal{G}$, $i + 1$ cannot be the first element of another interval of $\mathcal{G}$ since all the intervals are of size at most $\frac{n}{2}$. Therefore, the total number of intervals in $\mathcal{G}$ is at most $1 + (k - 1) = k \leq \frac{n}{2}$, as desired. $\square$

Let σ be canonical, and $\mathcal{G}$ be a collection of intervals along σ. If $\mathcal{G}$ contains only intervals of size j of the form $A_i^j, A_{i+1}^j, \ldots, A_{i+s}^j$, then we say that $\mathcal{G}$ is *contiguous*. If $\mathcal{G}$ is a collection consisting of intervals $A_i^j, A_{i+1}^j, \ldots, A_{i+s}^j, A_{i+s+1}^{j+1}, A_{i+s+2}^{j+1}, \ldots, A_{i-2}^{j+1}$, then we say $\mathcal{G}$ is *pair-contiguous*. Equivalently, $\mathcal{G}$ is pair-contiguous if it is an antichain, has size $n - 1$, and is the union of two contiguous collections of intervals spanning two consecutive levels. We extend these definitions to arbitrary cyclic permutations in the obvious way.

Lemma 3. *If $\mathcal{G}$ is an antichain of intervals along a cyclic permutation σ such that $|\mathcal{G}| = n - 1$ and $\mathcal{G}$ contains intervals of at least two sizes, then $\mathcal{G}$ is pair-contiguous.*

Proof. Assume that σ is canonical. Let us consider the canonical chain decomposition. Let $\mathcal{G}_{min}$ be the collection of those intervals in $\mathcal{G}$ of minimum size, say j^*. Since $\mathcal{G}_{min}$ is not a full level there must be an i such that $A_i^{j^*} \in \mathcal{G}_{min}$ but $A_{i-1}^{j^*} \notin \mathcal{G}_{min}$. Then, we know that $\mathcal{C}_{i-1}$ has no interval from $\mathcal{G}$, and if $|\mathcal{G}| = n - 1$ it must be that each chain $\mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{i-2}$ contains an interval from $\mathcal{G}$. Observe that if $\mathcal{G}$ contains an interval of size j_1 in $\mathcal{C}_{i_1}$ and an interval of size j_2 in $\mathcal{C}_{i_1+1}$, then $j_1 \leq j_2$ for otherwise we would not have an antichain. Finally, the interval from $\mathcal{G}$ in $\mathcal{C}_{i-2}$ must have size $j^* + 1$ for if it were any larger it would contain $A_i^{j^*}$. It follows that $\mathcal{G}$ is a pair-contiguous family contained in levels j^* and $j^* + 1$. $\square$

We call a member of $\mathcal{G}$ isolated, if it is comparable with no other member of $\mathcal{G}$.

Lemma 4. *Let $\mathcal{G}$ be a 2-Sperner family of intervals on a cyclic permutation with I isolated intervals, then there are at most $2n - I$ intervals in $\mathcal{G}$.*

Proof. Consider the canonical chain decomposition. Each isolated interval is found on a different one of the chains. The remaining chains can have at most 2 intervals each. It follows that the total number of intervals is at most $I + 2(n - I) = 2n - I$. $\square$

Lemma 5. *Let $\mathcal{G}$ be a 2-Sperner family of intervals on a cyclic permutation with I isolated intervals, where $1 \leq I \leq n - 1$. Then, there are at most $2n - I - 1$ intervals in total.*

Proof. Consider the canonical chain decomposition. If we do not have either an isolated interval or two intervals on each chain we are done because then the total number of intervals is at most $I + 1 + 2(n - I - 1)$, as desired. So suppose by contradiction we do. Let $\mathcal{A}$ be the set of inclusion minimal intervals. $\mathcal{A}$ is an antichain and so if it contains intervals of more than one size, then $|\mathcal{A}| \leq n - 1$ and we are done again. Thus, we may assume that $\mathcal{A}$ is uniform. But then we have a contradiction because we cannot have a chain with two intervals followed by a chain with an isolated interval of the same size as the smaller of the two intervals in the first chain. $\square$

3 Intersecting B-free families

In this section we prove Theorem 9 by determining the exact value of $\mathrm{La}_I(n, B)$ and classifying the extremal families. We may assume that $[n] \notin \mathcal{F}$. Indeed, if $[n] \in \mathcal{F}$, then $\mathcal{F} \setminus \{[n]\}$ contains no three sets A, B, C with $A, B \subset C$. In this case, Theorem 3 shows that such a family may have size at most $(1 + 2/n)\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Thus, for $n \geq 7$ the family will be too small.

Let $\mathcal{F}_m = \{F \in \mathcal{F} \mid \exists A, B \in \mathcal{F} \text{ such that } A \subset F \subset B\}$ (notice that A and B are unique since $\mathcal{F}$ is butterfly-free). We refer to $\mathcal{F}_m$ as the collection of *middle* sets in $\mathcal{F}$. Fix a cyclic permutation σ. We will distinguish four kinds of intervals in $\mathcal{F}^\sigma$ which we refer to as the *middle, isolated, top* and *bottom* intervals along σ.

$$\mathcal{M}_\sigma = \{F : F \in \mathcal{F}^\sigma \text{ and there exists } A, B \in \mathcal{F}^\sigma \text{ such that } A \subset F \subset B\};$$
$$\mathcal{I}_\sigma = \{F : F \in \mathcal{F}^\sigma \text{ and } F \text{ is comparable with no other interval in } \mathcal{F}^\sigma \};$$
$$\mathcal{T}_\sigma = \{F : F \in \mathcal{F}^\sigma \setminus \mathcal{I}_\sigma \text{ is inclusion maximal in } \mathcal{F}^\sigma\};$$
$$\mathcal{B}_\sigma = \{F : F \in \mathcal{F}^\sigma \setminus \mathcal{I}_\sigma \text{ is inclusion minimal in } \mathcal{F}^\sigma\}.$$

It is easy to see that these four sets of intervals form a partition of $\mathcal{F}^\sigma$. Importantly, note that the four collections are defined by their properties as intervals along σ, not in $\mathcal{F}$ itself. So we may have, for example, a set $F \in \mathcal{F}_m$ which is an interval along σ, but does not belong to $\mathcal{M}_\sigma$.

For any $F \in \mathcal{F}$, let α_F be the number of cyclic permutations containing F as a middle interval and β_F be the number of cyclic permutations containing F as an isolated interval. Our proof considers the tradeoffs associated with these two possibilities. We will need to know the relative frequency with which they occur. To this end, define

$$c = \max_{F \in \mathcal{F}_m} \frac{\alpha_F}{\beta_F}.$$

For a fixed cyclic permutation σ, let $m_\sigma, i_\sigma, t_\sigma$ and b_σ denote the weight of the collections $\mathcal{M}_\sigma, \mathcal{I}_\sigma, \mathcal{T}_\sigma$ and $\mathcal{B}_\sigma$ respectively. Define

$$R = n\Sigma_I(n, 2) = \begin{cases} n\binom{n}{\lfloor \frac{n}{2} \rfloor + 1} + n\binom{n}{\lfloor \frac{n}{2} \rfloor + 2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}\binom{n}{2} + n\binom{n}{\frac{n}{2} + 1} + \left(\frac{n}{2} - 2\right)\binom{n}{\frac{n}{2} + 2}, & \text{if } n \text{ is even.} \end{cases}$$

Thus, our aim is to show $|\mathcal{F}| \leq R/n$.

Lemma 6. *If for each cyclic permutation σ we have $t_\sigma + b_\sigma + (1 + c)i_\sigma \leq R$, then $|\mathcal{F}| \leq R/n$.*

Proof. It suffices to show that

$$n! \, |\mathcal{F}| = \sum_\sigma \sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} \leq (n - 1)! \, R.$$

For a given σ we have

$$\sum_{F \in \mathcal{F}^\sigma} \binom{n}{|F|} = t_\sigma + b_\sigma + i_\sigma + m_\sigma \leq R + m_\sigma - ci_\sigma. \tag{2}$$

Summing both sides of (2) over all cyclic permutations, we get

$$\sum_{\sigma}\sum_{F\in\mathcal{F}^{\sigma}}\binom{n}{|F|} \leq \sum_{\sigma}(R+m_{\sigma}-ci_{\sigma}) = (n-1)!\,R + \sum_{F\in\mathcal{F}_m}(\alpha_F-c\beta_F)\binom{n}{|F|} - \sum_{F\notin\mathcal{F}_m}c\beta_F\binom{n}{|F|}.$$

Now, since for every $F\in\mathcal{F}$ we have $\alpha_F-c\beta_F\leq 0$ (by the definition of c), our lemma follows. $\square$

Lemma 7. *If $\mathcal{F}$ is B-free and contains only sets of size at least 2 and at most $n-2$, then for each $F\in\mathcal{F}_m$ we have*

$$\frac{\beta_F}{\alpha_F} \geq \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1.$$

Proof. Assume that $A\subset F\subset B$. The number of cyclic permutations containing A, F and B is

$$\alpha_F = |A|\,!\,(|F|-|A|+1)!\,(|B|-|F|+1)!\,(n-|B|)!\,.$$

The number of cyclic permutations containing only F is (by inclusion/exclusion)

$$\beta_F = |F|\,!\,(n-|F|)! - |A|\,!\,(|F|-|A|+1)!\,(n-|F|)! - |F|\,!\,(|B|-|F|+1)!\,(n-|B|)! + \alpha_F.$$

So we have

$$\frac{\beta_F}{\alpha_F} = 1 + \frac{|F|\,!\,(n-|F|)!}{|A|\,!\,(|F|-|A|+1)!\,(|B|-|F|+1)!\,(n-|B|)!}$$
$$- \frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - \frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!}$$
$$= \left(\frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - 1\right)\left(\frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!} - 1\right)$$
$$\geq \min_{B}\left(\frac{(n-|F|)!}{(|B|-|F|+1)!\,(n-|B|)!} - 1\right)\cdot\min_{A}\left(\frac{|F|\,!}{|A|\,!\,(|F|-|A|+1)!} - 1\right).$$

The first term is minimized by taking $|B| = |F|+1$, and the second term is minimized by taking $|A| = |F|-1$. By substituting these values in the inequality above, we get

$$\frac{\beta_F}{\alpha_F} \geq \left(\frac{n-|F|}{2} - 1\right)\left(\frac{|F|}{2} - 1\right)$$
$$= \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1. \qquad \square$$

Note 3. *If the middle sets in $\mathcal{F}$ all have size at least 3 and at most $n-3$, then for each $F\in\mathcal{F}_m$,*

$$\frac{\beta_F}{\alpha_F} \geq \frac{|F|\,(n-|F|)}{4} - \frac{n}{2} + 1 \geq \frac{n-5}{4}.$$

Therefore,

$$c = \max_{F\in\mathcal{F}_m}\frac{\alpha_F}{\beta_F} \leq \frac{4}{n-5}.$$

Lemma 8. *If $\mathcal{F}$ is B-free and contains a set of size 1 or $n-1$, then $|\mathcal{F}| < \Sigma_I(n,2)$ for $n\geq 18$.*

Proof. Assume that $\mathcal{F}$ contains a set of size $n - 1$, say S. We define two subfamilies of $\mathcal{F}$. Denote by $\mathcal{F}_1$ the family of those sets in $\mathcal{F}$ which are properly contained in S and set $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Since $\mathcal{F}$ is B-free, it follows that $\mathcal{F}_1$ has no three sets A, B, C with $A, B \subset C$. Thus, using Theorem 3 applied to an $n - 1$ element ground set we have

$$|\mathcal{F}_1| \leq \left(1 + \frac{2}{n-1} \right) \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

Since every set in $\mathcal{F}_2$ contains a fixed element, we can use Theorem 4 applied to an $n - 1$ element ground set to show

$$|\mathcal{F}_2| \leq \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor + 1}.$$

One can easily verify this implies $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| < \Sigma_I(n, 2)$ for $n \geq 18$.

If $\mathcal{F}$ contains a set of size 1, a similar proof works by symmetry (note that we do not use the intersecting property here). $\square$

We will use the following special case of Lemma 11 which will be proved in Section 6:

Lemma 9. *Let $\mathcal{G}$ be an intersecting 2-Sperner collection of intervals along a cyclic permutation σ, then*

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\Sigma_I(n, 2). \tag{3}$$

Equality holds in (3) *if and only if:*

- *n is odd and $\mathcal{G} = \mathcal{H}_{0,n,2}^{\sigma}$;*

- *n is even and $\mathcal{G} = \mathcal{H}_{x,n,2}^{\sigma}$ for some $x \in [n]$.*

Now we are ready to prove our main theorem.

Proof of Theorem 9. Let σ be a cyclic permutation. By Lemma 6, it is enough to prove

$$t_\sigma + b_\sigma + (1 + c)i_\sigma \leq R. \tag{4}$$

If $i_\sigma = 0$, then our family of intervals is 2-Sperner and we are done by Lemma 9. Assume that n is even and $\mathcal{F}^\sigma$ has $I > 0$ isolated intervals. If $I > \frac{n}{2}$, then by Lemma 5, the total number of intervals along σ is less than $\frac{3n}{2} - 1$. Since isolated sets form an antichain, $I \leq n$ by Lemma 1. So the maximum weight of these intervals is at most $\left(\frac{n}{2} \binom{n}{\frac{n}{2}} + \frac{n}{2} \binom{n}{\frac{n}{2}+1} \right)(1 + c) + \left(\frac{n}{2} - 2 \right) \binom{n}{\frac{n}{2}+1} < R$, when $n \geq 18$ as desired.

Now, consider the case when there are $2 \leq I \leq \frac{n}{2}$ isolated intervals along σ. By Lemma 5 it follows that the total number of intervals along σ is at most $2n - I - 1$. Pairing off intervals with their complements and considering the maximum weight we can obtain with $2n - I - 1$ intervals, it is enough to show

$$(1 + c)I \binom{n}{\frac{n}{2}} + \left(\frac{n}{2} - I \right) \binom{n}{\frac{n}{2}} + n \binom{n}{\frac{n}{2}+1} + \left(\frac{n}{2} - I - 1 \right) \binom{n}{\frac{n}{2}+2} \leq R.$$

Simplifying,

$$cI\binom{n}{\frac{n}{2}} \leq (I-1)\binom{n}{\frac{n}{2}+2}.$$

Dividing through by $\binom{n}{\frac{n}{2}}$, we get

$$I \geq \frac{\frac{n(n-2)}{(n+2)(n+4)}}{\frac{n(n-2)}{(n+2)(n+4)} - c}.$$

By Note 3, we have $c \leq \frac{4}{n-5}$. Substituting this value of c in the above inequality, we get that the right-hand side is strictly less than 2 when $n \geq 18$, as desired. If n is odd, a similar calculation implies that $I > 0$ which settles the odd case completely.

So we may assume that $I = 1$ (n is even) and that the total number of intervals along σ is exactly $2n - 2$ (If we have less than $2n - 2$ intervals and $I = 1$, it can be checked easily that $t_\sigma + b_\sigma + (1+c)i_\sigma < R$ for $n \geq 18$). Now, the intervals in $\mathcal{T}_\sigma \cup \mathcal{B}_\sigma \cup \mathcal{I}_\sigma$ form a 2-Sperner family of intervals along σ. Let us call the subfamily of maximal intervals (i.e., those intervals that are not contained in any other interval) $\mathcal{U}$ and the subfamily of minimal intervals (i.e., those intervals that do not contain any other interval) $\mathcal{D}$. Now, if either $\mathcal{U}$ or $\mathcal{D}$ contains n intervals, then, since it is an antichain, by Lemma 1 it has to be a complete level $\mathcal{L}_r^\sigma$. As the family is intersecting, we have $r > \frac{n}{2}$.

If $\mathcal{D} = \mathcal{L}_r^\sigma$, then $\mathcal{U}$ consists of $n-2$ intervals of size at least $\frac{n}{2}+2$, and a simple calculation shows $t_\sigma + b_\sigma + (1+c)i_\sigma < R$ for $n \geq 18$. If $\mathcal{U} = \mathcal{L}_r^\sigma$ and $r \geq \frac{n}{2}+2$, a similar calculation shows again that $t_\sigma + b_\sigma + (1+c)i_\sigma < R$ for $n \geq 18$. If $\mathcal{U} = \mathcal{L}_r^\sigma$ and $r = \frac{n}{2}+1$, then $\mathcal{D}$ is an intersecting antichain of intervals of size at most $\frac{n}{2}$. Now by Lemma 2, we have $|\mathcal{D}| \leq \frac{n}{2}$ in this case, contradicting the fact that the total number of intervals is $2n - 2$.

So we can assume that both $\mathcal{U}$ and $\mathcal{D}$ contain at most $n - 1$ intervals. Since the interval in $\mathcal{I}_\sigma$ is both maximal and minimal we have $|\mathcal{U} \cap \mathcal{D}| \geq 1$. But then, the total number of intervals in our 2-Sperner family is $|\mathcal{U} \cup \mathcal{D}| = |\mathcal{U}| + |\mathcal{D}| - |\mathcal{U} \cap \mathcal{D}| \leq 2n - 3$, a contradiction.

We now establish the cases of equality. First let us notice that by Lemma 6, we have $|\mathcal{F}| = \frac{R}{n}$ if and only if we have equality in (4) for each σ. However, we just saw that if $I > 0$, the inequality (4) is never sharp when n is large enough (for both the n is even case and n is odd case). Thus, we have $I = 0$ for every σ. However, since any middle set of $\mathcal{F}$ appears as an isolated interval on some σ, we may conclude that $\mathcal{F}$ has no middle sets (i.e., $|\mathcal{F}_m| = 0$). Therefore, $\mathcal{F}$ is 2-Sperner and so the equality cases follow from Theorem 13. $\square$

4   Bollobás and Greene-Katona-Kleitman-type inequalities

In this section we will prove Theorem 11.

Proof. Following [1] we will use the weight function

$$w(F, \sigma) = \begin{cases} \frac{1}{|F|}, & \text{if } F \in \mathcal{F} \text{ and } F \text{ is an interval in } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

On the one hand, we have

$$\sum_{F \in \mathcal{F}} \sum_\sigma w(F, \sigma) = \sum_{F \in \mathcal{F}} (|F| - 1)! \, (n - |F|)! \, .$$

10

We will show
$$\sum_\sigma \sum_{F \in \mathcal{F}} w(F, \sigma) \le 2(n-1)!\,.$$

Fix a cyclic permutation σ. As before, let $\mathcal{I}_\sigma$ be the collection of isolated intervals along σ. Similarly, let $\mathcal{M}_\sigma$ be the collection of middle intervals along σ. Then, we claim that the following inequality holds:
$$w(\mathcal{F}^\sigma) \le 2 + w(\mathcal{M}_\sigma) - w(\mathcal{I}_\sigma). \tag{5}$$

Indeed, initially leave out all intervals in $\mathcal{M}_\sigma$ and $\mathcal{I}_\sigma$. The remaining intervals may be partitioned into two antichains along σ, say $\mathcal{A}_1$ and $\mathcal{A}_2$. Clearly $\mathcal{A}_1 \cup \mathcal{I}_\sigma$ is an antichain, as is $\mathcal{A}_2 \cup \mathcal{I}_\sigma$. By the argument from [1] we have $w(\mathcal{A}_1 \cup \mathcal{I}_\sigma) \le 1$ and $w(\mathcal{A}_2 \cup \mathcal{I}_\sigma) \le 1$. Thus, summing we have
$$w(\mathcal{A}_1) + w(\mathcal{A}_2) + 2w(\mathcal{I}_\sigma) \le 2. \tag{6}$$

Rearranging and adding $w(\mathcal{M}_\sigma)$ to both sides yields (5).

Since the only possible middle intervals along a cyclic permutation are middle sets in $\mathcal{F}$ (that is, $\mathcal{M}_\sigma \subseteq \mathcal{F}_m^\sigma$), summing up (5) over all cyclic permutations σ, we get
$$\sum_\sigma \sum_{F \in \mathcal{F}} w(F, \sigma) \le 2(n-1)! + \sum_{F \in \mathcal{F}_m} \left(\frac{\alpha_F}{|F|} - \frac{\beta_F}{|F|} \right) - \sum_{F \notin \mathcal{F}_m} \frac{\beta_F}{|F|}. \tag{7}$$

We have seen already by Lemma 7 that $\beta_F \ge \alpha_F$ for $F \in \mathcal{F}_m$, and the proof is complete. $\qquad\square$

The proof of Theorem 10 uses the exact same idea but with the weight function defined in [8], namely
$$w(F, \sigma) = \begin{cases} \frac{n - |F| - 1}{|F|}, & \text{if } F \in \mathcal{F}, |F| \le \frac{n}{2} \text{ and } F \text{ is an interval in } \sigma \\ 1, & \text{if } F \in \mathcal{F}, |F| > \frac{n}{2} \text{ and } F \text{ is an interval in } \sigma \\ 0, & \text{otherwise.} \end{cases}$$

5 Results for general posets P

In this section we prove Theorem 12. Before we start the proof we define the notion of a *double chain* introduced in [2].

Definition 1 (Double chain). *Let $\emptyset = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = [n]$ be a maximal chain (so $|A_i| = i$). The double chain associated to this chain is given by*
$$\mathcal{D} = \{A_0, A_1, \ldots, A_n, M_1, M_2, \ldots, M_{n-1}\},$$
where $M_i = A_{i-1} \cup \{A_{i+1} \setminus A_i\}$.

We will now introduce the notion of a *double chain-complement pair* which is the key ingredient of the proof.

Definition 2 (Double chain-complement pair). *Let $\mathcal{D}$ be a double chain. By taking the complement of all the sets in $\mathcal{D}$ we get another double chain $\mathcal{D}'$. We refer to $\mathcal{H} = \mathcal{D} \cup \mathcal{D}'$ as a double chain-complement pair.*

In the rest of this section we shall work with the double chain-complement pair $\mathcal{H}_0 = \mathcal{D}_0 \cup \mathcal{D}_0'$ where $\mathcal{D}_0$ is defined by taking $A_i = [i]$; other double chain-complement pairs are related to it by permutation. Let $\pi \in S_n$ be a permutation and $F \subseteq [n]$ be a set, then F^π denotes the set $\{\pi(a) : a \in F\}$. We define the double chain-complement pair $\mathcal{H}_0^\pi$ to be the collection $\{F^\pi : F \in \mathcal{H}_0\}$. Notice that this gives us $n!$ double chain-complement pairs in total.

Now we are ready to prove our theorem. Let $\mathcal{F}$ be an intersecting P-free family. We will use the collections $\mathcal{H}_0^\pi = \mathcal{D} \cup \mathcal{D}'$ for a weighted double counting argument described below.

Define a weight function $w(F, \mathcal{H}_0^\pi)$ by

$$
w(F, \mathcal{H}_0^\pi) = \begin{cases} \binom{n}{|F|}, & \text{if } F \in \mathcal{F}, \ F \neq [n] \text{ and } F \in \mathcal{H}_0^\pi \\ 4, & \text{if } F \in \mathcal{F}, \ F = [n] \text{ and } F \in \mathcal{H}_0^\pi \\ 0, & \text{otherwise.} \end{cases}
$$

We want to compute $\sum_F \sum_{\mathcal{H}_0^\pi} w(F, \mathcal{H}_0^\pi)$ in two different ways. First let us fix a $F \in \mathcal{F}$ and determine how many collections $\mathcal{H}_0^\pi$ contain F. If $F = [n]$ we know that all $n!$ collections $\mathcal{H}_0^\pi$ contain it. So let us assume $F \neq [n]$. Let H_1, H_2, H_3, H_4 be the four sets in $\mathcal{H}_0$ of size $|F|$ (our assumption $n \geq 4$ ensures there are four distinct sets of this size). The number of permutations π such that a given H_i (where $1 \leq i \leq 4$) is mapped to F is $|F|!\,(n - |F|)!$, since we can map the elements of H_i to F arbitrarily and the elements of $[n] \setminus H_i$ to $[n] \setminus F$ arbitrarily. So it follows that the number of permutations π such that $F \in \mathcal{H}_0^\pi$ is $4\,|F|!\,(n - |F|)!$. Thus, we have

$$
\sum_F \sum_{\mathcal{H}_0^\pi} w(F, \mathcal{H}_0^\pi) = 4\,|\mathcal{F}|\,n!\,. \tag{8}
$$

Now let us fix an $\mathcal{H}_0^\pi$. Since n is odd, there are 8 sets in $\mathcal{H}_0^\pi$ of maximal weight $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ and 8 sets of second largest weight $\binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$ and so on. The 8 sets of $\mathcal{H}_0^\pi$ of the same weight $\binom{n}{\lfloor \frac{n}{2} \rfloor + i}$ (where $i \geq 1$) consist of 4 sets and their respective complements. Thus, at most 4 of them can belong to our family $\mathcal{F}$ (since $\mathcal{F}$ is intersecting). Now let us recall a lemma due to Burcsi and Nagy [2].

Lemma 10 (Burcsi-Nagy [2])**.** *Let P be a poset. Any subset of size $|P| + h(P) - 1$ of a double chain contains P as a subposet.*

Since a P-free family has at most $|P| + h(P) - 2$ sets on a double chain, it follows that we can have at most $2(|P| + h(P) - 2)$ sets in $\mathcal{F} \cap \mathcal{H}_0^\pi$ for any π. Since we can have at most 4 sets of weight $\binom{n}{\lfloor \frac{n}{2} \rfloor + i}$ in $\mathcal{F} \cap \mathcal{H}_0^\pi$, the total weight of sets in $\mathcal{F} \cap \mathcal{H}_0^\pi$ is at most $\sum_{i=1}^{\frac{2(|P| + h(P) - 2)}{4}} 4\binom{n}{\lfloor \frac{n}{2} \rfloor + i}$. So we have

$$
\sum_{\mathcal{H}_0^\pi} \sum_F w(F, \mathcal{H}_0^\pi) \leq n! \left(\sum_{i=1}^{\frac{|P| + h(P)}{2} - 1} 4\binom{n}{\lfloor \frac{n}{2} \rfloor + i} \right). \tag{9}
$$

Combining (8) and (9), we have the desired bound.

6 Intersecting k-Sperner families

The aim of this section is to prove Theorem 13.

Lemma 11. *Let $\mathcal{G}$ be an intersecting k-Sperner collection of intervals along a cyclic permutation σ, then*

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\Sigma_I(n, k). \tag{10}$$

Assume $k \leq \frac{n}{2}$, then equality holds in (10) if and only if:

- *n is odd and $\mathcal{G} = \mathcal{H}^\sigma_{0,n,k}$;*

- *n is even, $k = 1$ and $\mathcal{G} = \mathcal{H}^\sigma_{0,n,1}$ or $\mathcal{G} = \mathcal{H}^\sigma_{x,n,1}$ for some $x \in [n]$;*

- *n is even, $k > 1$ and $\mathcal{G} = \mathcal{H}^\sigma_{x,n,k}$ for some $x \in [n]$.*

Proof. First, fix k and suppose that n is odd. Following an argument of Mirsky [15], $\mathcal{G}$ can be decomposed into k antichains in the following way. For $1 \leq i \leq k$ set

$$\mathcal{G}_i = \{G : G \in \mathcal{G} \text{ and the longest chain in } \mathcal{G} \text{ with maximal element } G \text{ has length } i\}.$$

By Lemma 1 each $\mathcal{G}_i$ can have size at most n and so we have $|\mathcal{G}| \leq kn$. For any interval G along σ, it is easy to see that $[n] \setminus G$ is also an interval along σ and has size $n - |G|$. Since our family is intersecting, by pairing off each G with $[n] \setminus G$, we see that $\mathcal{G}$ contains at most n intervals of size $\lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n}{2} \rfloor + 1$ and at most n intervals of size $\lfloor \frac{n}{2} \rfloor - 1$ or $\lfloor \frac{n}{2} \rfloor + 2$ and so on. Thus, the bound

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} \leq n\left(\binom{n}{\lfloor \frac{n}{2} \rfloor + 1} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 2} + \cdots + \binom{n}{\lfloor \frac{n}{2} \rfloor + k}\right) = n\Sigma_I(n, k)$$

is immediate. Assume now that $\mathcal{G}$ attains this weight and $k \leq \frac{n}{2}$, then $\mathcal{G}$ must contain n sets from each of $\mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor} \cup \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + 1}$, $\mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor - 1} \cup \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + 2}$, $\ldots$, $\mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor - k + 1} \cup \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + k}$. In particular, we must have $|\mathcal{G}| = kn$.

Observe that each $\mathcal{G}_i$ is an antichain and, since $|\mathcal{G}| = kn$, we have $|\mathcal{G}_i| = n$ for all i. Then, Lemma 1 implies that each $\mathcal{G}_i$ is equal to a level of intervals along σ. If $\mathcal{G}_i$ is a level, it must consist of intervals of size at least $\lfloor \frac{n}{2} \rfloor + 1$. Thus, assuming $\mathcal{G}$ is of maximal weight, we have

$$\mathcal{G}_i = \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + i}$$

for each i and so

$$\mathcal{G} = \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + 1} \cup \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + 2} \cup \ldots \cup \mathcal{L}^\sigma_{\lfloor \frac{n}{2} \rfloor + k} = \mathcal{H}^\sigma_{0,n,k}.$$

Next, we consider the case when n is even and $k = 1$. By Lemma 1, if $|\mathcal{G}| = n$, then $\mathcal{G}$ is a level $\mathcal{L}^\sigma_i$ for some i. By the intersection property we have $i \geq \frac{n}{2} + 1$ and so the weight of the family is bounded by $n\binom{n}{\frac{n}{2}+1}$ with equality only if $\mathcal{G} = \mathcal{L}^\sigma_{\frac{n}{2}+1}$. If $|\mathcal{G}| \leq n - 1$ then, since we can take at most $\frac{n}{2}$ intervals of size $\frac{n}{2}$, the weight is bounded by $\frac{n}{2}\binom{n}{\frac{n}{2}} + (\frac{n}{2} - 1)\binom{n}{\frac{n}{2}+1}$. This bound can only be attained if $|\mathcal{G}| = n - 1$, and it follows by Lemma 3 that G is pair-contiguous which, in the case $k = 1$, implies $\mathcal{G} = \mathcal{H}^\sigma_{x,n,1}$ for some $x \in [n]$. Since $\frac{n}{2}\binom{n}{\frac{n}{2}} + (\frac{n}{2} - 1)\binom{n}{\frac{n}{2}+1} = n\binom{n}{\frac{n}{2}+1}$, both the $|\mathcal{G}| = n - 1$ case and the $|\mathcal{G}| = n$ case yield optimal configurations.

Finally, we consider the case when n is even and $k > 1$. Suppose first that none of $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are levels. Then, by Lemma 1 we have $|\mathcal{G}_i| \leq n - 1$ for all i. We have $|\mathcal{G}| \leq kn - k$, and we will see

13

that if $\mathcal{G}$ has maximal weight, then in fact $|\mathcal{G}| \geq kn - k$. Indeed, by pairing off intervals with their complements, we can have at most $\frac{n}{2}$ intervals of size $\frac{n}{2}$, n intervals of size $\frac{n}{2} - 1$ or $\frac{n}{2} + 1$ and so on. Thus, the total weight we can achieve with $kn - k$ intervals is bounded by

$$\frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+k-1} + \left(\frac{n}{2} - k\right)\binom{n}{\frac{n}{2}+k} = n\Sigma_I(n, k),$$

and if we have fewer than $kn - k$ intervals the weight will be strictly less than this. It follows that we may assume $|\mathcal{G}| = kn - k$ and $|\mathcal{G}_i| = n - 1$ for all $1 \leq i \leq k$. By Lemma 3 each $\mathcal{G}_i$ is pair-contiguous on two levels j and $j + 1$. If $j < \frac{n}{2}$, then the corresponding $\mathcal{G}_i$ would have size at most $\frac{n}{2}$ by Lemma 2. Thus, we may assume that $j \geq \frac{n}{2}$. However, this combined with the fact that $\mathcal{G}$ has maximal weight already determines the structure of $\mathcal{G}$. Namely, $\mathcal{G}_1$ is pair-contiguous spanning levels $\frac{n}{2}$ and $\frac{n}{2} + 1$ with $\frac{n}{2}$ sets of size $\frac{n}{2}$ forming a star about some element x, $\mathcal{G}_2$ is pair-contiguous spanning levels $\frac{n}{2} + 1$ and $\frac{n}{2} + 2$ containing all remaining $\frac{n}{2} + 1$ elements of $\mathcal{L}_{\frac{n}{2}+1}^\sigma$ and a contiguous part of $\mathcal{L}_{\frac{n}{2}+2}^\sigma$ and so on. It follows that $\mathcal{G} = \mathcal{H}_{x,n,k}^\sigma$ for some $x \in [n]$.

Now, we will show that if $\mathcal{G}$ has maximal weight, then it cannot be that any of the $\mathcal{G}_i$ are levels. This will complete the proof since we have already classified the extremal families in the case that there are no levels. Suppose, by way of contradiction, that s is the smallest number such that $\mathcal{G}_s$ is a level, say $\mathcal{L}_t^\sigma$ ($t > \frac{n}{2}$). The weight of $\mathcal{G}_s \cup \mathcal{G}_{s+1} \cup \ldots \cup \mathcal{G}_k$ is clearly bounded by

$$n\binom{n}{t} + n\binom{n}{t+1} + \cdots + n\binom{n}{t+k-s}.$$

If $t > \frac{n}{2} + s - 1$, then, by the previous case (no full levels), the weight of $\mathcal{G}_1 \cup \ldots \mathcal{G}_{s-1}$ is maximized by taking

$$\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1} = \mathcal{H}_{x,n,s-1}^\sigma,$$

for some $x \in [n]$. The weight of $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}$ is

$$w(\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}) = \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+s-2} + \left(\frac{n}{2} - (s-1)\right)\binom{n}{\frac{n}{2}+s-1},$$

and it follows that the total weight of $\mathcal{G}$ is at most

$$\frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}+s-2}$$
$$+ \left(\frac{n}{2} - (s-1)\right)\binom{n}{\frac{n}{2}+s-1} + n\binom{n}{\frac{n}{2}+s} + n\binom{n}{\frac{n}{2}+s+1} + \cdots + n\binom{n}{\frac{n}{2}+k}. \quad (11)$$

Subtracting $w(\mathcal{H}_{x,n,k}^\sigma) - w(\mathcal{G})$ we obtain

$$w(\mathcal{H}_{x,n,k}^\sigma) - w(\mathcal{G}) \geq \left(\frac{n}{2} + s - 1\right)\binom{n}{\frac{n}{2}+s-1} - \left(\frac{n}{2} + k\right)\binom{n}{\frac{n}{2}+k}$$
$$= n\left(\binom{n-1}{\frac{n}{2}+s-1} - \binom{n-1}{\frac{n}{2}+k-1}\right) > 0, \quad (12)$$

which implies $\mathcal{G}$ is not of maximum weight.

Next, consider the case when $t \leq \frac{n}{2} + s - 1$. By pairing off intervals with their complement along σ, it follows that

$$w(\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_{s-1}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{t-1}.$$

Thus, the whole weight is

$$w(\mathcal{G}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{t-1} + n\binom{n}{t} + \cdots + n\binom{n}{t+k-s},$$

but $t - s \leq \frac{n}{2} - 1$ so

$$w(\mathcal{G}) \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + n\binom{n}{\frac{n}{2}+1} + \cdots + n\binom{n}{\frac{n}{2}-1+k} < w(\mathcal{H}^{\sigma}_{x,k,n}).$$

Thus, we may conclude that there is no full level. It follows that the only possible equality case is $\mathcal{G} = \mathcal{H}^{\sigma}_{x,k,n}$ for some $x \in [n]$.

$\square$

Proof of Theorem 13. By Lemma 11 we have that for every σ,

$$\sum_{F \in \mathcal{F}^{\sigma}} w(F, \sigma) \leq n\Sigma_I(n, k). \tag{13}$$

By the double counting outlined in Section 2 it is immediate that $|\mathcal{F}| \leq \Sigma_I(n, k)$. Thus, it remains to determine the possible extremal families. If $\mathcal{F}$ is extremal, then for every σ we have equality in (13) and so we are in an equality case given by Lemma 11.

Assume first that n is odd, then for every σ we have that $\mathcal{F}^{\sigma}$ is equal to $\mathcal{H}^{\sigma}_{0,n,k}$. In this case, it is immediate that $\mathcal{F} = \mathcal{H}_{0,n,k}$.

Suppose now that n is even and $k = 1$. There are two cases: either there exists a σ for which $\mathcal{F}^{\sigma} = \mathcal{H}^{\sigma}_{0,n,1}$ or there does not. Assume that we have $\mathcal{F}^{\sigma} = \mathcal{H}^{\sigma}_{0,n,1}$, and form a new cyclic permutation σ' by transposing two adjacent elements of σ. Observe that $\mathcal{F}^{\sigma'}$ still contains $n-2$ of the same intervals on level $\frac{n}{2} + 1$ (namely, those without exactly one of the transposed elements). Now, configurations of the form $\mathcal{H}^{\sigma}_{x,n,1}$, $x \in [n]$, have $\frac{n}{2} - 1$ intervals of size $\frac{n}{2} + 1$. Thus, we have that $\mathcal{F}^{\sigma'}$ must have the form $\mathcal{H}^{\sigma'}_{0,n,1}$. Since every permutation can be generated by transpositions of consecutive elements it follows that for all σ, $\mathcal{F}^{\sigma} = \mathcal{H}^{\sigma}_{0,n,1}$ and so $\mathcal{F} = \mathcal{H}_{0,n,1}$. Thus, we will assume that for all σ we have $\mathcal{F}^{\sigma} = \mathcal{H}^{\sigma}_{x,n,1}$, $x \in [n]$.

If n is even and $k > 1$ and $\mathcal{F}^{\sigma} = \mathcal{H}^{\sigma}_{0,n,k}$ for some σ, then in a completely analogous way to the above $k = 1$ case we can deduce that $\mathcal{F} = \mathcal{H}_{0,n,k}$. However, for $k > 1$ we have $|\mathcal{H}_{0,n,k}| < |\mathcal{H}_{x,n,k}|$. Indeed, simply observe

$$\begin{aligned}
|\mathcal{H}_{x,n,k}| - |\mathcal{H}_{0,n,k}| &= \binom{n-1}{\frac{n}{2}-1} + \binom{n-1}{\frac{n}{2}+k} - \binom{n}{\frac{n}{2}+k} \\
&= \binom{n-1}{\frac{n}{2}-1} - \binom{n-1}{\frac{n}{2}+k-1} \\
&> 0.
\end{aligned}$$

15

Thus, we may rule out the $\mathcal{H}_{0,n,k}$ case for $k > 1$ and conclude that $\mathcal{F}^\sigma \neq \mathcal{H}_{0,n,k}^\sigma$ for any σ.

So, finally, we may suppose that n is even and $k \geq 1$ and that for every σ, we have $\mathcal{F}^\sigma = \mathcal{H}_{x,n,k}^\sigma$ for some $x \in [n]$. We want to show that $\mathcal{F} = \mathcal{H}_{x,n,k}$ for some x. Each cyclic permutation contains $\frac{n}{2}$ intervals of size $\frac{n}{2}$ and n intervals of size $\frac{n}{2} + i$ for $1 \leq i \leq k - 1$ and $\frac{n}{2} - k$ intervals of size $\frac{n}{2} + k$. By the transposition argument that we used above, we can easily show that all the sets of $\binom{[n]}{\frac{n}{2}+i}$ for $1 \leq i \leq k - 1$ are in $\mathcal{F}$. It only remains to show that $\mathcal{F}$ contains all the sets of size $\frac{n}{2}$ that contain a fixed element and all the sets of size $\frac{n}{2} + k$ that don't contain that fixed element.

We supposed that for each σ, $\mathcal{F}^\sigma$ contains all of the $\frac{n}{2}$-element intervals containing some x and all the $\frac{n}{2} + k$-element intervals not containing that x. However, the x's corresponding to different σ's may be different. Our aim is to show that this is impossible. First, let us fix a cyclic permutation σ and notice that we have two sets A and B that are intervals along this σ such that $A \cap B = \{x\}$. Suppose by contradiction that there exists an $\frac{n}{2}$-element set C (in $\mathcal{F}$) not containing x. Observe that $|[n] \setminus (A \cup B)| = 1$, and let us define $[n] \setminus (A \cup B) = \{y\}$. If C contains y, then we can find a cyclic permutation σ where A, B and C are intervals, a contradiction. However, if C does not contain y, we can find a cyclic permutation where A, B and $C \cup \{y\}$ are intervals. Along this σ, since we have two intervals (namely, A and B) that intersect just in x, all the $\frac{n}{2}$-element intervals must also contain x and all the $\frac{n}{2} + k$-element intervals do not contain x. In particular, there is an $\frac{n}{2} + k$-element interval, say K which contains $C \cup \{y\}$ (this is because the interval $C \cup \{y\}$ doesn't contain x). Now, since all the intervals along σ of sizes $\frac{n}{2} + i$, $2 \leq i \leq k - 1$ are in $\mathcal{F}$, it is easy to find a $(k + 1)$-chain in $\mathcal{F}$ consisting of C, $C \cup \{y\}$ and K, a contradiction. Thus, we can conclude that every $\frac{n}{2}$-element set C in $\mathcal{F}$ must contain x.

By a standard double counting of pairs (F, σ) where $F \in \mathcal{F}$ and F is an interval along σ, we can see that $\mathcal{F}$ contains exactly $\binom{n-1}{\frac{n}{2}-1}$ sets of size $\frac{n}{2}$, and by the previous paragraph all the $\frac{n}{2}$-sets in $\mathcal{F}$ must contain a fixed element. Therefore, $\mathcal{F}$ contains every $\frac{n}{2}$-element set containing a fixed element and nothing else. But this means $\mathcal{F}$ cannot contain any set of size $\frac{n}{2} + k$ containing x because otherwise we will have a $(k + 1)$-chain in $\mathcal{F}$, a contradiction. But by the same double counting argument we can see that $\mathcal{F}$ contains $\binom{n-1}{\frac{n}{2}+k}$ sets of size $\frac{n}{2} + k$, and all these sets must not contain x. This shows that $\mathcal{F} = \mathcal{H}_{x,n,k}$, as desired and we have established all the cases of equality for intersecting k-Sperner families.

$\square$

References

[1] Béla Bollobás. Sperner systems consisting of pairs of complementary subsets. *Journal of Combinatorial Theory, Series A*, 15(3):363–366, 1973.

[2] Peter Burcsi and Daniel T Nagy. The method of double chains for largest families with excluded subposets. *Electronic Journal of Graph Theory and Applications (EJGTA)*, 1(1):40–49, 2013.

[3] Annalisa De Bonis, Gyula O.H. Katona, and Konrad J Swanepoel. Largest family without $A \cup B \subset C \cap D$. *Journal of Combinatorial Theory, Series A*, 111(2):331–336, 2005.

[4] Paul Erdős. On a lemma of Littlewood and Offord. *Bulletin of the American Mathematical Society*, 51(12):898–902, 1945.

[5] Paul Erdős, Chao Ko, and Richard Rado. Intersection theorems for systems of finite sets. *The Quarterly Journal of Mathematics*, 12(1):313–320, 1961.

[6] Péter L Erdős, Peter Frankl, and Gyula O.H. Katona. Intersecting Sperner families and their convex hulls. *Combinatorica*, 4(1):21–34, 1984.

[7] Dániel Gerbner. Profile polytopes of some classes of families. *Combinatorica*, 33(2):199–216, 2013.

[8] Curtis Greene, Gyula O.H. Katona, and Daniel J. Kleitman. Extensions of the Erdős-Ko-Rado theorem. *Recent Advances in Graph Theory, Proc. Symp. held in Prague, June 1974 (Academia Prague, 1975)*, pages 223–231, 1976.

[9] Jerrold R Griggs and Wei-Tian Li. The partition method for poset-free families. *Journal of Combinatorial Optimization*, 25(4):587–596, 2013.

[10] Jerrold R Griggs, Wei-Tian Li, and Linyuan Lu. Diamond-free families. *Journal of Combinatorial Theory, Series A*, 119(2):310–322, 2012.

[11] Gyula O.H. Katona. A simple proof of the Erdős-Chao Ko-Rado theorem. *Journal of Combinatorial Theory, Series B*, 13(2):183–184, 1972.

[12] Gyula O.H. Katona. A simple proof of a theorem of Milner. *Journal of Combinatorial Theory, Series A*, 83(1):138–140, 1998.

[13] Gyula O.H. Katona and Tamás Gy Tarján. Extremal problems with excluded subgraphs in the n-cube. In *Graph Theory*, pages 84–93. Springer, 1983.

[14] EC Milner. A combinatorial theorem on systems of sets. *Journal of the London Mathematical Society*, 1(1):204–206, 1968.

[15] L. Mirsky. A dual of Dilworth's decomposition theorem. *The American Mathematical Monthly*, 78(8):876–877, 1971.

[16] Alex D Scott. Another simple proof of a theorem of Milner. *Journal of Combinatorial Theory, Series A*, 87(2):379–380, 1999.

[17] Emanuel Sperner. Ein satz über untermengen einer endlichen menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928.