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ON THE DUALIZATION OF SUBDIRECT EMBEDDINGS

By

F. SZÁSZ and R. WIEGANDT (Budapest)

To Professor G. ALEXITS *on his 70th birthday*

§ 1. Introduction

In the algebra there are several kinds of structure theorems which can be formulated without operations, using only homological tools. For instance, the well-known fact that any universal algebra can be subdirectly embedded in a direct product of subdirectly irreducible algebras, can be formulated in a pure category-theoretical manner. Now the question arises what its dual statement asserts. Our purpose is to give such a category which satisfies certain selfdual conditions, and making use of these, to prove structure theorems and their dual statements. The structure theorems themselves are, of course, well-known statements for algebraic stuctures. However, their duals yield some theorems of unusual type. About the possibility of the dualization there occurs some trouble. The most of the difficulties is at finding selfdual conditions being necessary to prove the theorems. So we must not make use of the condition 'every epimorphism is a normal one' which is fulfilled for groups, since its dual is false. Further the lattice of all congruence-relations of any universal algebra is a so-called compactly generated lattice. This fact plays a very important role in the proof of the theorem according to subdirect embeddings of universal algebras, nevertheless compactly generating is not a selfdual notion.

Applying the theorems proved for certain categories, we establish some particular theorems for rings, groups, modules, respectively.

In § 2 we give a detailed enumeration of the usual notions and assertions of the theory of categories with respect to the importance of the dual notions and assertions, moreover, we form a system of selfdual conditions which will be satisfied by the category we are dealing with. § 3 is devoted to the investigation of subdirect embeddings, subdirect irreducibility and to the dualization of those. In § 4 we applicate the results developed before for rings, groups, modules and abelian groups. Most of the applications are concerned with rings.

§ 2. Preliminaries

Let $\mathscr{C}$ be a category. The objects and maps of $\mathscr{C}$ will be denoted by small Latin and small Greek letters, respectively. By definition $\mathscr{C}$ satisfies the following conditions:

(C_1) *If $\alpha:a\to b$ and $\beta:b\to c$ are maps, then there is a uniquely defined map $\alpha\beta:a\to c$, which is called the product of the maps α and β;*

(C_2) *If $\alpha:a\to b$, $\beta:b\to c$, $\gamma:c\to d$ are maps, then $(\alpha\beta)\gamma=\alpha(\beta\gamma)$ holds.*

(C_3) *For each object $a\in\mathscr{C}$ there is a map $\varepsilon_a:a\to a$, called the identity map of a such that for each $\alpha:b\to a$ and $\beta:a\to c$ we have $\alpha\varepsilon_a=\alpha$, $\varepsilon_a\beta=\beta$.*

The *dual category* of the category $\mathscr{C}$, denoted by $\mathscr{C}^*$, consists of the same objects as $\mathscr{C}$, and $\alpha^*:b\to a$ is a map of $\mathscr{C}^*$ if and only if $\alpha:a\to b$ is a map of $\mathscr{C}$. Clearly $(\mathscr{C}^*)^*=\mathscr{C}$, and if a statement P is true for category $\mathscr{C}$, then there is a dual statement P^* which will be true for $\mathscr{C}^*$. In what follows we shall assume that the category $\mathscr{C}$ satisfies some additional assumptions. *These requirements will be selfdual which means that both of $\mathscr{C}$ and $\mathscr{C}^*$ satisfy them. So any statement P which can be proved for $\mathscr{C}$, will be true for $\mathscr{C}^*$ too. Hence statement P^* is true for $(\mathscr{C}^*)^*=\mathscr{C}$.*

Let $H(a, b)$ denote the class of all maps of $\mathscr{C}$ which map a into b. An object $o\in\mathscr{C}$ is said to be a *zero object* if for any object a of $\mathscr{C}$ both of the classes $H(a, o)$ and $H(o, a)$ contain only one map.

We assume that

(C_4) *$\mathscr{C}$ possesses zero objects.*

Obviously also $\mathscr{C}^*$ contains zero objects. We shall say that $\mathscr{C}$ is a *category with zero maps*, if for any ordered pair of objects a, b there is a map $\omega_{ab}:a\to b$ such that for any $\alpha:c\to a$, $\beta:b\to d$ we have $\alpha\omega_{ab}=\omega_{cb}$ and $\omega_{ab}\beta=\omega_{ad}$. If $\mathscr{C}$ possesses zero objects, then $\mathscr{C}$ is a category with zero maps (cf. KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8]). If there is no doubt between which objects the zero map operates, then that zero map will be shortly denoted by ω.

A map $\alpha:a\to c$ will be called a *monomorphism*, if for any maps $\varrho:b\to a$, $\sigma:b\to a$ from $\varrho\alpha=\sigma\alpha$ it follows $\varrho=\sigma$.

A map $\alpha:c\to a$ will be called an *epimorphism*, if for any maps $\varrho:a\to b$, $\sigma:a\to b$ from $\alpha\varrho=\alpha\sigma$ it follows $\varrho=\sigma$.

The notion of epimorphism is dual to that of monomorphism in the sense that α is a monomorphism of $\mathscr{C}$ if and only if α^* is an epimorphism of $\mathscr{C}^*$.

The product of two monomorphism (if it exists) is again a monomorphism. If $\alpha\beta$ is a monomorphism, then α is also a monomorphism.

The product of two epimorphisms (if it exists) is again an epimorphism. If $\beta\alpha$ is an epimorphism, then α is also an epimorphism.

The statements are well-known (cf. KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8], or MITCHELL [9]). Now we are going to give the definitions of some usual notions together with their duals.

Let $\beta_1:b_1\to a$ and $\beta_2:b_2\to a$ be monomorphisms. We shall say that $(b_2, \beta_2)\leqq(b_1, \beta_1)$, if there exists a map ϱ (which has to be a monomorphism) such that $\varrho\beta_1=\beta_2$. If both of $(b_2, \beta_2)\leqq(b_1, \beta_1)$ and $(b_1, \beta_1)\leqq(b_2, \beta_2)$ hold then the pairs (b_1, β_1) and (b_2, β_2) are said to be equivalent. If $(b_2, \beta_2)\leqq(b_1, \beta_1)$ but they are not equivalent, then we shall write $(b_2, \beta_2)<(b_1, \beta_1)$. The equivalence classes of the relation thus defined will be called the *subobjects* of a. For

Let $\beta_1:a\to b_1$ and $\beta_2:a\to b_2$ be epimorphisms. We shall say that $(\beta_2, b_2)\leqq(\beta_1, b_1)$ if there exists a map ϱ (which has to be an epimorphism) such that $\beta_1\varrho=\beta_2$. If both of $(\beta_2, b_2)\leqq(\beta_1, b_1)$ and $(\beta_1, b_1)\leqq(\beta_2, b_2)$ hold, then the pairs (β_1, b_1) and (β_2, b_2) are said to be equivalent. If $(\beta_2, b_2)\leqq(\beta_1, b_1)$ but they are not equivalent, then we shall write $(\beta_2, b_2)<(\beta_1, b_1)$. The equivalence classes of the relation thus defined will be called the *factor-objects* of a. For convenience the convenience the equivalence class represented by the pair (b, β) will also be denoted by (b, β).

A commutative diagram

$$\begin{tikzcd} k \arrow[r, "\varkappa_1"] \arrow[d, "\varkappa_2"'] & d_1 \arrow[d, "\delta_1"] \\ d_2 \arrow[r, "\delta_2"'] & a \end{tikzcd}$$

is called a *pullback* for δ_1 and δ_2, if for any object $c \in \mathscr{C}$ and commutative diagram

$$\begin{tikzcd} c \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & d_1 \arrow[d, "\delta_1"] \\ d_2 \arrow[r, "\delta_2"] & a \end{tikzcd}$$

there exists a unique map $\gamma: c \to k$ such that the diagram

$$\begin{tikzcd} c \arrow[rr, "\gamma_1"] \arrow[dd, "\gamma_2"'] \arrow[dr, "\gamma"] & & d_1 \arrow[dd, "\delta_1"] \\ & k \arrow[ur, "\varkappa_1"'] \arrow[dl, "\varkappa_2"] & \\ d_2 \arrow[rr, "\delta_2"] & & a \end{tikzcd}$$

is again commutative.

A subobject $(k, \varkappa)$ of an object $a \in \mathscr{C}$ is said to be a *kernel* of the map $\alpha: a \to b$, if

$$\begin{tikzcd} k \arrow[r] \arrow[d, "\varkappa"'] & 0 \arrow[d] \\ a \arrow[r, "\alpha"] & b \end{tikzcd}$$

is a pullback diagram. Here the map $\varkappa$ has to be a monomorphism. Equivalently, the subobject $(k, \varkappa)$ is the kernel of α if (i) $\varkappa\alpha = \omega$; (ii) for each $\gamma: c \to a$ satisfying $\gamma\alpha = \omega$, there is a unique map $\gamma': c \to k$ such that $\gamma'\varkappa = \gamma$. If $(k, \varkappa)$ is a kernel of α, then we shall write $\operatorname{Ker} \alpha = (k, \varkappa)$, or only $\operatorname{Ker} \alpha = k$. The map $\varkappa$ is called a normal monomorphism and the subobject $(k, \varkappa)$ is a *normal subobject or* an *ideal* of a.

equivalence class represented by the pair (β, b) will also be denoted by (β, b).

A commutative diagram

$$\begin{tikzcd} a \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & d_1 \arrow[d, "\varkappa_1"] \\ d_2 \arrow[r, "\varkappa_2"] & k \end{tikzcd}$$

is called a *pushout* for δ_1 and δ_2, if for any object $c \in \mathscr{C}$ and commutative diagram

$$\begin{tikzcd} a \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] & d_1 \arrow[d, "\gamma_1"] \\ d_2 \arrow[r, "\gamma_2"] & c \end{tikzcd}$$

there exists a unique map $\gamma: k \to c$ such that

$$\begin{tikzcd} a \arrow[rr, "\delta_1"] \arrow[dd, "\delta_2"'] & & d_1 \arrow[dd, "\gamma_1"] \arrow[dl, "\varkappa_1"'] \\ & k \arrow[dr, "\gamma"] & \\ d_2 \arrow[ur, "\varkappa_2"] \arrow[rr, "\gamma_2"] & & c \end{tikzcd}$$

is again commutative.

A factorobject $(\varkappa, k)$ of an object $a \in \mathscr{C}$ is said to be a *cokernel* of the map $\alpha: b \to a$ if

$$\begin{tikzcd} b \arrow[r, "\alpha"] \arrow[d] & a \arrow[d, "\varkappa"] \\ 0 \arrow[r] & k \end{tikzcd}$$

is a pusout diagram. Here the map $\varkappa$ has to be an epimorphism. Equivalently, the factorobject $(\varkappa, k)$ is the cokernel of α if (i) $\alpha\varkappa = \omega$; (ii) for each $\gamma: a \to c$ satisfying $\alpha\gamma = \omega$, there is a unique map $\gamma': k \to c$ such that $\varkappa\gamma' = \gamma$. If $(\varkappa, k)$ is a cokernel of α, then we shall write $\operatorname{Coker} \alpha = (\varkappa, k)$ or only $\operatorname{Coker} \alpha = k$. The map $\varkappa$ is called a *normal epimorphism* and the factorobject $(\varkappa, k)$ is a *normal factorobject* of a.

These definitions correspond to those of MITCHELL [9] and SULIŃSKI [13]. In KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8] ideals and normal subobjects (and so their duals) are not the same notions, but under conditions supposed below they coincide.

In the category of groups every epimorphism is a normal one, but not every monomorphism is a normal one (i.e. not every subgroup is a normal subgroup). The product of two normal monomorphisms need not be a normal one. Moreover, if α is a monomorphism, then Ker $\alpha=(o, \omega)$, but the converse statement does not hold.

If $\alpha\beta$ is a normal monomorphism and β is a monomorphism, then α is a normal monomorphism. (Cf. [8] § 8.3). The dual statement also holds for normal epimorphisms.

We assume that

(C_5) *Every map has a kernel and a cokernel.*

PROPOSITION 1. Ker Coker Ker $\alpha=$ Ker α.

PROOF. Let $\alpha: a \to b$ be a map, and put Ker $\alpha=(k, \varkappa)$ and Coker $\varkappa=(\lambda, l)$. We have to prove Ker $\lambda=(k, \varkappa)$. (i) Since $(\lambda, l)=$ Coker $\varkappa$, so by definition $\varkappa\lambda=\omega$ holds. (ii) Let $\gamma: c \to a$ be a map with $\gamma\lambda=\omega$. By definition of Coker $\varkappa$ there is a unique map $\gamma': c \to k$ such that $\gamma'\varkappa=\gamma$. Thus from $\gamma\lambda=\omega$ we get the existence of a unique map γ' satisfying $\gamma'\varkappa=\gamma$. Hence Ker $\lambda=(k, \varkappa)$ is valid.

Dualizing we get

PROPOSITION 1*. Coker Ker Coker $\alpha=$ Coker α.

We suppose that

(C_6) *The class of all subobjects and factorobjects of any object a is a set, and it forms a complete lattice L_a and L_a^* with respect to the relation $\leqq$ defined for subobjects and factorobjects, respectively.*

(C_7) *For each object $a \in \mathscr{C}$ the set of all normal subobjects and normal factorobjects, forms a complete sublattice of L_a and L_a^*, respectively.*

The intersection $\cap$ and union $\cup$ in the lattices L_a and L_a^* of the ideals and normal factorobjects of the objects a can be defined in the following way.

The intersection $(k, \varkappa)$ of two ideals (d_1, δ_1), (d_2, δ_2) is an ideal such that

$$\begin{tikzcd} k \arrow[r] \arrow[d] \arrow[dr, "\varkappa"] & d_1 \arrow[d, "\delta_1"] \\ d_2 \arrow[r, "\delta_2"] & a \end{tikzcd}$$

is a pullback diagram.

The intersection $(\varkappa, k)$ of two normal factorobjects (δ_1, d_1), (δ_2, d_2) is a normal factorobject such that

$$\begin{tikzcd} a \arrow[r, "\delta_1"] \arrow[d, "\delta_2"'] \arrow[dr, "\varkappa"] & d_1 \arrow[d] \\ d_2 \arrow[r] & k \end{tikzcd}$$

is a pushout diagram.

The union (l, λ) of two ideals (d_1, δ_1), (d_2, δ_2) means an ideal for which

$$\begin{tikzcd} l \arrow[drr, "\lambda"] & & d_1 \arrow[ll] \arrow[d, "\delta_1"] \\ d_2 \arrow[u] \arrow[rr, "\delta_2"] & & {} \end{tikzcd}$$

is a commutative diagram, and for any monomorphism $\gamma: c \to a$ and diagram

$$\begin{tikzcd} l & & d_1 \arrow[ll] \arrow[dl] \\ & C \arrow[dr, "\gamma"] & \\ d_2 \arrow[uu] \arrow[ur] & & a \end{tikzcd}$$

there is a monomorphism $\lambda': l \to c$ such that $\lambda'\gamma = \lambda$, and the diagram becomes commutative.

The union of two normal factorobjects is defined in the dual way.

These definitions of the unions correspond to the relation $\leqq$ defined on L_a and L_a^*, respectively. However, in MITCHELL [9] the unions are defined in a somewhat different manner.

PROPOSITION 2. *The lattice L_a of the ideals of an object a is dually isomorphic to the lattice L_a^* of the normal factorobjects of a in the following sense. Any ideal $(k, \varkappa)$ of L_a is a kernel* Ker α *of a map α. The correspondence* Ker $\alpha \to$ Coker Ker $\alpha =$ $= (\lambda, l)$ *is one-to-one, further the relation $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$ holds if and only if $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$ is valid for their cokernels in L_a^*.*

PROOF. Proposition 1 implies that Ker $\alpha \to$ Coker Ker α is a one-to-one correspondence.

Assume $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2) \in L_a$, and put Coker $\varkappa_i = (\lambda_i, l_i)$, $i = 1, 2$. By definition

$$\begin{tikzcd} k_1 \arrow[r, "\varkappa_1"] \arrow[d] & a \arrow[d, "\lambda_1"] \\ 0 \arrow[r] & l_1 \end{tikzcd} \tag{1}$$

is a pushout diagram. Since $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$, so there is a map $\varkappa': k_1 \to k_2$ such that $\varkappa'\varkappa_2 = \varkappa_1$. Thus

$$\begin{tikzcd} k_1 \arrow[r, "\varkappa'\varkappa_2"] \arrow[d] & a \arrow[d, "\lambda_2"] \\ 0 \arrow[r] & l_2 \end{tikzcd}$$

is a commutative diagram, and since (1) is a pushout, therefore there is a map $\lambda': l_1 \to l_2$ such that $\lambda_1\lambda' = \lambda_2$. This means $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$.

Dualizing, $(\lambda_1, l_1) \geqq (\lambda_2, l_2)$ implies $(k_1, \varkappa_1) \leqq (k_2, \varkappa_2)$. Thus proposition 2 is proved. One can formulate this statement as follows:

$$(k_1, \varkappa_1) \cap (k_2, \varkappa_2) = (c, \gamma),$$

$$(k_1, \varkappa_2) \cup (k_2, \varkappa_2) = (d, \delta)$$

are valid if and only if

$$(\lambda_1, l_1) \cup (\lambda_2, l_2) = \text{Coker } \gamma,$$

$$(\lambda_1, l_1) \cap (\lambda_2, l_2) = \text{Coker } \delta$$

are valid.

Let $\alpha : a \to b$ be a map. If $\mu : a \to m$ is an epimorphism and $\nu : m \to b$ a monomorphism with $\mu\nu = \alpha$, then the subobjects (m, ν) of b will be called *an image of* α (with the epimorphism μ), (m, ν) is said to be a *normal image*, if μ is a normal epimorphism.

Let $(k, \varkappa)$ be a subobject of the object a and let $\alpha : a \to b$ be an epimorphism. If (m, ν) is an image of $\varkappa\alpha$, then (m, ν) will be called an *image of* $(k, \varkappa)$ *by the epimorphism* α.

Let $\alpha : b \to a$ be a map. If $\mu : m \to a$ is a monomorphism and $\nu : b \to m$ is an epimorphism with $\nu\mu = \alpha$, then the factorobject (ν, m) of b will be called a *coimage of* α (with the monomorphism μ), (ν, m) is said to be a *normal coimage*, if μ is a normal monomorphism.

Let $(\varkappa, k)$ be a factorobject of the object a and let $\alpha : b \to a$ be a monomorphism. If (ν, m) is a coimage of αk, then (ν, m) *will be called a coimage of* $(\varkappa, k)$ *by the monomorphism* α.

A normal image (and normal coimage) is uniquely determined, but image (and coimage) is not (cf. Kuroš—Livšits—Šulgeifer—Tsalenko [8]). If (m, ν) is an image of b such that for every image (m', ν'), of b $(m, \nu) \leqq (m', \nu')$, then (m, ν) will be denoted by Im α. Coim α will denote the dual notion.

In the category of groups or rings, for any map α both of Im α and Coim α does exist, moreover, Im α is always a normal image, but Coim α need not be a normal coimage.

Let us assume that

(C_8) *For any map* α *there exist* Im α *and* Coim α *(they need not be normal).*

(C_9) *An im ageof an ideal by a normal epimorphism is always a normal ideal, and a coimage of a normal factorobject by a normal monomorphism is always a normal factorobject.*

Obviously all axioms (C_1)—(C_8) are satisfied in the category of groups or rings. This category satisfies clearly the first condition of axiom (C_9). Also the second condition is fulfilled. Consider the coimage (ν, M) of a normal factorobject $(\varkappa, K)$ by a monomorphism $\alpha : B \to A$. Now the group (or ring) K is a factorgroup A/C and B is a subgroup of A. By the Second Isomorphism Theorem $B/B \cap C$ is isomorphic to a subgroup B'/C of A/C, and if B is a normal subgroup of A, then B'/C is also a normal one of A/C.

Proposition 3. *If the map* α *has a normal image and* $\text{Ker } \alpha = (o, \omega)$, *then* α *is a monomorphism.*

This statement is proved in KUROŠ—LIVŠITS—ŠULGEIFER—TSALENKO [8] § 10,6.

Let a_i $(i \in I)$ be a family of objects of the category $\mathscr{C}$.

An object g is said to be a *direct product* of the objects a_i $(i \in I)$, if there are maps $\pi_i : g \to a_i$ $(i \in I)$ (called the *projections* of g onto a_i) such that for each object $h \in \mathscr{C}$ and for any system of maps $\alpha_i : h \to \alpha_i$ $(i \in I)$, there is a unique map (called the *canonical map*) $\gamma : h \to g$ such that $\gamma\pi_i = \alpha_i$ for all $i \in I$. g will be denoted by $g = \prod_{i \in I} a_i(\pi_i)$.

An object f is said to be a *free product* of the objects a_i $(i \in I)$ if there are maps $\varrho_i : a_i \to f$ $(i \in I)$ (called the *injections* of a_i into f) such that for each object $h \in \mathscr{C}$ and for any system of maps $\alpha_i : a_i \to h$ $(i \in I)$ there is a unique map (called the *canonical* map) $\gamma : f \to h$ such that $\varrho_i \gamma = \alpha_i$ for all $i \in I$. f will be denoted by $f = \sum_{i \in I} a_i(\varrho_i)$.

Assume that

(C_{10}) *Every family of objects has a direct product and a free product.*

Axiom (C_4) implies that all the projections π_i (injections ϱ_i) of a direct product $g = \prod_{i \in I} a_i(\pi_i)$ $\left(\text{free product } f = \sum_{i \in I} a_i(\varrho_i)\right)$ are epimorphism (monomorphisms). Moreover, to every projection π_i there is a normal monomorphism $\sigma_i : a_i \to g$ such that $\sigma_i \pi_i = \varepsilon_{a_i}$ and $\sigma_i \pi_j = \omega\,(i \neq j)$ hold, and so (a_i, σ_i) is an ideal of g (dually: to every injection ϱ_i there is a normal epimorphism $\tau_i : f \to a_i$ satisfying $\varrho_i \tau_i = \varepsilon_{a_i}$, $\varrho_j \tau_i = \omega\,(i \neq j)$). These facts are proved in [8].

PROPOSITION 4. *Let $(k_i, \varkappa_i)$ $(i \in I)$ be a family of ideals of an object $a \in \mathscr{C}$, and let $\alpha_i : a \to a_i$ be epimorphisms with* $\operatorname{Ker} \alpha_i = (k_i, \varkappa_i)$ $(i \in I)$. *Consider the direct product $g = \prod_{i \in I} a_i(\pi_i)$, and the canonical map $\gamma : a \to g$ $(\gamma\pi_i = \alpha_i, i \in I)$. Then* $\operatorname{Ker} \gamma = \bigcap_{i \in I} (k_i, \varkappa_i)$ *is valid.*

For the proof we refer to SULIŃSKI [13], Proposition 2. 1. We omit to formulate the dual statement.

An object $a \in \mathscr{C}$ is said to be *subdirectly embedded* in the direct product $g = \prod_{i \in I} a_i(\pi_i)$ if there exists a monomorphism $\gamma : a \to g$ such that all maps $\alpha_i = \gamma\pi_i : a \to a_i$ $(i \in I)$ are normal epimorphisms (cf. [13]).

An object $a \in \mathscr{C}$ is said to be a *transfree image* of the free product $f = \sum_{i \in I} a_i(\varrho_i)$, if there exists an epimorphisms $\sigma : f \to a$ such that all maps $\beta_i = \varrho_i \sigma_i : a_i \to a$ $(i \in I)$ are normal monomorphisms.

Let us remark that according to this definition generally g can not be embedded subdirectly in itself, for the projections π_i need not be normal epimorphisms. The dual consideration holds for transfree images.

PROPOSITION 5. *An object $a \in \mathscr{C}$ can be subdirectly embedded in the direct product $g = \prod_{i \in I} a_i(\pi_i)$ if and only if there is a family of ideals $(k_i, \varkappa_i)$ $(i \in I)$ of a such that each of them is the kernel of the normal epimorphism $\alpha_i : a \to a_i$ $(i \in I)$ and $\bigcap_{i \in I} (k_i, \varkappa_i) = (o, \omega)$ holds.*

Dualizing we obtain

PROPOSITION 5*. *An object $a \in \mathscr{C}$ is a transfree image of the free product $f = \sum_{i \in I} a_i(\varrho_i)$ if and only if there is a family of normal factorobjects (λ_i, l_i) $(i \in I)$, of a such that each of them is the cokernel of the normal monomorphism $\beta_i : a_i \to a$ $(i \in I)$ and $\bigcap_{i \in I} (\lambda_i, l_i) = (\omega, o)$ holds.*

The statement of Proposition 5 is proved in SULIŃSKI [13] (Theorem 2, 3), assumed that every epimorphism is a normal one. Thus we give a modificated proof of this assertion.

Let a be subdirectly embedded in g by a monomorphism $\gamma : a \to g$. Now every $\alpha_i = \gamma\pi_i$ $(i \in I)$ is a normal epimorphism. If $(k_i, \varkappa_i) = \operatorname{Ker} \alpha_i$, then by Proposition 4 we get $\operatorname{Ker} \gamma = \bigcap_{i \in I} (k_i, \varkappa_i)$. Since γ is a monomorphism, therefore $\bigcap_{i \in I} (k_i; \varkappa_i) = (o, \omega)$ is valid.

Conversely, let $(k_i, \varkappa_i)$ be a family of ideals of a such that $(k_i, \varkappa_i) = \operatorname{Ker} \alpha_i$ where $\alpha_i : a \to a_i$ are normal epimorphisms and $\bigcap_{i \in I} (k_i, \varkappa_i) = (o, \omega)$ holds. Then there is a map $\gamma : a \to g$ such that $\gamma\pi_i = \alpha_i$ for $i \in I$. Applying Proposition 4, we get $\operatorname{Ker} \gamma = \bigcap_{i \in I} (k_i, \varkappa_i) = (o, \omega)$. By Proposition 2 we obtain $\bigcup_{i \in I} (\alpha_i, a_i) = \operatorname{Coker} \omega = (\varepsilon_a, a)$. Consider $\operatorname{Im} \gamma = (m, \nu)$ with the epimorphism μ (i.e. ν is a monomorphism and $\gamma = \mu\nu$). Since $\alpha_i = \mu\nu\pi_i$ and α_i $(i \in I)$ is an epimorphism, so $\nu\pi_i$ is also an epimorphism. Thus $(\mu, m) \geqq (\alpha_i, a_i)$ holds for every $i \in I$. Therefore we have $(\mu, m) \geqq \bigcup_{i \in I} (\alpha_i, a_i) = (\varepsilon_a, a)$. So (μ, m) is equivalent to (ε_a, a), and μ is a normal epimorphism. Therefore Proposition 3 implies that γ is a monomorphism, and Proposition 5 is proved.

An object $a \in \mathscr{C}$ is said to be *subdirectly irreducible*, if the intersection all of its non-zero ideals is a non-zero ideal.

An object $a \in \mathscr{C}$ is said to be *transfreely irreducible*, if the intersection all of its non-zero normal factorobjects is a non-zero normal factorobject.

According to Proposition 2, an object $a \in \mathscr{C}$ is transfreely irreducible if and only if the join of all its ideals $\neq (a, \varepsilon_a)$ differs from (a, ε_a).

Finally, let us mention that the categories of all rings and groups, respectively, and their dual categories fulfill axioms (C_1)—(C_{10}).

§ 3. Subdirect embeddings and transfree images

It is well-known that any universal algebra can be subdirectly embedded in a direct product of subdirectly irreducible universal algebras (G. BIRKHOFF [4]). In the proof there is making use of the fact that the lattice of congruence-relations of any universal algebra is compactly generated.

Let L be a complete lattice. An element $k \in L$ is said to be a *compact element*, if $k \leqq \bigcup_{i \in I} l_i$ implies $k \leqq \bigcup_{j \in J} l_j$ for some finite $J \subseteqq I$. The lattice is called *compactly generated*, if L is complete and every element of L is a union of (an infinite number of) compact elements.

In his paper [13] SULIŃSKI asked whether every object of a category satisfying somewhat stronger conditions than (C_1)—(C_{10}), can be subdirectly embedded in a direct product of subdirectly irreducible objects. Concerning this problem for a category $\mathscr{C}$ satisfying axioms (C_1)—(C_{10}) we present

THEOREM 1. *If the lattice L_a of all ideals of an object $a \in \mathscr{C}$ is compactly generated, then a can be subdirectly embedded in a direct product $g = \prod_{i \in I} a_i(\pi_i)$ by a monomorphism γ such a way that every $\gamma\pi_i = \alpha_i$ $(i \in I)$ is a normal epimorphism. A normal factorobject a_i of this decomposition is subdirectly irreducible if and only if the following condition holds:*

(I) *For any normal factorobject $(\chi, m) \neq (\varepsilon_{a_i}, a_i)$ of a_i (which is clearly a factorobject (χ_1, m) of a) there exists a normal factorobject (δ, d) of a such that $(\alpha_i, a_i) > > (\delta, d) \geqq (\chi_1, m)$.*

REMARK. Condition (I) seems to be complicated, but in the category of groups and rings, respectively, (I) is trivially fulfilled, for $\operatorname{Im} \alpha$ is always a normal image. However, its dual will be a rather natural condition in Theorem 1*. By Proposition 2 condition (I) means that for any ideal $(m', \chi') \neq (o, \omega)$ of a_i, there exists an ideal $(d', \delta') \geqq \operatorname{Ker} \alpha_i$ of a such that for its image (n', v') by α_i we have $(o, \omega) \neq \neq (n', v') \leqq (n', \chi')$.

PROOF. Let $(k, \varkappa) \neq (o, \omega)$ be a compact element of the lattice L_a of all ideals of an object $a \in \mathscr{C}$. Consider the set $S_k = \{(l_j, \lambda_j)\}_{j \in J}$ of all ideals of a for which $(k, \varkappa) \cap (l_j, \lambda_j) < (k, \varkappa)$. Let $(l_1, \lambda_1) < (l_2, \lambda_2) < \cdots < (l_n, \lambda_n) < \cdots$ an ascending chain of ideals from S_k, and denote $\bigcup_n (l_n, \lambda_n)$ by (l_0, λ_0). We will show that $(k, \varkappa) \cap \cap (l_0, \lambda_0) < (k, \varkappa)$. Otherwise it would be $(k, \varkappa) = (l_0, \lambda_0)$ and since $(k, \varkappa)$ is a compact element of L_a, so for an index n_0 a relation $(k, \varkappa) \leqq (l_{n_0}, \lambda_{n_0})$ would hold in contradiction to the assumption. Making use of Zorn's lemma we obtain the existence of a maximal element $(\bar{l}, \bar{\lambda})$ of S_k.

To any compact element $(k_i, \varkappa_i)$ $(i \in I)$ of L_a, consider a maximal element $(\bar{l}_i, \bar{\lambda}_i)$ of S_{k_i}. Now we shall show $\bigcap_{i \in I} (\bar{l}_i, \bar{\lambda}_i) = (o, \omega)$. On the contrary, suppose $(l', \lambda') = = \bigcap_{i \in I} (\bar{l}_i, \bar{\lambda}_i) \neq (o, \omega)$. Since L_a is compactly generated, so (l', λ') is a union $\bigcup_{t \in T} (k_t, \varkappa_t)$ of compact elements $(k_t, \varkappa_t) \neq (o, \omega)$. The maximal elements $(\bar{l}_t, \bar{\lambda}_t)$ of S_{k_t} belonging to $(k_t, \varkappa_t)$ occur in the intersection representation of (l', λ'). Thus we get $(k_t, \varkappa_t) \leqq$

$\leqq(l', \lambda')\leqq(\bar{l}_t, \bar{\lambda}_t)$ which implies $(k_t, \varkappa_t)\cap(\bar{l}_t, \bar{\lambda}_t)=(k_t, \varkappa_t)$ contradicting the choice of $(\bar{l}_t, \bar{\lambda}_t)$.

Now, consider $(\alpha_i, a_i)=\operatorname{Coker}\bar{\lambda}_i$. Since $(\bar{l}_i, \bar{\lambda}_i)$ is an ideal, therefore by Proposition 1 we have $\operatorname{Ker}\alpha_i=(\bar{l}_i, \bar{\lambda}_i)=\operatorname{Ker}\operatorname{Coker}\bar{\lambda}_i$, further α_i is a normal epimorphism for all $i\in I$. Hence by virtue of Proposition 5 a can be subdirectly embedded in a direct product $g=\prod_{i\in I} a_i(\pi_i)$ by a monomorphism $\gamma: a\to g$ such that every map $\gamma\pi_i=\alpha_i$ is a normal epimorphism.

Finally, assume (I) for an object a_i. Since $(\alpha_i, a_i)\succ(\delta, d)$ so by Proposition 2 for their kernels we obtain $(\bar{l}_i, \bar{\lambda}_i)=\operatorname{Ker}\alpha_i<\operatorname{Ker}\delta=(d', \delta')$. By the choice of $(\bar{l}_i, \bar{\lambda}_i)$ it follows $(k_i, \varkappa_i)\leqq(d', \delta')$ where $(k_i, \varkappa_i)$ denotes the compact element of L_a belonging to $(\bar{l}_i, \bar{\lambda}_i)$. Thus for the intersection (d_0', δ_0') of all ideals $(d', \delta')\succ(\bar{l}_i, \bar{\lambda}_i)$ we have $(k_i, \varkappa_i)\leqq(d_0', \delta_0')$. Again, by Proposition 2 for $\operatorname{Coker}\varkappa_i=(\varkappa_0, k_0)$ and $\operatorname{Coker}\delta_0'=(\delta_0, d_0)$ we get $(\varkappa_0, k_0)\geqq(\delta_0, d_0)$ and $(\alpha_i, a_i)\geqq(\delta_0, d_0)$. Hereby

$$(\varkappa_0, k_0)\cup(\alpha_i, a_i)\succ(\varkappa_0, k_0)\geqq(\delta_0, d_0)$$

and so $(\alpha_i, a_i)\succ(\delta_0, d_0)$ follows. On the other hand for any normal factorobject (χ, m) of a_i being a factorobject (χ_1, m) of a the relation

$$(\chi_1, m)\leqq(\delta, d)\leqq(\delta_0, d_0)<(\alpha_i, a_i)$$

is valid. Therefore the union of all normal factorobjects differs from (ε_{a_i}, a_i) and so a_i is indeed subdirectly irreducible. If a_i is subdirectly irreducible, then (I) is trivially fulfilled.

REMARK. From Theorem 1 one can easily obtain BIRKHOFF's well-known theorem mentioned at the beginning of this chapter.

Dualizing Theorem 1 we obtain

THEOREM 1*. *If the lattice L_a^* of all normal factorobjects of an object $a\in\mathscr{C}$ is compactly generated, then a is a transfree image of a free product $f=\sum_{i\in I} a_i(\varrho_i)$ by an epimorphism γ such a way that every map $\varrho_i\gamma=\alpha_i: a_i\to a$ $(i\in I)$ is a normal monomorphism. A factor a_i of this decomposition is transfreely irreducible if and only if the following condition holds:*

(I*) *For any ideal $(m, \chi)\neq(a_i, \varepsilon_{a_i})$ of a_i (which is clearly a subobject (m, χ_1) of a) there exists an ideal (d, δ) of a such that $(m, \chi_1)\leqq(d, \delta)<(a_i, \alpha_i)$.*

Condition (I*) means that *for the ideal (a_i, α_i) the object a_i has exactly one maximal ideal (d, δ') (and (d, δ) is an ideal of a $(\delta=\delta'\alpha_i)$)*.

To give an interpretation of Theorem 1* we introduce the following concept. An element k of a complete lattice L is called a *co-compact element*, if $k\geqq\bigcap_{i\in I} l_i$ implies $k\geqq\bigcap_{j\in J} l_j$ for some finite $J\subseteq I$. The lattice L is said to be co-compactly generated, if L is complete and every element of L is an intersection of co-compact elements. Hence by Proposition 2 the condition 'the lattice L_a^* of all normal factorobjects of a is compactly generated' should read '*the lattice L_a of all ideals of a is co-compactly generated*'. For comparison we mention that the lattice of all ideals of a ring need not be co-compactly generated and the same holds for groups too.

§ 4. Some applications

In what follows $\mathscr{C}_R$ will denote the category of rings. As it was mentioned before, $\mathscr{C}_R$ satisfies axioms (C_1)—(C_{10}) and condition (I), furter the lattice L_A of all ideals of a ring $A \in \mathscr{C}_R$ is compactly generated. Moreover, in $\mathscr{C}_R$ every map has a normal image. This means that Theorems 1 and 1* hold for $\mathscr{C}_R$. On the other hand, L_A has not to be co-compactly generated and condition (I*) is generally not fulfilled. However, there are some special but usual conditions which involve the validity of (I*) or that L_A is co-compactly generated. Thus Theorem 1* yields some theorems of unusual type for rings.

First of all we remark that instead of a free product of rings we speak about a *free sum* of rings. Further, if A_i $(i \in I)$ is a family of rings, then their free sum F is defined as the ring F consisting of all formal finite sums $\sum n_r \varphi_r$ where n_r is an integer and φ_r is a product of a finite number of elements from some A_i. For commutative rings, as it is well-known, free sum means the tensor product.

First, we show the existence of a ring the ideals of which do not form a co-compactly generated lattice.

EXAMPLE. Let A be a commutative principal ideal-ring with unity and without divisors of zero. (Such a ring is e.g. the ring of rational integers.) For any ideal $J \neq 0$ of A there exists an element $a \in A$ with $(a) = I$. According to RÉDEI [10], Satz 188 and 189 in A there exist g.c.d. and irreducible elements. Let $p \neq 1$ be an irreducible element with $(a, p) = 1$. Now $(a) \supset 0 = \bigcap_{k=1}^{\infty} (p^k)$ is valid, but obviously $(a) \not\supseteq (p^k)$ for any finite k. Thus $(a) = I$ is not a co-compact element of the lattice L_A of all ideals of A. Since I was chosen arbitrarily, so L_A is not co-compactly generated.

Now we give some sufficient conditions which guarantee that a lattice L should be co-compactly generated.

PROPOSITION 6. *Every element l of a lattice L is co-compact if and only if L satisfies the descending chain condition. In particular, the lattice L_A of a ring, abelian group and R-module A satisfying the minimum condition for ideals, subgroups and R-modules, respectively, is co-compactly generated.*

PROOF. Assume that each element of L is co-compact, and consider a descending chain $l_1 \supseteq l_2 \supseteq l_3 \supseteq \ldots$ in L. Since also $l_0 = \bigcap l_n$ is co-compact, so there exists an index n_0 with $l_0 = l_{n_0}$ and the chain is finite. The inverse statement is trivial.

PROPOSITION 7. *If the ring A is a discrete direct sum of rings A_i $(i \in I)$ with minimum condition for ideals and each A_i has either a left or a right unity, then the ideal lattice L_A of A is co-compactly generated.*

PROOF. At first we prove that any ideal B of A is a discrete direct sum $B = \sum_{i \in I} \oplus B_i$ where B_i is an ideal of A_i for all $i \in I$. Let b be an arbitrary element of B, then b is a finite sum $b = \sum_{i \in J} a_i$ of elements $a_i \in A_i$. Let e_i be , for instance, a left unity of A_i. Then we obtain $e_i b = a_i \in B \cap A_i$ and obviously $B_i = B \cap A_i$ is an ideal of A. Thus we have $B = \sum \oplus B_i$. Since A_i fulfils the minimum condition for ideals, so there is a finite number of ideals of A containing $K_i = B_i + \sum_{i \neq j \in I} \oplus A_j$. Therefore K_i is a co-compact element of the lattice L_A. Further we have $B = \sum_{i \in I} \oplus B_i = \bigcap_{i \in I} K_i$ which means that L_A is co-compactly generated.

We remark that a ring satisfying the condition of this proposition need not fulfil the minimum condition for ideals.

Let us list some types of rings which fulfil condition (I*).

1) *Every accessible subring of the ring is an ideal.* A subring S is called accessible in the ring A, if there exists a finite ascending chain of subrings $S = S_1 \subseteq \subseteq S_2 \subseteq \ldots \subseteq S_n = A$ where each S_i is an ideal of S_{i+1} $(i = 1, 2, \ldots, n-1)$. Since any ideal of an ideal is any accessible subring, so this condition involves condition (I*) trivially. (Cf. ANDERSON—DIVINSKY—SULIŃSKI [1]).

2) *Every subring of the ring is an ideal* (Cf. RÉDEI [11]).

3) *The ring is completely reducible*, i.e. it is a discrete direct sum of simple rings. In such a ring every ideal is a direct summand. Since any ideal of a direct summand is an ideal also in the ring, so it follows condition (I*) (Cf. JACOBSON [7] Chapter IV. 1).

4) *Every subring of the ring is a direct summand* (cf. F. SZÁSZ [14]).

5) *Every ideal of the ring is idempotent.* Let A be, namely, such a ring and K an ideal of the ideal I of A. By a varied form of a lemma of ANDRUNAKIEVIČ [2] (see also DIVINSKY [6], Lemma 61), we obtain

$$\bar{K} = \bar{K}^3 \subseteq I \cdot \bar{K} \cdot I = I(K + KA + AK + AKA)I \subseteq K \subseteq \bar{K},$$

where $\bar{K}$ denotes the ideal of A generated by the subring K. Thus K is an ideal in A too.

Important subcases of 5) are the following:

6) *The ring A is regular in the sense of* VON NEUMANN, i.e. for any $a \in A$ there exists an element $x \in A$ with $a = axa$. By definition, it is clear that the ideals of such a ring are idempotent.

7) *The ring A is weakly regular*, i.e. every right ideal of A is idempotent (Cf. BROWN—MCCOY [5]).

8) *The ring A is biregular*, i.e. every principal two-sided ideal of A can be generated by a central idempotent element (Cf. ARENS—KAPLANSKY [3], BROWN—MCCOY [5] and ANDRUNAKIEVIČ [2]). If I is an arbitrary ideal of the ring A and $a \in I$, then there is a central idempotent element $c \in A$ such that $a \in (a) = (c)$. Hence from $c \in (c)^2 = (a)^2 \subseteq I^2$ we obtain $a \in I^2$ for every $a \in I$. Thus the ideals of A are idempotent.

Theorem 1* yields immediately

THEOREM 2. *Let A be a ring of one of the types* 1)—8). *If the ring A is either a ring with minimum condition for ideals or a discrete direct sum of rings with left or right unity elements and the direct components satisfy the minimum condition for ideals, then there exist ideals A_i $(i \in I)$ of A such that*

(i) *A_i has exactly one maximal ideal which is an ideal also of A for each $i \in I$.*

(ii) *every A_i is of the same type as A,*

(iii) *A is a homomorphic image of a free sum $\sum'_{i \in I} B_i$, where $B_i \cong A_i$ holds for all $i \in I$.*

The statement that rings having one of the properties 1)—8) satisfy condition (ii), is almost trivial.

As another application, consider the category $\mathscr{C}_G$ of all groups. Now conditions (C_1)—(C_{10}) are satisfied. Condition (I^*) is fulfilled, for instance, if any normal subgroup of a normal subgroup is a normal subgroup of the group, or briefly: normality is a transitive relation among the subgroups of a group. (Cf. D. S. ROBINSON [12]). From Theorem 1* it follows immediately

THEOREM 3. *Let L_G denote the lattice of all normal subgroups of a group G. If L_G is co-compactly generated, and normality of subgroups of G is a transitive relation, then there exist normal subgroups G_i $(i\in I)$ of G such that*

(i) *each G_i has exactly one maximal normal subgroup,*

(ii) *G is a homomorphic image of a free product $\prod_{i\in I}^* F_i$ where $F_i\cong G_i$ holds for every $i\in I$.*

Let R be a ring, and consider the category $\mathscr{C}_R$ of all R-modules. $\mathscr{C}_R$ fulfils conditions (C_1)—(C_{10}) as well as (I) and (I^*). In $\mathscr{C}_R$ free sum means discrete direct sum. Hence from Theorem 1* we obtain

THEOREM 4. *If the lattice L_M of submodules of an R-modul M is co-compactly generated, then there exist submodules M_i $(i\in I)$ of M such that M is a homomorphic image of a discrete direct sum $\sum_{i\in I}\oplus N_i$ where N_i is isomorphic to M_i and N_i has exactly one maximal submodule for each $i\in I$.*

Since any abelian group can be regarded as a *module over the integers, so the analogous statement to that of Theorem 4 is valid for abelian groups too.*

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MTA MATEMATIKAI KUTATÓ INTÉZETE,
BUDAPEST, V., REÁLTANODA U. 13—15

References

[1] R. ANDERSON—N. DIVINSKY—A. SULIŃSKI, Lower radical properties for associative and alternative rings, *J. London Math. Soc.*, **41**, pp. 417—424.
[2] B. A. ANDRUNAKIEVIČ, Biregular rings (Russian), *Math. Sbornik*, **39** (1956), pp. 447—464.
[3] R. F. ARENS—J. KAPLANSKY, Topological representation of algebras, *Trans. Amer. Math. Soc.*, **63** (1948), pp. 457—481.
[4] G. BIRKHOFF, Subdirect unions in universal algebras, *Bull. Amer. Math. Soc.*, **50** (1944), pp. 764—768.
[5] B. BROWN—N. H. MCCOY, Some theorems on groups with application to ring theory, *Trans. Amer. Math. Soc.*, **66** (1950), pp. 302—311.
[6] N. DIVINSKY, *Rings and radicals* (London, 1965).
[7] N. JACOBSON, *Structure of rings* (Providence, 1956).
[8] A. G. KUROSCH—A. CH. LIWSCHITZ—E. G. SCHULGEIFER—M. S. ZALENKO, *Zur Theorie der Kategorien* (Berlin, 1963).
[9] B. MITCHELL, *Theory of categories* (New York—London, 1965).
[10] L. RÉDEI, *Algebra I* (Leipzig, 1959).

[11] L. Rédei, Vollidealringe im weiteren Sinn. I, *Acta Math. Acad. Sci. Hung.*, **3** (1952), pp. 243—268.
[12] D. S. Robinson, Groups in which normality is a transitive relation, *Proc. Cambridge Phil. Soc.*, **60** (1964), pp. 21—38.
[13] A. Suliński, The Brown-McCoy radical in categories, *Fund. Math.*, **59** (1966), pp. 23—41.
[14] F. Szász, On rings whose every subring is a direct summand of the ring (Russian), *Mat. Sbornik*, **40** (1956), pp. 269—272.