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Generalized biideals of rings. I

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Throughout this paper by a ring A we mean an associative ring. For the arbitrary subsets C and D of a ring A, by the product CD we mean the additive subgroup of A^+, generated by all $c.\ d$ with $c \in C$ and $d \in D$. Thus CD is always an additive subgroup, but this product fails generally to be a subring A even for the subrings C and D of A.

By a generalized biideal B of the ring A we mean an *additive subgroup* B of A^+ satisfying $BAB \subseteqq B$. For a generalized biideal B of A holds always $B^n \subseteqq B$, if $n \geqq 3$.

Following the terminology of the joint paper [10] of S. LAJOS and of author, by a biideal B of a ring A we mean a *subring* of A satisfying

$$BAB \subseteqq B.$$

For a biideal B of A also the subring generated by the additive subgroup BAB lies in B. The concept of the biideal for semigroups was introduced in the joint paper [3] of R. A. GOOD and D. R. HUGHES. The biideal is obviously a special case of the (m, n)-ideals, dis cussed by S. LAJOS [7] for semigroups.

An important particular case of biideals is the quasiideal of O. STEINFELD [13]. This is an additive subgroup Q of A^+, satisfying

$$QA \cap AQ \subseteqq Q.$$

By J. CALAIS [1] is given a biideal, which is not a quasiideal. Furthermore in the ring $A = \{a\}$ with defining relations $a^3 = p\,a = 0$ (p is a prime number) the additive group, generated by a, is a generalized biideal of A, but this fails to be a biideal of A. A collection of general statements on biideals of rings is given by S. LAJOS and author [10]. Furthermore minimal biideals of rings were discussed by author [16].

Now we wish to publish some results on the generalized biideals of rings.

Firstly we discuss some general properties of generalized biideals of rings.

Proposition 1. *The intersection of an arbitrary set of generalized biideals $B_\lambda (\lambda \in \wedge)$ of a ring A is again a generalized biideal of A.*

Proof. Suppose that $B = \bigcap_{\lambda \in \wedge} B_\lambda$. Then B is an additive subgroup of A^+. Then $B_\lambda A B_\lambda \subseteq B_\lambda$ and $B \subseteq B_\lambda$ imply

$$BAB \subseteq B_\lambda A B_\lambda \subseteq B_\lambda$$

and therefore $BAB \subseteq B$.

Proposition 2. *The intersection of a generalized biideal B and of an additive subgroup S of A^+ with $S^3 \subseteq S$ is a generalized biideal in the group S.*

Proof. Evidently $C = B \cap S$ is an additive subgroup of A^+. By $S^3 \subseteq S$ and $C \subseteq S$ we have $CSC \subseteq S^3 \subseteq S$. Furthermore

$$CSC \subseteq BCB \subseteq BAB \subseteq B$$

consequently $CSC \subseteq B \cap S = C$.

Proposition 3. *For an arbitrary subset T and for a generalized biideal B of a ring A the products BT and TB are generalized biideals of A.*

Proof. By the definition of the product BT and TB are subgroups of A^+. By $TA \subseteq A$ and $BAB \subseteq B$ holds $B(TA)\, B \subseteq BAB \subseteq B$, and by the monotony property of the product also $(BT)A(BT) \subseteq BT$. Therefore BT is a generalized biideal of A. The proof is for TB similar to the above.

Proposition 4. *Let B be a generalized biideal of a ring A and C be a generalized biideal in the group B such that $C^2 = C$. Then C is a generalized biideal of the ring A.*

Proof. C is an additive subgroup of A^+. Furthermore $BAB \subseteq B$, $C^2 = C$, $CBC \subseteq C$ imply

$$CAC = C^2 AC^2 \subseteq C\,(BAB)\,C \subseteq CBC \subseteq C.$$

Proposition 5. *An arbitrary ring A contains no nontrivial generalized biideals if and only if A either is a zero ring of prime order, or it is a division ring.*

Proof. If A contains no nontrivial generalized biideals, then A contains no nontrivial right ideals. Consequently A satisfies the minimum condition on right ideals. If A is semisimple, then it is a division ring by the WEDDERBURN-ARTIN structure theorem. If A is a radical ring, then it is a zero ring of prime order. Conversely, if A is a zero ring of prime order, then A has no nontrivial generalized biideals. Furthermore if A is a division ring, then for $B \neq 0$ we have $BA = A$ and $AB = A$, consequently $BAB = A \subseteq B$ and $B = A$, therefore also the division ring A has no nontrivial generalized biideals.

Remark 1. In the author's paper [14] can be found a short elementary proof of the fact that rings without nontrivial right ideals are division rings or zero rings of prime order.

Proposition 6. *The generalized biideal of a ring A, generated by a nonempty subset T, has the form $T_{(1,1)} = I \cdot T + TAT$ where I is the ring of rational integers, and $I \cdot T$ is the additive subgroup of A^+ generated by T. Furthermore $T_{(1,1)}$ coincides with the intersection of all generalized biideals of A, containing T.*

Proof is trivial.

Remark 2. A generalized biideal generated by an additive subgroup T has the form $T_{(1,1)} = T + TAT$. Furthermore a generalized biideal generated by a single idempotent element $e (= e^2)$ is of form $(e)_{(1,1)} = eAe$.

Proposition 7. *For any ring A denote $\bar{A}$ the set of all additive subgroups of A^+ and $\bar{\bar{A}}$ the set of all generalized biideals of A. Then $\bar{A}$ and $\bar{\bar{A}}$ are semigroups under the multiplication (defined in the introduction of this paper). furthermore $\bar{\bar{A}}$ is a twosided ideal of $\bar{A}$.*

Proof follows from Proposition 3.

Remark 3. The multiplicative semigroup of all nonempty subsets of a (multiplicative) semigroup S was earlier discussed by S. LAJOS [8]. By J. CALAIS [1] is shown an example of a semigroup having two quasiideals. whose product fails to be a quasiideal. But for the particular case of regular semigroups and regular rings the product of quasiideals is again a quasiideal. (See S. LAJOS [9])

Remark 4. In spite of Theorem 1 of the joint paper [10] of S. LAJOS and author, which states that any biideal is a left ideal of a right ideal of the ring, and also a right ideal of a left ideal of the ring, a generalized biideal generally fails to have such a representation. An example for this is the ring $A = \{a\}$ with $a^3 = p\,a = 0$, where $I\,a$ is a generalized biideal of A, but $I\,a$ fails to be an ideal of the ideal, generated by a since

$$(a).\ I\,a = (I\,a + aA)\,I\,a = I\,a^2 \subsetneqq I\,a.$$

In this ring A the biideal $B = I\,a$ is evidently minimal, and satisfies $B^3 = 0$ with $B^2 \neq 0$ (cf. Theorem 2 of this paper).

Proposition 8. *Let $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots$ be a transfinite ascending chain of generalized biideals of a ring A, and B be the union of this chain. Then B is again a generalized biideal of A.*

Proof. Obviously B is an additive subgroup of A^+. We consider an arbitrary element $\Sigma\, b'\, a\, b''$ of the product BAB. Since the sum Σ has only finite number of summands $b'a\,b''$, there is a generalized biideal B_α of A

such that B_x contains any b' and any b'' of the sum Σ. Then

$$\Sigma b' a b'' \in B_x A B_x \subseteq B_x \subseteq B$$

consequently also $BAB \subseteq B$.

Problem 1. Investigate the class of all rings A, for which $B_1 + B_2$ is a generalized biideal of A for any generalized biideal B_1 and B_2 of A!

Problem 2. Investigate the intersection of all maximal generalized biideals of a ring A! (cf. F. Szász [18]).

Problem 3. Investigate the intersection of all maximal biideals of a semigroup S having zero element! (cf. H. J. Hoehnke [4, 5], H. Seidel [12] and F. Szász [17]).

Now we discuss the minimal generalized biideals of rings.

Theorem 1. *If the generalized biideal B of a ring A is a division ring, then B is a minimal generalized biideal of A.*

Proof. If C is an arbitrary generalized biideal of A, satisfying $C \subseteq B$, then by $CAC \subseteq C$ holds also $CBC \subseteq C$, consequently C is a generalized biideal of the division ring B. But by Proposition 5 we have either $C = 0$ or $C = B$, that is the minimality of B in A.

Theorem 2. *Any minimal generalized biideal B of a ring A either is nilpotent satisfying $B^3 = 0$ or it is a division ring.*

Proof. By definition $B^3 \subseteq BAB \subseteq B$, and by Proposition 3 also B^3 is a generalized biideal of A. The minimality of B implies then either $B^3 = 0$ or $B^3 = B$. Assume now $B^3 = B$. The there exist elements $a, b \in B$ with $a B b \neq 0$, and by Proposition 3 and $a B b \subseteq B$ also with $a B b = B$. Consequently there exist elements $b_1,\ b_2 \in B$, satisfying $a = a b_1 b$ and $b = a b_2 b$. Now $B^3 = a B \cdot b a \cdot B b a B b = B \neq 0$ implies $b a \neq 0$, whence

$$0 \neq b a = a b_2 b \cdot a b_1 b = b a \cdot b_1 b = a b_2 \cdot b a$$
$$= b a \cdot b_1 (b b_1) b = a (b_2 a) b_2 \cdot b a \in b a B \cap B b a = D.$$

But being $b a B \subseteq B^3 \subseteq B$, $B b a \subseteq B^3 \subseteq B$, $D \neq 0$ the minimality of B yields at once $D = B$, that is $B = B b a \cap b a B$. Consequently there exist four elements $b_3, b_4, b_5, b_6 \in B$ satisfying

$$a = b_3 \cdot b a = b a \cdot b_4 \neq 0 \quad \text{and} \quad b = b_5 b a = b a b_6 \neq 0.$$

Discussing now the element $e = b_3 b a b_6$ one sees by $a \neq 0$ and $b \neq 0$ that

$$e = b_3 b a b_6 = a b_6 = b_3 b \neq 0 \quad \text{and} \quad e^2 = (b_3 b)(a b_6) = e.$$

Being $e \in B$, $e B e \subseteq B$, by Proposition 3 and minimality of B holds $B = e B e$. This implies $(e B)^2 = e B$ and $B^2 = e B e \cdot e B e = e B e = B$. Therefore, being $e b e$ an arbitrary nonzero element of B, we have $B e b e = B$ by $e.\ e b e \neq 0$ and the Proposition 3. Consequently there exists an element $e b' e \in B$ satisfying $e b' e \cdot e b e = e$, which means that $e B e = B$ is a division ring, indeed.

Theorem 3. *If a minimal generalized biideal B of a ring A contains an element b such that b is neither a left divisor of zero, nor a right divisor of zero in A, then A must have a twosided unity element.*

Proof. One has evidently $b^3 \neq 0$ and $b^3 \in bAb \subseteq BAB \subseteq B$. Proposition 3 and minimality of B imply $bAb = B$. Hence there exists an element $a \in A$ with $b = b a b$, and therefore $x b = x b a b$ and $b y = b a b y$ for any $x, y \in A$. By the twosided cancelling rule we obtain $x = x b a$ and $y = a b y$ for any $x, y \in A$. Consequently $e = a b = a b \cdot b a = b a$ is the twosided unity of A, indeed.

Theorem 4. *If R is a minimal right ideal and L a minimal left ideal of a ring A, then either is $R \cdot L = 0$, or RL is a minimal generalized biideal of A.*

Proof. Obviously RL is an additive subgroup of A^+. Assume $RL \neq 0$, and let be $B = R \cdot L$. If C is any generalized biideal of A with $0 \neq C \subseteq B$, then by $C \subseteq RL \subseteq R$ holds also $CA \subseteq R$, which by the minimality of R implies either $CA = 0$ or $CA = R$. If $CA = 0$, then C is also a right ideal of A, that is $C = R$ and $R \cdot L = 0$, which is impossible. Therefore $CA = R$ and similarly $AC = L$, which imply $B = CA \cdot AC \subseteq CAC \subseteq C \subseteq B$, that is $C = B$.

For some particular cases holds also the converse statement to Theorem 4, as follows:

Theorem 5. *Any minimal generalized biideal B of a ring A without nonzero nilpotent ideals can be represented in the form $R.\ L$ with minimal right ideal R and minimal left ideal L of A.*

Proof. By $BAB \subseteq B$, Proposition 3 and minimality of B holds either $BAB = 0$ or $BAB = B$. If $BAB = 0$, then $(BA)^2 = 0$, and by our assumption $BA = 0$, which implies $B = 0$ for the right ideal B with $B^2 = 0$. Therefore $B = BAB$ which yields also $B = BABAB$, consequently by $B = BABAB \subseteq BA^2B = BA \cdot AB \subseteq BAB \subseteq B$, also $B = BA \cdot AB$. We prove that $R = BA$ is a minimal right ideal and $L = AB$ is a minimal left ideal of A. If R' is a right ideal of A with $0 \neq R' \subseteq BA$, then $B' = R'AB$ is by Proposition 3 a generalized biideal of A, satisfying

$$B' = R'AB \subseteq BAAB \subseteq B.$$

By the minimality of B holds either $B' = 0$ or $B' = B$. But $B' = 0$ implies $R'AR' = (R'A)^2 = R'A = 0$, and $R' = 0$, being then R' a nilpotent right ideal of A. Consequently holds $B' = B$, which completes the proof for $R = BA$.

The proof is for the minimality of $L = AB$ similar.

Theorem 6. *Any ring A without nonzero nilpotent ideals and with minimum condition on principal right ideals is a sum of minimal generalized biideals.*

Proof. By author [15] we have $A = \sum_{\alpha} R_{\alpha} = \sum_{\beta} L_{\beta}$ and $A^2 = A$, where R_{α} and L_{β} are minimal right ideals and minimal right ideals, respectively. Now $A = \sum_{\alpha,\beta} R_{\alpha}L_{\beta}$ and application of Theorem 4 completes the proof.

References

[1] J. Calais, Demi-groupes quasi-inversifs, C. r. Acad. Sci. Paris **252**, 2357–2359 (1961)

[2] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, I, II. Providence (1961, 1967).

[3] R. A. Good and D. R. Hughes, Associated groups for a semigroup, Bull. Amer. Math. Soc. **58**, 624–625 (1952).

[4] H. J. Hoehnke, Zur Strukturtheorie der Halbgruppe, Diese Nachr. **26**, 1–13 (1963).

[5] –, Structure of semigroups, Canadian J. Math. **18**, 449–491 (1966).

[6] N. Jacobson, Structure of rings, Providence (1964).

[7] S. Lajos, Generalized ideals in semigroups, Acta Sci. Math. Szeged **22**, 217–222 (1961).

[8] –, On ideal theory of semigroups (in Hungarian), Magyar Tudományos Akadémia Mat. Fiz. Tud. Oszt. Közleményei **11**, 57–66 (1961).

[9] –, On quasiideals of regular rings, Proc. Japan Acad. **38**, 210–211 (1962).

[10] S. Lajos and F. Szász, Bi-ideals in associative rings, Acta Sci. Math. Szeged (to appear).

[11] –, Some characterizations of twosided regular rings, Acta Sci. Math. Szeged (to appear).

[12] H. Seidel, Über das Radikal einer Halbgruppe, Diese Nachr. **29**, 255–269 (1963).

[13] O. Steinfeld, Über die Quasiideale von Ringen, Acta Sci. Math. Szeged **17**, 170–180 (1956).

[14] F. Szász, Note on rings in which every proper left ideal is cyclic, Fund. Math. **44**, 330–332 (1957).

[15] –, Über Ringe mit Minimalbedingung für Hauptrechtsideale, II, Acta Math. Acad. Sci. Hungar. **12**, 417–439 (1961).

[16] –, On minimal biideals of rings, Acta Sci. Math. Szeged (to appear).

[17] –, Radikalbegriffe für Halbgruppen, die dem Jacobsonschen ringtheoretischen Radikal ähnlich sind, Diese Nachr. **34**, 157–161 (1967).

[18] –, On Frattini one-sided ideals and subgroups, Diese Nachr. (to appear).