THE SHARPENING OF A RESULT CONCERNING PRIMITIVE IDEALS OF AN ASSOCIATIVE RING

F. SZÁSZ

The importance of the concept of primitive ideals of associative rings consists in the well-known theorem stating that every semisimple ring A is a subdirect sum of primitive rings B_ν, where a ring A is called semisimple (in the sense of Jacobson) if the Jacobson radical, i.e. the intersection of all primitive ideals, coincides with the zero ideal (0), and a ring B_ν is called primitive if the ideal (0) is a primitive ideal of B_ν. (Cf. N. Jacobson, *Structure of rings*, Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.)

Some new characterizations were recently given for the Jacobson radical of a ring A. For instance, A. Kertész [**3**] has shown in these Proceedings (generalizing an observation of L. Fuchs [**1**]) that the Jacobson radical J of a ring A consists of exactly those elements x of A for which the product yx lies with every $y \in A$ in the Frattini A-submodule of the ring A, as of an A-right module A for itself (cf. also Hille [**2**]). Furthermore A. Kertész [**4**] has shown that J is the intersection of all those maximal right ideals R of A for which there must exist, for any element $x \notin R$ $(x \in A)$, a second element $y \in A$ with $yx \notin R$; that is, those right ideals for which $A^{-1}R \subseteq R$ holds, where $X^{-1}R = \{y;\, y \in A,\, Xy \subseteq R\}$ for an arbitrary subset X of A. Furthermore, let $L \cdot Y^{-1}$ denote the subset $\{z;\, z \in A,\, zY \subseteq L\}$.

Every modular right ideal R of A is *quasi-modular* in the sense that $A^{-1}R \subseteq R$ holds. The concept of quasi-modularity of right ideals R was introduced in [**6**]. Solving a problem proposed by Kertész [**4**] I have shown in [**6**] the existence of an associative ring which has a quasi-modular maximal but not a modular right ideal. In my other paper [**7**] a two-sided ideal Q of A is called *quasi-primitive* if there exists a quasi-modular maximal right ideal R of A with $Q = A^{-1}R \subseteq R$. Obviously every primitive ideal is also quasi-modular in A, and almost trivially every artin ring with (0) quasi-primitive ideal is a total matrix ring over a skew field. Furthermore, any quasi-primitive ideal is clearly a prime ideal, and any commutative ring with (0) quasi-primitive ideal is a field.

Solving a problem of my colleague Dr. Steinfeld, I have proved in [**7**] that the Jacobson radical J of A must coincide with the intersec-

Received by the editors July 20, 1966.

tion of all quasi-primitive ideals. There are two proofs of this fact in [7], an entirely elementary proof without quasi-regular element and irreducible modules, and (in a footnote) a second short proof with quasi-regular elements too.

In my note [7] some open problems on quasi-primitive ideals are mentioned, which have recently been solved completely by Dr. Steinfeld. He has shown that the concepts of primitivity and quasi-primitivity of ideals of arbitrary associative rings must coincide. Using a lemma which is proved but not explicitly announced in [7], Dr. Steinfeld has proved that *there exists* for every fixed quasi-modular maximal right ideal R of A an element X of A for which the right ideal quotient $R_x = \{x\}^{-1}R$ is a modular maximal right ideal of A such that $A^{-1}R = A^{-1}R_x = (xA)^{-1}R$, which means that every quasi-primitive ideal $Q = A^{-1}R$ is by $Q = A^{-1}R_x$ also primitive in A.

This result of Dr. Steinfeld can be sharpened as follows:

THEOREM. *If R is a quasi-modular maximal right ideal of an arbitrary associative ring, and if $x \in A$ is an arbitrary element of A with the condition $x \notin R$, then the quasi-primitive ideal $Q = A^{-1}R$ coincides with the primitive ideal $P_x = A^{-1}R_x = (xA)^{-1}R$ of A (instead of a single x for any $x \notin R$).*

PROOF. In my note [7] it is shown that $R_x = \{x\}^{-1}R$ is a modular maximal right ideal of A for every quasi-modular maximal right ideal R of A and for every $x \in A$ with $x \notin R$. Namely, $R_x = \{x\}^{-1}R$ is a right ideal of A. By the quasi-modularity of R, $A^2 + R = A$, and therefore we obtain $RA^{-1} = R$; that is, $xA + R = A$ for any $x \notin R$, $x \in A$. Since there exists for $x \notin R$ an element $y \in A$ with $xy \notin R$, the right ideal R_x has the property $y \notin R_x$, i.e. $R_x \neq A$. If $z \in A$ is any element with $z \notin R_x$, one has by $xz \notin R$ obviously $xzA + R = A$, and thus for any $b \in A$ the existence of $a \in A$ and $r \in R$ with $xza + r = xb$, and thereby also $x(b - za) = r \in R$, $b - za \in R_x$, $b \in zA + R_x$ and $A = zA + R_x$, which means the maximality of R_x in A. Moreover, one has $xza_1 + r_1 = x$ with some $a_1 \in A$ and $r_1 \in R$, which implies $x(1 - za_1)A \subseteq R$, consequently $(1 - za_1)A \subseteq R_x$ and the modularity of the maximal right ideal R_x of A.

By $xA + R = A$ and $A((xA)^{-1}R) = (xA + R)((xA)^{-1}R) \subseteq R$ one has on one side $(xA)^{-1}R \subseteq A^{-1}R$. On the other hand the condition $y \in A^{-1}R$ implies by $A^{-1}R = (xA + R)^{-1}R$ obviously $xAy \subseteq R$, that is $y \in (xA)^{-1}R$, and thus holds $A^{-1}R = (xA)^{-1}R$ for every $x \notin R$ $(x \in A)$. But one has almost trivially $(xA)^{-1}R = A^{-1}\{x\}^{-1}R = A^{-1}R_x$ too, which means that $Q = A^{-1}R = (xA)^{-1}R$ and $P_x = A^{-1}R_x$ must be for every $x \notin R$ the same primitive ideals of A. Q.E.D.

References

1. L. Fuchs, *A remark on the Jacobson radical*, Acta Sci. Math. Szeged **14** (1952), 167–168.

2. E. Hille, *Functional analysis and semi-groups*, Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R. I., 1948; esp. p. 486 and Theorem 22.15.3.

3. A. Kertész, *A characterization of the Jacobson radical*, Proc. Amer. Math. Soc. **14** (1963), 595–597.

4. ———, *Über Artinsche Ringe*, Akadémiai Kiadó, Budapest, 1966.

5. O. Steinfeld, *Eine Bemerkung über die primitiven Ideale eines Ringes*, Acta Math. Acad. Sci. Hungar. (to appear).

6. F. Szász, *Die Lösung eines Problemes bezüglich einer Charakterisierung des Jacobsonschen Radikales*, Acta Math. Acad. Sci. Hungar. (to appear).

7. ———, *Eine Charakterisierung des Jacobsonschen Radikales eines Ringes*, Bull. Acad. Polon. Sci. **152** (1967), 53–56.

8. ———, *Radikalbegriffe für Halbgruppen mit Nullelement, die dem Jacobsonschen ringtheoretischen Radikal ähnlich sind*, Math. Nachr. (to appear).