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Generalized biideals of rings. II

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In this note, which is a continuation of author's paper [12], we discuss the generalized biideals of regular rings [6] and of strongly regular rings [3].

By a ring we mean here an associative ring. The ring A is called regular (respectively strongly regular) if $a \in a\,A\,a$ (respectively $a \in a^2A$) for any element $a \in A$ holds. Furthermore, A is twosided subcommutative, if one has $a\,A = A\,a$ for any $a \in A$. (cf. D. Barbilian's book*.)

Following [12], by a generalized biideal B of a ring A we mean an additive subgroup of A^+ satisfying $BAB \subseteqq B$. Then for $n \geqq 3$ holds $B^n \subseteqq B$. In the example $A = \{a\}$ with $a^3 = p\,a = 0$ (where p is a prime number) the generalized biideal $B = I\,a$ (I is the ring of rational integers) is cyclic, and shows $B^2 \nsubseteqq B$.

If $B^2 \subseteqq B$ is satisfied for a generalized biideal B of a ring A, then B is a subring of A, and then B is called a biideal (see [8] and [11]). An important special case of biideal is the quasiideal Q (see [10]), when for the additive subgroup Q of A^+ also $QA \cap AQ \subseteqq Q$ holds. For semigroups the biideal is a special case of the (m, n)-ideals (see [7]).

The following four important assertations are well-known:

1) Any principal one-sided ideal of a regular ring is generated by an idempotent element e. — Namely $a = a\,x\,a$ implies $(a\,x)^2 = a\,x$, $(x\,a)^2 = x\,a$, $(a)_r = (a\,x)_r$ and $(a)_l = (x\,a)_l$ (see [6]).

2) Any strongly regular ring A has no nonzero nilpotent elements. — Namely $a = a^2x$ and $a^n = 0$ imply $a = 0$.

3) Any strongly regular ring A is also regular. — Namely $a = a^2x$ and $y = a - a\,x\,a$ imply $y^2 = 0$, consequently $y = 0$.

4) Every idempotent element e of any ring A without nonzero nilpotent elements lies in the center Z of A. — Namely for any $x \in A$ holds

$$(e\,x - e\,x\,e)^2 = 0 \quad \text{and} \quad (x\,e - e\,x\,e)^2 = 0,$$

which imply $e\,x = e\,x\,e = x\,e$ (see [1]).

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* Grupuri cu Operatori, București (România, 1960)

Now we discuss two preliminary statements:

Proposition 1. *Any generalized biideal B of a regular ring A coincides with the intersection of the right ideal $R = B + BA$ and of the left ideal $L = B + AB$, that is $B = D = (B + BA) \frown (B + AB)$, consequently $B^2 \subseteqq B$, therefore B is a subring.*

Proof. Evidently $a \in aAa$ implies also $B \subseteqq BAB$, which by $BAB \subseteqq B$ yields $B = BAB$. Furthermore by $B \subseteqq BA$, $B \subseteqq AB$, $A^2 = A$ and L. G. Kovács [4] we have

$$B = BAB = BA^2B = BA \cdot AB = (B + BA) \cdot (B + AB)$$
$$= (B + BA) \frown (B + AB) = B,$$

and therefore $B^2 \subseteqq B$.

Example. Let A be the total matrix ring of type 2×2 over a division ring, $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
Then $B = e_{11} A e_{22} = e_{11} A \frown A e_{22}$ is a generalized biideal of the regular ring A satisfying $0 = B^2 \neq B$.

Proposition 2. *Any generalized biideal B of a strongly regular ring A is a twosided ideal of A.*

Proof. By Proposition 1 and well-known assertation 3) holds

$$B = (B + BA) \frown (B + AB),$$

and therefore it is sufficient to verify that every one-sided ideal of A is twosided. But *e. g.* the right ideal R is the union of all principal right ideals $(a)_r$ with $a \in R$, where by the well-known assertations 1), 2) and 4) every $(a)_r$ is generated by a central idempotent element e_a, which implies that $(a)_r$ and R are twosided ideals. The proof is similar for left ideals L.

Theorem 1. *The following twelve conditions for a ring A are mutually equivalent:*

(I) *A is regular*

(II) $R \frown L = R \cdot L$ *for any right ideal R and for any left ideal L of A*

(III) $(a)_r \frown (b)_l = (a)_r \cdot (b)_l$, *for any $a, b \in A$*

(IV) $(a)_r \frown (a)_l = (a)_r \cdot (a)_l$, *for any $a \in A$*

(V) $(a)_q = (a)_r \cdot (a)_l$, *for any $a \in A$, denoting $(a)_q$ the principal quasiideal generated by $a \in A$*

(VI) $\overline{(a)_{(1,1)}} = (a)_r \cdot (a)_l$, *for any $a \in A$, denoting $\overline{(a)_{(1,1)}}$, the principal biideal, generated by $a \in A$*

(VII) $(a)_{(1,1)} = (a)_r \cdot$
zed biideal, ge

(VIII) $\overline{(a)_{(1,1)}} = aAa$

(IX) $(a)_{(1,1)} = aAa$

(X) $QAQ = Q$ *for*

(XI) $\bar{B} \cdot A \cdot \bar{B} = \bar{B}$

(XII) $BAB = B$ *for*

Proof is an imme
the joint paper [8]
author's paper [12].

Theorem 2. *The fo*
equivalent:

(I) *A is strongly*

(II) *A is a twosid*
ideals are two

(III) *A is a subcon*

(IV) $B^2 = B$ *for a*

(V) $(\bar{B})^2 = \bar{B}$ *for*

(VI) $Q^2 = Q$ *for an*

(VII) $RL = R \frown L$

(VIII) $L \frown R = LR$

(IX) $L_1 \frown L_2 = L$
any right idea

(X) $L \frown T = LT$
ideal R and

(XI) *A is regular a*

(XII) *A is a regula*

(XIII) $L_1 \frown L_2 = L_1$

(XIV) $R_1 \frown R_2 = R$

(XV) $L \frown T = LT$

(XVI) $R \frown T = TR$
of A

(XVII) $Q_1 \frown Q_2 = Q_1$

(VII) $(a)_{(1,1)} = (a)_r \cdot (a)_l$ *for any* $a \in A$, *denoting* $(a)_{(1,1)}$ *the principal generalized biideal, generated by* $a \in A$

(VIII) $\overline{(a)}_{(1,1)} = aAa$ *for any* $a \in A$

(IX) $(a)_{(1,1)} = aAa$ *for any* $a \in A$

(X) $QAQ = Q$ *for any quasiideal* Q *of* A

(XI) $\bar{B} \cdot A \cdot \bar{B} = \bar{B}$ *for any biideal* $\bar{B}$ *of* A

(XII) $BAB = B$ *for any generalized biideal* B *of* A.

Proof is an immediate consequence of Proposition 1, of Theorem 2 of the joint paper [8] S. Lajos' and author's, furthermore of results of author's paper [12].

Theorem 2. *The following twenty-one conditions for a ring* A *are mutually equivalent*:

(I) A *is strongly regular*

(II) A *is a twosided regular ring (i. e. regular ring in which all one-sided ideals are twosided ideals)*

(III) A *is a subcommutative regular ring*

(IV) $B^2 = B$ *for any generalized biideal* B *of* A

(V) $(\bar{B})^2 = \bar{B}$ *for any biideal* $\bar{B}$ *of* A

(VI) $Q^2 = Q$ *for any quasiideal* Q *of* A

(VII) $RL = R \cap L \subseteq LR$ *for any left ideal* L *and any right ideal* R *of* A

(VIII) $L \cap R = LR$ *for any left ideal* L *and any right ideal* R *of* A

(IX) $L_1 \cap L_2 = L_1 L_2$ *and* $R_1 \cap R_2 = R_1 R_2$ *for any left ideals* L_i *and any right ideals* R_i *of* A

(X) $L \cap T = LT$ *and* $R \cap T = TR$ *for any left ideal* L *for any right ideal* R *and any twosided ideal* T *of* A

(XI) A *is regular and it is a subdirect sum of division rings*

(XII) A *is a regular ring without nonzero nilpotent elements*

(XIII) $L_1 \cap L_2 = L_1 L_2$ *for any left ideals* L_i *of* A

(XIV) $R_1 \cap R_2 = R_1 R_2$ *for any right ideals* R_i *of* A

(XV) $L \cap T = LT$ *for any left ideal* L *and for any twosided ideal* T *of* A

(XVI) $R \cap T = TR$ *for any right ideal* R *and for any twosided ideal* T *of* A

(XVII) $Q_1 \cap Q_2 = Q_1 \cdot Q_2$ *for any quasiideals* Q_i *of* A

(XVIII) $\bar{B}_1 \cap \bar{B}_2 = \bar{B}_1 \cdot \bar{B}_2$ *for any biideals* $\bar{B}_i$ *of* A

(XIX) $B_1 \cap B_2 = B_1 \cdot B_2$ *for any generalized biideals* B_i *of* A

(XX) *The multiplicative semigroup of* A *is a semilattice of groups* G_i

(XXI) *Every principal one-sided ideal is generated by a central idempotent element.*

Remark. Combinatorically could be formulated yet other equivalent conditions for A, building intersections and products of (principal) generalized biideals, biideals, quasiideals, right ideals, left ideals and twosided ideals.

Proof of Theorem 2 follows from Proposition 2, from Theorem 3 of the joint paper [8] and from [9] S. Lajos' and author's and from author's paper [12].

References

[1] A. Forsythe and N. H. McCoy, On the commutativity of certain rings, Bull. Amer. Math. Soc. **52**, 523–526 (1946).
[2] N. Jacobson, Structure of rings, Providence (1964).
[3] T. Kando, Strong regularity in arbitrary rings, Nagoya Math. J. **4**, 51–53 (1952).
[4] L. G. Kovács, A note on regular rings, Publ. Math. Debrecen **4**, 465–468 (1956).
[5] J. Luh, A characterization of regular rings; Proc. Japan Acad. **39**, 741–742 (1963).
[6] J. von Neumann, On regular rings, Proc. nat. Acad. Sci. USA **22**, 707–713 (1936).
[7] S. Lajos, Generalized ideals in semigroups, Acta Sci. Math. Szeged **22**, 217–222 (1961).
[8] S. Lajos and F. Szász, Bi-ideals in associative rings, Acta Sci. Math. Szeged (to appear).
[9] —, Characterizations of strongly regular rings, II, Proc. Japan Acad. **46**;3, 287–289 (1970).
[10] O. Steinfeld, Über die Quasiideale von Ringen, Acta Sci. Math. Szeged **17**, 170–180 (1956).
[11] F. Szász, On minimal biideals of rings, Acta Sci. Math. Szeged (to appear).
[12] —, Generalized biideals of rings, I, Diese Nachr. **47**, 355–360 (1970).