EXPANDERS WITH SUPERQUADRATIC GROWTH

ANTAL BALOG, OLIVER ROCHE-NEWTON AND DMITRY ZHELEZOV

Abstract. We will prove several expanders with exponent strictly greater than 2. For any finite set $A \subset \mathbb{R}$, we prove the following six-variable expander results

$$|(A-A)(A-A)(A-A)| \gg \frac{|A|^{2+\frac{1}{8}}}{\log^{\frac{17}{16}}|A|},$$

$$\left|\frac{A+A}{A+A}+\frac{A}{A}\right| \gg \frac{|A|^{2+\frac{2}{17}}}{\log^{\frac{16}{17}}|A|},$$

$$\left|\frac{AA+AA}{A+A}\right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log|A|},$$

$$\left|\frac{AA+A}{AA+A}\right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log|A|}.$$

1. Introduction

Let A be a finite[1] set of real numbers. The *sum set* of A is the set $A+A=\{a+b : a,b \in A\}$ and the product set AA is defined analogously. The Erdős-Szemerédi sum-product conjecture[2] states that, for any such A and all $\epsilon > 0$ there exists an absolute constant $c_\epsilon > 0$ such that

$$\max\{|A+A|,|AA|\} \geq c_\epsilon |A|^{2-\epsilon}.$$

In other words, it is believed that at least one of the sum set and product set will always be close to the maximum possible size $|A|^2$, suggesting that sets with additive structure do not have multiplicative structure, and vice versa.

A familiar variation of the sum-product problem is that of showing that sets defined by a combination of additive and multiplicative operations are large. A classical and beautiful result of this type, due to Ungar [21], is the result that for any finite set $A \subset \mathbb{R}$

$$\left|\frac{A-A}{A-A}\right| \geq |A|^2-2, \tag{1.1}$$

where

$$\frac{A-A}{A-A}=\left\{\frac{a-b}{c-d} : a,b,c,d \in A, c \neq d\right\}.$$

[1] From now on, A, B, C etc. will always be finite sets.

[2] In fact, the conjecture was originally stated for all $A \subset \mathbb{Z}$, but it is also widely believed to be true for all $A \subset \mathbb{R}$.

This notation will be used with flexibility to describe sets formed by a combination of additive and multiplicative operations on different sets. For example, if A, B and C are sets of real numbers, then $AB + C := \{ab + c : a \in A, b \in B, c \in C\}$. We use the shorthand kA for the k-fold sum set; that is $kA := \{a_1 + a_2 + \cdots + a_k : a_1, \dots, a_k \in A\}$. Similarly, the k-fold product set is denoted $A^{(k)}$; that is $A^{(k)} := \{a_1 a_2 \cdots a_k : a_1, \dots, a_k \in A\}$.

We refer to sets such as $\frac{A-A}{A-A}$, which are known to be large, as *expanders*. To be more precise, we may specify the number of variables defining the set; for example, we refer to $\frac{A-A}{A-A}$ as a *four variable expander*.

Recent years have seen new lower bounds for expanders. For example, Roche-Newton and Rudnev [16] proved[3] that for any $A \subset \mathbb{R}$

$$|(A-A)(A-A)| \gg \frac{|A|^2}{\log |A|}, \tag{1.2}$$

and Balog and Roche-Newton [2] proved that for any set A of strictly positive real numbers

$$\left| \frac{A+A}{A+A} \right| \geq 2|A|^2 - 1. \tag{1.3}$$

Note that equations (1.1), (1.2) and (1.3) are optimal up to constant (and in the case of (1.2), logarithmic) factors, as can be seen by taking $A = \{1, 2, \dots, N\}$. More generally, any set A with $|A + A| \ll |A|$ is extremal for equations (1.1), (1.2) and (1.3).

With these results, along with others in [5], [6], [11] and [14], we have a growing collection of near-optimal expander results with a lower bound $\Omega(|A|^2)$ or $\Omega(|A|^2/\log |A|)$. All of the near-optimal expanders that are known have at least 3 variables. The aim of this paper is to move beyond this quadratic threshold and give expander results with relatively few variables and with lower bounds of the form $\Omega(|A|^{2+c})$ for some absolute constant $c > 0$.

1.1. **Statement of results.** It was conjectured in [2] that for any $A \subset \mathbb{R}$ and any $\epsilon > 0$, $|(A-A)(A-A)(A-A)| \gg |A|^{3-\epsilon}$. In this paper, a small step towards this conjecture is made in the form of the following result.

Theorem 1.1. *Let $A \subset \mathbb{R}$. Then*

$$|(A-A)(A-A)(A-A)| \gg \frac{|A|^{2+\frac{1}{8}}}{\log^{17/16} |A|}.$$

This result is the first improvement on the bound $|(A-A)(A-A)(A-A)| \gg |A|^2/\log |A|$ which follows trivially from (1.2). The proof uses some beautiful ideas of Shkredov [18].

The following theorem gives partial support for the aforementioned conjecture from a slightly different perspective.

[3]Throughout the paper, this standard notation $\ll, \gg$ and respectively $O(\cdot), \Omega(\cdot)$ is applied to positive quantities in the usual way. Saying $X \gg Y$ or $X = \Omega(Y)$ means that $X \geq cY$, for some absolute constant $c > 0$. All logarithms in this paper are base 2.

Theorem 1.2. *Let $A \subset \mathbb{R}$. Then for any $\epsilon > 0$ there is an integer $k > 0$ such that*
$$|(A-A)^{(k)}| \gg_\epsilon |A|^{3-\epsilon}.$$

We also prove the following six variables expanders have superquadratic growth.

Theorem 1.3. *Let $A \subset \mathbb{R}$. Then*
$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg \frac{|A|^{2+2/17}}{\log^{16/17}|A|}.$$

Theorem 1.4. *Let $A \subset \mathbb{R}$. Then*
$$\left|\frac{AA+AA}{A+A}\right| \gg \frac{|A|^{11/8}|AA|^{3/4}}{\log|A|}.$$
In particular, since $|AA| \geq |A|$,
$$\left|\frac{AA+AA}{A+A}\right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log|A|}.$$

Theorem 1.5. *Let $A \subset \mathbb{R}$. Then*
$$\left|\frac{AA+A}{AA+A}\right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log|A|}.$$

The proofs of these three results make use of the results and ideas of Lund [10].

In fact, a closer inspection of the proof of Theorem 1.5 reveals that we obtain the inequality
$$\left|\left\{\frac{ab+c}{ad+e} : a,b,c,d,e \in A\right\}\right| \gg \frac{|A|^{2+\frac{1}{8}}}{\log|A|}.$$
Therefore, Theorem 1.5 actually gives a superquadratic five variable expander.

2. Preliminary Results

For the proof of Theorem 1.1 we will require the Ruzsa Triangle Inequality. See Lemma 2.6 in Tao-Vu [20].

Lemma 2.1. *Let A, B and C be subsets of an abelian group $(G,+)$. Then*
$$|A-B||C| \leq |A-C||B-C|.$$

A closely related result is the Plünnecke-Ruzsa inequality. A simple proof of the following formulation of the Plünnecke-Ruzsa inequality can be found in [13].

Lemma 2.2. *Let A be a subset of an abelian group $(G,+)$. Then*
$$|kA - lA| \leq \frac{|A+A|^{k+l}}{|A|^{k+l-1}}.$$

We will also use the following variant, which is Corollary 1.5 in Katz-Shen [9]. The result was originally stated for subsets of the additive group $\mathbb{F}_p$, but the proof is valid for any abelian group.

Lemma 2.3. *Let $X, B_1, \dots, B_k$ be subsets of an abelian group $(G, +)$. Then there exists $X' \subset X$ such that $|X'| \geq |X|/2$ and*

$$|X' + B_1 + \dots + B_k| \ll \frac{|X + B_1||X + B_2| \cdots |X + B_k|}{|X|^{k-1}}.$$

We will need various existing results for expanders. The first is due to Garaev and Shen [4].

Lemma 2.4. *Let $X, Y, Z \subset \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then*

$$|XY||(X + \alpha)Z| \gg |X|^{3/2}|Y|^{1/2}|Z|^{1/2}.$$

In particular,

$$|X(X + \alpha)| \gg |X|^{5/4} \tag{2.1}$$

and

$$\max\{|XY|, |(X + \alpha)Y|\} \gg |X|^{3/4}|Y|^{1/2}. \tag{2.2}$$

Note that Lemma 2.4 was originally stated only for $\alpha = 1$, but the proof extends without alteration to hold for an arbitrary non-zero real number α. A similar and earlier result of Elekes, Nathanson and Ruzsa [3] will also be used.

Lemma 2.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex or concave function and let $X, Y, Z \subset \mathbb{R}$. Then*

$$|f(X) + Y||X + Z| \gg |X|^{3/2}|Y|^{1/2}|Z|^{1/2}.$$

Define

$$R[A] := \left\{ \frac{a - b}{a - c} : a, b, c \in A \right\}.$$

The following result is due to Jones [6]. An alternative proof can be found in [15].

Lemma 2.6. *Let $A \subset \mathbb{R}$. Then*

$$|R[A]| \gg \frac{|A|^2}{\log |A|}.$$

Each of the three latter results come from simple applications of the Szemerédi-Trotter Theorem.

Note that the proof of Lemma 2.6 also implies that there exists $a, b \in A$ such that

$$|(A - a)(A - b)| \gg \frac{|A|^2}{\log |A|}. \tag{2.3}$$

See [15] for details. In particular, this gives a shorter proof of inequality (1.2), requiring only a simple application of the Szemerédi-Trotter Theorem. The inequality (1.2) will also be used in the proof of Theorem 1.1.

An important tool in this paper is the following result of Lund [10], which gives an improvement on (1.3) unless the ratio set A/A is very large.

Lemma 2.7. *Let $A \subset \mathbb{R}$. Then*

$$\left|\frac{A+A}{A+A}\right| \gg \frac{|A|^2}{\log|A|}\left(\frac{|A|^2}{|A/A|}\right)^{1/8}.$$

In fact, a closer examination of the proof of Lemma 2.7 reveals that it can be generalised without making any meaningful changes to give the following statement.

Lemma 2.8. *Let $A, B \subset \mathbb{R}$. Then*

$$\left|\frac{A+A}{B+B}\right| \gg \frac{|A||B|}{\log|A|+\log|B|}\left(\frac{|A||B|}{|A/B|}\right)^{1/8}.$$

The proofs of Theorems 1.3 and 1.4 use Lemma 2.8 as a black box. However, for the proof of Theorem 1.5 we need to dissect the methods from [10] in more detail and reconstruct a variant of the argument for our problem. To do this, we will also need the following tools which were used in [10]. The first is a generalisation of the Szemerédi-Trotter Theorem to certain well-behaved families of curves. A more general version of this result can be found in Pach-Sharir [12].

Lemma 2.9. *Let $\mathcal{P}$ be an arbitrary point set in $\mathbb{R}^2$. Let $\mathcal{L}$ be a family of curves in $\mathbb{R}^2$ such that*

- *any two distinct curves from $\mathcal{L}$ intersect in at most two points and*
- *for any two distinct points $p, q \in \mathcal{P}$, there exist at most two curves from $\mathcal{L}$ which pass through both p and q.*

Let $K \geq 2$ be some parameter and define $\mathcal{L}_K := \{l \in \mathcal{L} : |l \cap \mathcal{P}| \geq K\}$. Then

$$|\mathcal{L}_K| \ll \frac{|\mathcal{P}|^2}{K^3} + \frac{|\mathcal{P}|}{K}.$$

We will need the following version of the Lovász Local Lemma. This precise statement is Corollary 5.1.2 in [1].

Lemma 2.10. *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent from all but at most d of the events A_j with $j \neq i$. Suppose also that the probability of the event A_i occuring is at most p for all $1 \leq i \leq n$. Finally, suppose that*

$$ep(d+1) \leq 1.$$

Then, with positive probability, none of the events $A_1, \ldots, A_n$ occur.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Write $D = A - A$ and apply Lemma 2.3 in the multiplicative setting with $k = 2$, $X = DD$ and $B_1 = B_2 = D$. We obtain a subset $X' \subseteq DD$ such that $|X'| \gg |DD|$ and

$$|X'DD| \ll \frac{|DDD|^2}{|DD|}. \tag{3.1}$$

Then apply Lemma 2.1, again in the multiplicative setting, with $A = B = DD$ and $C = (X')^{-1}$. This bounds the left hand side of (3.1) from below, giving

$$|DD/DD|^{1/2}|X'|^{1/2} \leq |X'DD| \ll \frac{|DDD|^2}{|DD|}. \tag{3.2}$$

Recall the observation of Shkredov [18] that $R[A] - 1 = -R[A]$. Indeed, for any $a, b, c \in A$

$$\frac{a-b}{a-c} - 1 = \frac{a-b-(a-c)}{a-c} = -\frac{c-b}{c-a}.$$

Therefore, by Lemmas 2.4 and 2.6,

$$|DD/DD| \geq |R[A] \cdot R[A]| = |R[A] \cdot (R[A]-1)| \gg |R[A]|^{5/4} \gg \frac{|A|^{5/2}}{\log^{5/4}|A|}.$$

Putting this bound into (3.2) yields

$$\frac{|A|^{5/4}}{\log^{5/8}|A|}|X'|^{1/2} \ll \frac{|DDD|^2}{|DD|}. \tag{3.3}$$

Finally, since $|X'| \gg |DD| \gg \frac{|A|^2}{\log|A|}$ by (1.2), it follows that

$$|DDD|^2 \gg \frac{|A|^{5/4}}{\log^{5/8}|A|}|DD|^{3/2} \gg \frac{|A|^{5/4}}{\log^{5/8}|A|}\left(\frac{|A|^2}{\log|A|}\right)^{3/2} = \frac{|A|^{17/4}}{\log^{17/8}|A|}. \tag{3.4}$$

and thus

$$|DDD| \gg \frac{|A|^{2+\frac{1}{8}}}{\log^{17/16}|A|}$$

as claimed. □

We now turn to the proof of Theorem 1.2, which exploits similar ideas to the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $R := R[A]$ and $D = A - A$. Further, define $X_0 = D/D$ and recursively X_i to be either $X_{i-1}R$ or $X_{i-1}(R-1)$ such that

$$|X_i| = \max\{|X_{i-1}R|, |X_{i-1}(R-1)|\}.$$

We are going to prove by induction on k that

$$|X_k| \gg_k \frac{|A|^{3-\frac{1}{2^k}}}{\log^{\frac{3}{2}}|A|}.$$

Indeed, the base case $k=0$ follows from (1.1). Now, let $k \geq 1$. Then applying inequality (2.2) in Lemma 2.4, Lemma 2.6 and the inductive hypothesis

$$|X_{k+1}| \gg |X_k|^{1/2}|R|^{3/4} \gg_k \left(\frac{|A|^{3-\frac{1}{2^k}}}{\log^{\frac{3}{2}}|A|}\right)^{1/2}\left(\frac{|A|^2}{\log|A|}\right)^{3/4} = \frac{|A|^{3-\frac{1}{2^{k+1}}}}{\log^{\frac{3}{2}}|A|}.$$

Now fix $\epsilon > 0$ and choose k sufficiently large so that $\frac{1}{2^k} < \epsilon$. It was already noted earlier, $R \subseteq D/D$ and $R-1 \subseteq -D/D$, and so

$$|A|^{3-\epsilon} \leq \frac{|A|^{3-\frac{1}{2^k}}}{\log^{\frac{3}{2}}|A|} \ll_k |X_k| \leq \left|\frac{D^{(k+1)}}{D^{(k+1)}}\right|.$$

Applying Lemma 2.1 multiplicatively with $A = B = D^{(k+1)}$ and $C = 1/D^{(k+1)}$ we obtain that

$$|D^{(k+1)}||A|^{3-\epsilon} \ll_\epsilon |D^{(2k+2)}|^2,$$

so $|D^{(2k+2)}| \gg_\epsilon |A|^{3-\epsilon}$. Since k depends on ϵ only, it completes the proof. □

3.1. **Remarks, improvements and conjectures.** An improvement to Lemma 2.4 was given in [7], in the form of the bound

$$|A(A+\alpha)| \gg \frac{|A|^{24/19}}{\log^{2/19}|A|}.$$

Inserting this into the previous argument, we obtain the following small improvement:

$$|DDD| \gg \frac{|A|^{2+\frac{5}{38}}}{\log^{\frac{83}{76}}|A|}.$$

Furthermore, a small modification of the previous arguments can also give the bound

$$|DD/D| \gg \frac{|A|^{2+\frac{5}{38}}}{\log^{\frac{83}{76}}|A|}.$$

In the spirit of Theorem 1.2, it is reasonable to conjecture the following.

Conjecture 3.1. *For any $l > 0$ there exists $k > 0$ such that*

$$|(A-A)^{(k)}| \gg_{k,l} |A|^l$$

uniformly for all sets $A \subset \mathbb{R}$.

Even the case $l = 3$ is of interest as it is seemingly beyond the limit of the methods of the present paper. An alternative form of Conjecture 3.1 is as follows.

Conjecture 3.2. *For any $\epsilon > 0$ there exists $\delta > 0$ such that for any real set X with*

$$|XX| \leq |X|^{1+\delta}$$

the following holds: if $A \subset \mathbb{R}$ is such that

$$A - A \subset X,$$

then

$$|A| \ll_\delta |X|^\epsilon.$$

For comparison with Conjecture 3.1, we note that a similar sum-product estimate with many variables was proven in [2], in the form of the inequality

$$|4^{k-1}A^{(k)}| \gg |A|^k.$$

We also note that Corollary 4 in [19] verifies Conjecture 3.2 for any $\epsilon > 1/2 - c$, where $c > 0$ is some unspecified (but effectively computable) absolute constant.

It is not hard to see that Conjecture 3.2 is indeed equivalent to Conjecture 3.1. Assume that Conjecture 3.1 is true and fix $\epsilon > 0$. Next, take $l = \lfloor 1/\epsilon \rfloor + 3$. Assuming that Conjecture 3.1 holds, there is $k(\epsilon)$ such that

$$|(A-A)^{(k)}| \gg_{k,l} |A|^l \tag{3.5}$$

holds for real sets A.

Now, in order to deduce Conjecture 3.2, take $\delta = \epsilon/10k$ and assume that there are sets X, A such that $|XX| \leq |X|^{1+\delta}$ and $A - A \subset X$. If we now also assume for contradiction that $|A| \geq |X|^\epsilon$, then by the Plünnecke-Ruzsa inequality (2.2)

$$|(A-A)^{(k)}| \leq |X^{(k)}| \leq |X|^{1+\delta k} \leq |A|^{\frac{1+\delta k}{\epsilon}} \leq |A|^{l-1},$$

which contradicts (3.5) if $|A|$ is large enough (depending on ϵ), which we can safely assume.

Now let us assume that Conjecture 3.2 holds true. Let $l > 0$ be fixed and $\epsilon = \frac{1}{l+1}$. Let A be an arbitrary real set. Consider the set $X_0 = (A - A)$ and define recursively

$$X_{i+1} = X_i X_i.$$

Note that by construction

$$X_i = (A-A)^{(2^i)}.$$

Let c be an arbitrary non-zero element in $A - A$. Observe that

$$c^{2^i-1} \cdot A - c^{2^i-1} \cdot A = c^{2^i-1} \cdot (A-A) \subset (A-A)^{(2^i)} = X_i,$$

and so $A_i - A_i \subset X_i$ where $A_i := c^{2^i-1} \cdot A$. Thus, we are in position to apply the assumption that Conjecture 3.2 holds true. In particular, there is $\delta(\epsilon) > 0$ such that $|A| \ll_\delta |X|^\epsilon$ whenever $A - A \subset X$ and $|XX| \leq |X|^{1+\delta}$.

Now consider X_i for $i = 1, \ldots, \lfloor l/\delta \rfloor + 1 := j$. For each i, if $|X_{i+1}| \leq |X_i|^{1+\delta}$ it follows from Conjecture 3.2 that $|A| = |A_i| \ll_\delta |X_i|^\epsilon$, so

$$|(A-A)^{(2^i)}| = |X_i| \gg_\delta |A|^{1/\epsilon} \geq |A|^l$$

and we are done. Otherwise, if for each $1 \leq i \leq j$ holds $|X_{i+1}| \geq |X_i|^{1+\delta}$, one has

$$|(A-A)^{(2^j)}| = |X_j| \geq |X_0|^{1+j\delta} \geq |A|^l.$$

Thus, Conjecture 3.1 holds uniformly in A with

$$k(l) := 2^j = 2^{\lfloor l/\delta(l) \rfloor + 1}.$$

For a further support, let us remark that Conjecture 3.2 holds true if one replaces the condition $|XX| \leq |X|^{1+\delta}$ with the more restrictive one $|XX| \leq K|X|$ where $K > 0$ is an arbitrary but fixed absolute constant. In this setting Conjecture 3.2 can be proved by combining the Freiman Theorem and the Subspace Theorem and then applying almost verbatim the arguments of [17]. We leave the details to the interested reader.

4. Proofs of Theorems 1.3 and 1.4

4.1. Proof of Theorem 1.3.

We will first prove the following lemma.

Lemma 4.1. *Let $A \subset \mathbb{R}$. Then*

$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg \frac{|A|^{54/32}|A/A|^{13/32}}{\log^{3/4}|A|}.$$

Proof. Apply Lemma 2.5 with $f(x) = 1/x$, $X = (A+A)/(A+A)$ and $Y = Z = A/A$. Note that $f(X) = X$ and so

$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg \left|\frac{A+A}{A+A}\right|^{3/4} |A/A|^{1/2}.$$

Then applying Lemma 2.7, it follows that

$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg \frac{|A|^{3/2}}{\log^{3/4}|A|}\left(\frac{|A|^2}{|A/A|}\right)^{\frac{3}{32}} |A/A|^{1/2} = \frac{|A|^{54/32}|A/A|^{13/32}}{\log^{3/4}|A|}.$$

□

This immediately implies that

$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg |A|^{2+\frac{3}{32}-\epsilon}.$$

However, by optimising between Lemma 4.1 and Lemma 2.7 we can get a slight improvement in the form of Theorem 1.3.

Proof of Theorem 1.3. Let $|A/A| = K|A|$. If $K \geq \frac{|A|^{\frac{1}{17}}}{\log^{\frac{8}{17}}|A|}$ then Lemma 4.1 implies that

$$\left|\frac{A+A}{A+A} + \frac{A}{A}\right| \gg \frac{|A|^{67/32}K^{13/32}}{\log^{3/4}|A|} \gg \frac{|A|^{2+2/17}}{\log^{16/17}|A|}.$$

On the other hand, if $K \leq \frac{|A|^{\frac{1}{17}}}{\log^{\frac{8}{17}}|A|}$ then Lemma 2.7 implies that

$$\left|\frac{A+A}{A+A}+\frac{A}{A}\right| \geq \left|\frac{A+A}{A+A}\right| \gg \frac{|A|^2}{\log|A|}\left(\frac{|A|}{K}\right)^{1/8} \gg \frac{|A|^{2+2/17}}{\log^{16/17}|A|}.$$

□

4.2. **Proof of Theorem 1.4.** Apply Lemma 2.8 with $B = AA$. This yields

$$\left|\frac{AA+AA}{A+A}\right| \gg \frac{|A||AA|}{\log|A|}\left(\frac{|A||AA|}{|A/AA|}\right)^{1/8}.$$

By applying Lemma 2.2 in the multiplicative setting, we have

$$|AA/A| \leq \frac{|AA|^3}{|A|^2}$$

and so

$$\left|\frac{AA+AA}{A+A}\right| \gg \frac{|A||AA|}{\log|A|}\left(\frac{|A||AA|}{|A/AA|}\right)^{1/8} \geq \frac{|A||AA|}{\log|A|}\left(\frac{|A|^3}{|AA|^2}\right)^{1/8} = \frac{|A|^{11/8}|AA|^{3/4}}{\log|A|}$$

as required.

5. Proof of Theorem 1.5

Consider the point set $A \times A$ in the plane. Without loss of generality, we may assume that A consists of strictly positive reals, and so this point set lies exclusively in the positive quadrant. We also assume that $|A| \geq C$ for some sufficiently large absolute constant C. For smaller sets, the theorem holds by adjusting the implied multiplicative constant accordingly.

For $\lambda \in A/A$, let $\mathcal{A}_\lambda$ denote the set of points from $A \times A$ on the line through the origin with slope λ and let A_λ denote the projection of this set onto the horizontal axis. That is,

$$\mathcal{A}_\lambda := \{(x,y) \in A \times A : y = \lambda x\}, \qquad A_\lambda := \{x : (x,y) \in \mathcal{A}_\lambda\}.$$

Note that $|\mathcal{A}_\lambda| = |A_\lambda|$ and

$$\sum_\lambda |A_\lambda| = |A|^2.$$

We begin by dyadically decomposing this sum and applying the pigeonhole principle in order to find a large subset of $A \times A$ consisting of points which lie on lines of similar richness. Note that

$$\sum_{\lambda : |A_\lambda| \leq \frac{|A|^2}{2|A/A|}} |A_\lambda| \leq \frac{|A|^2}{2},$$

and so

$$\sum_{\lambda : |A_\lambda| \geq \frac{|A|^2}{2|A/A|}} |A_\lambda| \geq \frac{|A|^2}{2}.$$

Dyadically decompose the sum to get

$$\sum_{j\geq 1}^{\lceil \log |A| \rceil} \sum_{\lambda : 2^{j-1}\frac{|A|^2}{2|A/A|} \leq |A_\lambda| < 2^j \frac{|A|^2}{2|A/A|}} |A_\lambda| \geq \frac{|A|^2}{2}.$$

Therefore, there exists some $\tau \geq \frac{|A|^2}{2|A/A|}$ such that

$$\tau |S_\tau| \gg \sum_{\lambda \in S_\tau} |A_\lambda| \gg \frac{|A|^2}{\log |A|}, \tag{5.1}$$

where $S_\tau := \{\lambda : \tau \leq |A_\lambda| < 2\tau\}$. Using the trivial bound $\tau \leq |A|$, it also follows that

$$|S_\tau| \gg \frac{|A|}{\log |A|}. \tag{5.2}$$

For a point $p = (x, y)$ in the plane with $x \neq 0$, let $r(p) := y/x$ denote the slope of the line through the origin and p. For a point set $P \subseteq \mathbb{R}^2$ let $r(P) := \{r(p) : p \in P\}$. The aim is to prove that

$$|r((AA + A) \times (AA + A))| = |r((A \times A) + (AA \times AA))| \gg \frac{|A|^{2+\frac{1}{8}}}{\log |A|}. \tag{5.3}$$

Since $r((AA + A) \times (AA + A)) = \frac{AA+A}{AA+A}$, inequality (5.3) implies the theorem.

Write $S_\tau = \{\lambda_1, \lambda_2, \ldots, \lambda_{|S_\tau|}\}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_{|S_\tau|}$ and similarly write $A = \{x_1, \ldots, x_{|A|}\}$ with $x_1 < x_2 < \cdots < x_{|A|}$. For each slope λ_i, arbitrarily fix an element $\alpha_i \in A_{\lambda_i}$. Note that, for any $1 \leq i \leq |S_\tau| - 1$,

$$\begin{aligned}
\lambda_i < r((\alpha_i, \lambda_i\alpha_i) + (\alpha_{i+1}x_1, \lambda_{i+1}\alpha_{i+1}x_1)) &< r((\alpha_i, \lambda_i\alpha_i) + (\alpha_{i+1}x_2, \lambda_{i+1}\alpha_{i+1}x_2)) \\
&< \ldots \\
&< r((\alpha_i, \lambda_i\alpha_i) + (\alpha_{i+1}x_{|A|}, \lambda_{i+1}\alpha_{i+1}x_{|A|})) < \lambda_{i+1}.
\end{aligned}$$

Since $\lambda_i\alpha_i$ and $\lambda_{i+1}\alpha_{i+1}$ are elements of A, this gives $|A|$ distinct elements of $R((AA + A) \times (AA + A))$ in the interval $(\lambda_i, \lambda_{i+1})$. Summing over all i, it follows that

$$|r((AA + A) \times (AA + A))| \geq \sum_{i=1}^{|S_\tau|-1} |A| = |A|(|S_\tau| - 1) \gg |A||S_\tau|. \tag{5.4}$$

If $|S_\tau| \geq \frac{c|A|^{9/8}}{\log |A|}$ for any absolute constant $c > 0$ then we are done. Therefore, we may assume for the remainder of the proof that this is not the case. In particular, by (5.1), we may assume that

$$\tau \geq C|A|^{7/8} \tag{5.5}$$

holds for any absolute constant C.

Next, the basic lower bound (5.4) will be enhanced by looking at larger clusters of lines, a technique introduced by Konyagin and Shkredov [9] and utilised again by Lund [10]. We will largely adopt the notation from [10].

Let $2 \leq M \leq \frac{|S_\tau|}{2}$ be an integer parameter, to be determined later. We partition S_τ into clusters of size $2M$, with each cluster split into two subclusters of size M, as follows. For each $1 \leq t \leq \left\lfloor \frac{|S_\tau|}{2M} \right\rfloor$, let

$$\begin{aligned} f_t &= 2M(t-1) \\ T_t &= \{\lambda_{f_t+1}, \lambda_{f_t+2}, \dots, \lambda_{f_t+M}\} \\ U_t &= \{\lambda_{f_t+M+1}, \lambda_{f_tM++2}, \dots, \lambda_{f_t+2M}\}. \end{aligned}$$

For the remainder of the proof we consider the first cluster with $t = 1$, but the same arguments work for any $1 \leq t \leq \left\lfloor \frac{|S_\tau|}{2M} \right\rfloor$. We simplify the notation by writing $T_1 = T$ and $U_1 = U$.

Let $1 \leq i, k \leq M$ and $M+1 \leq j, l \leq 2M$ with at least one of $i \neq k$ or $j \neq l$ holding. For $a_i \in A_{\lambda_i}$ and $a_k \in A_{\lambda_k}$. Define

$$\mathcal{E}(a_i, j, a_k, l) = |\{(x, y) \in A \times A : r((a_i, \lambda_i a_i) + (\alpha_j x, \lambda_j \alpha_j x)) = r((a_k, \lambda_k a_k) + (\alpha_l y, \lambda_l \alpha_l y))|.$$

Lemma 5.1. *Let i, j, k, l satisfy the above conditions and let $K \geq 2$. Then there are $O(|A|^4/K^3 + |A|^2/K)$ pairs $(a_i, a_k) \in A_{\lambda_i} \times A_{\lambda_k}$ such that*

$$\mathcal{E}(a_i, j, a_k, l) \geq K.$$

Proof. We essentially copy the proof of Lemma 2 in [10], and so some details are omitted. Let $l_{a,b}$ be the curve with equation

$$(\lambda_i a + \lambda_j \alpha_j x)(b + \alpha_l y) = (\lambda_k b + \lambda_l \alpha_l y)(a + \alpha_j x).$$

Let $\mathcal{L}$ be the set of curves

$$\mathcal{L} = \{l_{a,b} : a \in A_{\lambda_i}, b \in A_{\lambda_k}\}$$

and let $\mathcal{P} = A \times A$. Note that $(x, y) \in l_{a_i, a_k}$ if and only if

$$r((a_i, \lambda_i a_i) + (\alpha_j x, \lambda_j \alpha_j x)) = r((a_k, \lambda_k a_k) + (\alpha_l y, \lambda_l \alpha_l y)).$$

Hence $\mathcal{E}(a_i, j, a_k, l) \geq K$ if and only if $|l_{a_i, a_k} \cap \mathcal{P}| \geq K$.

We can verify that the set of curves $\mathcal{L}$ satisfies the conditions of Lemma 2.9. One can copy this verbatim from the corresponding part of of the proof of Lemma 2 in [10]. Therefore, there are most

$$O\left(\frac{|\mathcal{P}|^2}{K^3} + \frac{|\mathcal{P}|}{K}\right) = O\left(\frac{|A|^4}{K^3} + \frac{|A|^2}{K}\right)$$

curves $l \in \mathcal{L}$ such that $|l \cap \mathcal{P}| \geq K$. The lemma follows. $\square$

Now, for each (i,j) such that $1 \le i \le M$ and $M+1 \le j \le 2M$ choose an element $a_{ij} \in A_{\lambda_i}$ uniformly at random. Then, for any $1 \le i,k \le M$ and $M+1 \le j,l \le 2M$, define $X(i,j,k,l)$ to be the event that

$$\mathcal{E}(a_{ij}, j, a_{kl}, l) \ge B,$$

where B is a parameter to be specified later. By Lemma 5.1, the probability that the event $X(i,j,k,l)$ occurs is at most

$$\frac{C}{\tau^2}\left(\frac{|A|^4}{B^3} + \frac{|A|^2}{B}\right),$$

where $C > 0$ is an absolute constant.

Furthermore, note that the event $X(i,j,k,l)$ is independent of the event $X(i',j',k',l')$ unless $(i,j) = (i',j')$ or $(k,l) = (k',l')$. Therefore, the event $X(i,j,k,l)$ is independent of all but at most of $2M^2$ of the other events $X(i',j',k',l')$. With this information, we can apply Lemma 2.10 with

$$n = M^4 - M^2, \qquad d = 2M^2, \qquad p = \frac{C}{\tau^2}\left(\frac{|A|^4}{B^3} + \frac{|A|^2}{B}\right).$$

It follows that there is a positive probability that none of the the events $X(i,j,k,l)$ occur, provided that

$$\frac{eC}{\tau^2}\left(\frac{|A|^4}{B^3} + \frac{|A|^2}{B}\right)(2M^2+1) \le 1. \tag{5.6}$$

The validity of (5.6) is dependent on our subsequent choice of the value of B. For now we proceed under the assumption that this condition is satisfied.

Let

$$Q = \bigcup_{1 \le i \le M, M+1 \le j \le 2M} \{(a_{ij}, \lambda_i a_{ij}) + (\alpha_j a, \lambda_j \alpha_j a) : a \in A\}.$$

Crucially,

$$r(Q) \ge M^2|A| - \sum_{1 \le i,k \le M, M+1 \le j,l \le 2M : \{i,j\} \ne \{k,l\}} \mathcal{E}(a_{ij}, j, a_{kl}, k). \tag{5.7}$$

In (5.7), the first term is obtained by counting the $|A|$ slopes in Q coming from all pairs of lines in $U \times T$. The second error term covers the overcounting of slopes that are counted more than once in the first term.

Since $\mathcal{E}(a_{ij}, j, a_{kl}, k) \le B$ for all quadruples (i,j,k,l) satisfying the aforementioned conditions, it follows that

$$r(Q) \ge M^2|A| - M^4 B. \tag{5.8}$$

Choosing $B = \frac{|A|}{2M^2}$, it follows that

$$r(Q) \ge \frac{M^2|A|}{2}. \tag{5.9}$$

This choice of B is valid as long as

$$\frac{eC}{\tau^2}(8M^6|A|+2M^2|A|)(2M^2+1)\leq 1. \tag{5.10}$$

This will certainly hold if

$$\frac{30eC}{\tau^2}M^8|A|\leq 1$$

and so we choose

$$M=\left\lfloor\left(\frac{\tau^2}{30eC|A|}\right)^{1/8}\right\rfloor.$$

In particular, by (5.5) we have $M\geq 2$ and so

$$M\gg\frac{\tau^{1/4}}{|A|^{1/8}}. \tag{5.11}$$

It is also true that $M\leq\frac{|S_\tau|}{2}$. This is true for all sufficiently large A since

$$|S_\tau|\geq\frac{c|A|}{\log|A|}\geq|A|^{1/8}\geq 2M.$$

Therefore

$$\left\lfloor\frac{|S_\tau|}{2M}\right\rfloor\gg\frac{|S_\tau|}{M}. \tag{5.12}$$

Next, note that $r(Q)$ is a subset of the interval (λ_1,λ_{2M}). We can repeat this argument for the next cluster to find at least $M^2|A|/2$ elements of $r((AA+A)\times(AA+A))$ in the interval $(\lambda_{2M+1},\lambda_{4M})$ and then so on for each of the $\left\lfloor\frac{|S_\tau|}{2M}\right\rfloor$ clusters of size $2M$. It then follows from (5.12) and (5.11) that

$$\begin{aligned}\left|\frac{AA+A}{AA+A}\right|&=|r((AA+A)\times(AA+A))|\\&\geq\sum_{j=1}^{\left\lfloor\frac{|S_\tau|}{2M}\right\rfloor}\frac{M^2|A|}{2}\\&\gg|S_\tau|M|A|\\&\gg(|S_\tau|\tau)^{1/4}|A|^{7/8}|S_\tau|^{3/4}.\end{aligned}$$

Applying (5.1) and (5.2), we conclude that

$$\left|\frac{AA+A}{AA+A}\right|\gg\frac{|A|^{2+\frac{1}{8}}}{\log|A|}$$

as required.

Acknowledgement

The research of Antal Balog was supported by the Hungarian National Science Foundation Grants NK104183 and K109789. Oliver Roche-Newton was supported by the Austrian Science Fund FWF Project F5511-N26, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

References

[1] N. Alon and J. Spencer, 'The probabilistic method' *John Wiley and Sons* (2008).
[2] A. Balog and O. Roche-Newton, 'New sum-product estimates for real and complex numbers', *Comput. Geom.* 53 (2015), no. 4, 825-846.
[3] G. Elekes, M. Nathanson and I. Ruzsa, 'Convexity and sumsets', *J Number Theory.* **83** (1999), 194-201.
[4] M. Z. Garaev and C.-Y. Shen, 'On the size of the set $A(A+1)$', *Math. Z.* 265 (2010), no. 1, 125-132.
[5] L. Guth and N. H. Katz, 'On the Erdős distinct distance problem in the plane', *Ann. of Math.* (2) 181 (2015), no. 1, 155-190.
[6] T. G. F. Jones, 'New quantitative estimates on the incidence geometry and growth of finite sets', PhD thesis, available at *arXiv:1301.4853* (2013).
[7] T. G. F. Jones and O. Roche-Newton, 'Improved bounds on the set $A(A+1)$', *J. Combin. Theory Ser. A* 120 (2013), no. 3, 515-526.
[8] N. H. Katz and C.-Y. Shen, 'A slight improvement to Garaev's sum product estimate', *Proc. Amer. Math. Soc.* 136 (2008), no. 7, 2499-2504.
[9] S. Konyagin and I. Shkredov, 'On sum sets of sets, having small product set', *Proc. Steklov Inst. Math.* **290** (2015), 288-299.
[10] B. Lund, 'An improved bound on $(A+A)/(A+A)$', *Electron. J. Combin.* 23, no. 3 (2016).
[11] B. Murphy, O. Roche-Newton and I. Shkredov, 'Variations on the sum-product problem', *SIAM J. Discrete Math.* 29 (2015), no. 1, 514-540.
[12] J. Pach and M. Sharir, 'On the number of incidences between points and curves', *Combinat. Probat. Comput.* **7** (1998), 121-127.
[13] G. Petridis, 'New proofs of Plünnecke-type estimates for product sets in groups', *Combinatorica* **32** (2012), no. 6, 721-733.
[14] O. Roche-Newton,'A new expander and improved bounds for $A(A+A)$', *arXiv:1603.06827* (2016).
[15] O. Roche-Newton, 'A short proof of a near-optimal cardinality estimate for the product of a sum set' *31st International Symposium on Computational Geometry*, 74-80 (2015), 10.4230/LIPIcs.SOCG.2015.74.
[16] O. Roche-Newton and M. Rudnev, 'On the Minkowski distances and products of sum sets', *Israel J. Math.* 209 (2015), no. 2, 507-526.
[17] O. Roche-Newton and D. Zhelezov, 'A bound on the multiplicative energy of a sum set and extremal sum-product problems', *Mosc. J. Comb. Number Theory* 5 (2015), no. 1-2, 52-69.
[18] I. Shkredov, 'Difference sets are not multiplicatively closed', *Discrete Analysis* **17** (2016).
[19] I. Shkredov and D. Zhelezov, 'On additive bases of sets with small product set', *Int. Math. Res. Notices*, forthcoming.
[20] T. Tao, V. Vu, 'Additive combinatorics' *Cambridge University Press* (2006).
[21] P. Ungar, '$2N$ non collinear points determine at least $2N$ directions', *J. Combin. Theory Ser. A* **33** (1982), 343-347.

A. Balog: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

E-mail address: balog.antal@renyi.mta.hu

O. Roche-Newton: 69 Altenberger Strasse, Johannes Kepler Universität, Linz, Austria

E-mail address: o.rochenewton@gmail.com

D. Zhelezov: Mathematical Sciences, Chalmers University of Technology, Göteborg, Sweden

E-mail address: dzhelezov@gmail.com