On Frattini one-sided ideals and subgroups

Dedicated to the memory of Iván Seres (1907—1966)

By Ferenc A. Szász (Budapest)

The purpose of this paper is threefold.

Firstly it is given a greater (but not complete) survey on Frattini substructures of groups, semigroups and rings (see "References").

Secondly, we disprove some (proved) conjectures of Homer Bechtell [3] on Frattini right ideals, and we correct also a partially invalid reference by Timon Anderson [1] on Bechtell's ringtheoretical results.

Finally a solution of a problem, raised verbally by Otto Steinfeld, solved earlier by the present author [40] (in Hungarian), is here communicated in connection with the Frattini right ideal Φ_r of a ring A.

All rings considered in this paper are associative rings (see N. Jacobson [17]). The notions, used in this paper, can be found in the books D. Barbilian [2], N. Divinsky [5], M. Hall [11], E. Hille [12], N. Jacobson [17], A. G. Kurosh [19], J. Lambek [23], N. H. McCoy [24], L. Rédei [28], and H. Zassenhaus [43].

As well known, for an arbitrary group G, the Frattini subgroup $\Phi(G)$ is the intersection of all maximal subgroups of G, or $\Phi(G) = G$ for groups without maximal proper subgroups. (Cf. A. G. Kurosh [19].) Originally this was defined by G. Frattini [9] only for finite groups G, for which always $\Phi(G) \neq G$ holds. For the theory of the Frattini subgroup $\Phi(G)$ we refer yet to V. Dlab [6], [7], V. Dlab-V. Korinek [8], W. Gaschütz [10], G. A. Miller [25], B. H. Neumann [26], and in the case of special groups e. g. to Gh. Pic [27] and H. Wielandt [42]. It may be remarked that by B. H. Neumann [26] the subgroup $\Phi(G)$ coincides with the subset of all elements x of G, such that x can be abandoned from every generating system of G.

For semigroups S the Frattini subsemigroup $\Phi(S)$, being the intersection of all maximal subsemigroups and $\Phi(S) = S$, when a maximal subsemigroup fails to be contained in S has been discussed by S. Lajos [20] and H. J. Weinert [41]. Obviously $\Phi(S)$ can be also empty.

For semigroups S having zero F. Szász [39] had investigated the intersection of all maximal right ideals of S, and a part of the problems raised in [39] has been solved by H. Seidel [31]. It can be observed that in the particular case, when the semigroup S with zero element is the multiplicative semigroup of a ring A, and when we consider only those one-sided ideals of S, which are one-sided ideals also of the ring A, then all six radicals of S, discussed by F. Szász [39], coincide with the Jacobson radical of A. For semigroups S having zero element earlier H. J. Hoehnke has introduced and discussed in his fundamental papers [13], [14], [15] a radical, which is in a strong connection with the Frattini right ideal of S. H. J. Hoehnke [13], [14], [15] used principally congruence relations of S, instead of one-sided ideals of the semigroups S with zero. A list of a sequence of further earlier important and interesting papers of H. J. Hoehnke on maximal one-sided congruences of a semigroup with zero is given in the bibliography of the paper F. Szász [39].

Generalizing more characterizations, mentioned above, for a property of the Frattini subelement of an element of an ordered (non-associative) groupoid, we refer to F. Szász [36], Sätze 1.2 (see yet „Bemerkung" and „Beispiel" of [36]).

A survey on maximal right ideals of rings is given by F. Szász [40].

For a ring A let Φ, Φ_r and Φ_l, respectively, denote the intersection of the maximal twosided, right and left ideals of A, or A itself, when A is without maximal ideals of the corresponding type. As well known, E. Hille [12] (Theorem 22.15.3, page 486) proved $AI \subseteqq \Phi_r$ for the Jacobson radical I of a ring A. Sharpening this Hille's result, A. Kertész [18] has shown, that I is the set of all elements x of the ring A satisfying $Ax \subseteqq \Phi_r$. For this see yet F. Szász [38]. By the result for

$$I = \{x; \ x \in A, \ Ax \subseteqq \Phi_r\},$$

mentioned above, and by B. H. Neumann [26], a characterization of the Jacobson radical I of a ring A can be formulated by the terms of non-generators.

In the particular case of a semisimple Artinian ring A, obviously all ideals Φ, Φ_r and Φ_l coincide with zero. In connection with this important elementary fact we yet mention a good summarizing of results, written by O. Steinfeld and R. Wiegandt, [32], on the different well known generalizations of the Wedderburn-Artin structure theorem for rings, semirings and semigroups.

Another particular case is when the ring is twosided regular. For these rings the twosided ideals Φ, Φ_r and Φ_l all equal with (0), and a characterization of this class of rings by a sequence of mutually equivalent conditions

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can be found in the joint paper of S. LAJOS and F. SZÁSZ [21]. Any regular ring is a common generalization of the semisimple ARTINIAN rings and of twosided regular rings, and a characterization of regular rings by means of more pairwise equivalent conditions is given by another joint paper of S. LAJOS and F. SZÁSZ [22]. For regular rings we have always $\Phi_r = \Phi_l = 0$, but by „Beispiel" of F. SZÁSZ [36] also $|\Phi| = \aleph_z$ can occur for every $\aleph_z$.

The right ideal R_1 of a ring A is called small, if $R_1 + R_2$ is a proper right ideal for every proper right ideal R_2 of A. Then one has evidently $R_1 \subseteqq \Phi_r$ (cf. LAMBEK [23]).

A collection of statements for the sets of nongenerators of an arbitrary ring A, that is for Φ, Φ_r and Φ_l, defined above, has been formulated in the interesting note of HOMER BECHTELL [3], referred in Mathematical Reviews by TIMON ANDERSON [1]. As already T. ANDERSON remarks in his reference, Theorem 4 of BECHTELL's note fails to be correct and he yet remarks that "the fact that the theorem is not correct, does not really affect the main results of the paper" (of H. BECHTELL).

KAPLANSKY's famous example, in a special case, disproves namely Theorem 4 from [3], being this particular example now, a dense ring A of linear transformations of the form of infinite matrices

$$\begin{bmatrix} U & 0 & 0 & 0 \cdots \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \\ \vdots & & & \end{bmatrix}$$

Here U is a finite matrix over a ring D, furthermore $d \in D$ is in the diagonal, zeros elsewhere, denoting D such a subring of a field, that D is a radical ring in the sense of JACOBSON. This D can be taken e. g. as the ring of all rationals with even numerator and odd denominator. By a routine calculation we assert that the center Z of A is diag $(d, d, \ldots)$, being isomorphic to D. Moreover, A is primitive in the sense of JACOBSON, but not semisimple in the sense of BROWN and MCCOY [4]. As T. ANDERSON [1] remarks, this ring A was mentioned also by N. JACOBSON [17], page 36, example (3).

For the canonical ring extension B of A with unity element we have obviously $B^2 = B$, but by ANDERSON [1] the relation

$$N(B) \cap Z(B) \not\subseteqq \Phi_r(B) \cap \Phi(B)$$

holds, denoting $N(B)$ and $Z(B)$, respectively, the BROWN-MCCOY radical of B and the center of B. Therefore by [1] Theorem 4 of [3] is incorrect.

Although it can be true that the invalid Theorem 4 from [3] "does not really affect the main results" of the note [3], but unfortunately, there

is a part of further statements from [3], which are incorrect and there is another part of assertations from [3] which are well- known or almost trivial. Now we discuss these:

Example. Let A be the ring, generated by the matrices of the form

$$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

over a finite prime field of order p, being p a prime number. Then A is a noncommutative ring of p^2 elementss, with the multiplication:

	x	y
x	0	x
y	0	y

By a routine calculation $(x)_r + (y)_r = A$, $(x)_r \cap (y)_r = 0$ holds, denoting $(z)_r$ the principal right ideal of A generated by the element z. Moreover we obtain evidently $A^2 = A$, having A a right unity y, and $(y)_r$ is a maximal right ideal, which cannot be modular in A. For this finite ring A, which is therefore also ARTINIAN, the KERTÉSZ radical (cf. F. SZÁSZ [33]) being (0). is properly smaller than the JACOBSON radical I, which coincides with $(x)_r$ and $(x)_r$ modular in A.

It may be remarked that this example was discussed also in the proof of the theorem of F. SZÁSZ [33].

We now observe:

Remark 1. *The Theorem 5 from [3] is false,* because this asserts that $\Phi_r = \Phi_l = I$ would be valid for all rings A satisfying $A^2 = A$. But the example, discussed above, disproves this conjecture of [3]. The mistake of the proof of this conjecture is in the place, where [3] points out that "$(I^*)^2 \Theta = 0$ implies that $R^* M^*$ is annihilated by R^*".

Remark 2. *But also Theorem 8 from [3] fails to be correct,* asserting that $\Phi_r = \Phi_l = \Phi$ would be true for any right ARTINIAN ring A. Namely, for the ring A from the above example holds obviously:

$$0 = \Phi_r + \Phi_l = \Phi = I = (x)_r + A.$$

Furthermore every finite ring A is ARTINIAN, contradicting to Theorem 8 in [3]. In his proof [3] has made a mistake, asserting: "then Φ^* is in the annihilator of M^*. This implies that M^* is an ideal of R^*".

Remark 3. T. ANDERSON mentions with R instead of A in [1] that "if $A^2 = A$ or if A satisfies the descending chain condition for right ideals, then Φ, Φ_r, Φ_l and I_l would coincide", according to BECHTELL [3]. But,

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when $A^2 = A$, then [3] speaks nothing on Φ (Theorem 5); and in the AR-TINian case there is no assertation by [3] on the JACOBSON radical (Theorem 8). Therefore J. ANDERSON's conjectures on the FRATTINI ideals of a ring, together with his reference on BECHTELL's results, are incorrect.

Remark 4. Let be observed that, contradicting to the sentence "Whenever the ring has unity" in lines 6 and 7 of page 241 of [3], no existence of unity in [18] was assumed.

Remark 5. The upper BAER radical has been denoted by N, in line 11 of page 242 of [3], but this situation contradicts to the assertation in lines 13 and 15 of page 242 of [3], being BAER's radical and BROWN-MCCOY's radical generally different.

Remark 6.
Theorem 2 of [3] is true, but well- known.
Theorem 3 of [3] is completely trivial.
Theorem 6 of [3] can be easily deduced from earlier known results or methods, see N. JACOBSON [17], or F. SZÁSZ [35], [34].
Generalizing the notion of the modularity of a maximal right ideal R of a ring A, by OTTO STEINFELD has been earlier introduced (at a verbal algebraic discussion) the following

Definition. The maximal right ideal R of a ring A satisfies condition (*), if there exists for every element $a \in A$ with $a \notin R$ and for every element $c \in A$ an element $b \in A$, depending generally on a and c, such that

$$a b c - c^2 \in R$$

holds.

Then O. STEINFELD has also asked, what are all maximal right ideals with condition (*) of a ring. Denoting by D the intersection of all right ideals with condition (*) in A, then STEINFELD's problem can be answered by the following

Proposition. Every maximal right ideal R of a ring A must satisfy the condition (*). Furthermore $D = \Phi_r$, and $A I \subseteq D \subseteq I$ holds for every ring A. Moreover condition $AI = I$ implies $D = I$.

Proof. Let R be an arbitrary maximal right ideal of an arbitrary ring A. Assume that $A^2 \subseteq R$. Then by $A^3 \subseteq R$ obviously we have

$$a b c - c^2 \in R$$

for every $a \in A$, $b \in A$, $c \in A$, which yields condition (*).
Supposing now $A^2 \not\subseteq R$ for R, the right A-module A/R has only trivial right A-submodules, and A/R is not annihilated by his operator domain A. Thus A/R is irreducible, consequently strictly cyclic by the Proposition 1 (2)

of N. Jacobson [17], page 5. From this yet does not follow the modularity of R, only the existence of a modular maximal right ideal M of A satisfying $A/M \cong A/R$, where $\cong$ denotes an isomorphy of the mentioned right A-modules. Then obviously $aA + R = A$ holds for every element $a \in A$ with $a \notin R$, because $A^2 \nsubseteq R$ and the subset $T = \{x; x \in A, xA \subseteq R\}$ is a right ideal of A satisfying $T \supseteq R$ and $TA \subseteq R$, consequently we have $T^2 \subseteq R$. Therefore, in fact $T \neq A$ and $T = R$ holds, which implies the desired equality $aA + R = A$ for any $a \notin R = T$. But now we can conclude to the relation

$$c^2 \in A\,c = (a\,A + R)\,c = a\,A\,c + R\,c \subseteq a\,A\,c + R$$

for every element $c \in A$.

By this inclusion there exists an element b of A and an element r of R with $c^2 = a\,b\,c + r$ consequently with

$$a\,b\,c - c^2 = -r \in R,$$

which exactly means that R satisfies (*).

Therefore it is obtained $D = \Phi_r$ for the above D and Φ_r. By [18] or [36] we have also $AI \subseteq D \subseteq I$, which completes the proof of our Proposition.

Problem 1. Must every quasiprimitive ideal of a multiplicative semigroup with zero element be primitive? (For the definitions see F. Szász [39], Bemerkungen 2. and yet [38]. [37].)

Problem 2. Give a necessary and sufficient condition that the one-sided ideals I_5 and I_6 of a multiplicative semigroup S with zero defined in F. Szász [37], are two-sided ideals! (As well-known Φ_r is a twosided ideal in S).

Problem 3. If S is a commutative multiplicative semigroup without zero element such that the intersection of all maximal subsemigroups of S is empty, then must every subset of S be a semigroup? (Cf. yet S. Lajos [20].)

Problem 4. Does there exist an infinite simple group without proper maximal subgroups? (For this group G holds obviously $\Phi(G) = G$) (cf. Dlab-Korinek [8]).

Problem 5. Is the class C_1 of rings closed under building subrings, homomorphic images and twosided discrete direct sums, for which every simple ring from C_1 is semisimple in the sense of Jacobson? (By E. Sasiada [29] this C_1 is evidently a proper subclass of the class of all associative rings, and by F. Szász [34] this C_1 contains the class of all rings with minimum condition on principal right ideals.)

Problem 6. Deter $I = \Phi_r$! (Cf. „Satz 5. $AI = I$ for every rin

Problem 7. In rel TELL [3] discuss the

$$\Phi_r(A \oplus B)$$

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Problem 6. Determine explicitly the class C_2 of all rings satisfying $I = \Phi_r$! (Cf. „Satz 5.1" of [38].) What is the subclass C_3 of C_2, for which $AI = I$ for every ring $A \in C_3$ holds?

Problem 7. In relation with the "twosided" statement (IX) of BECH-TELL [3] discuss the validity of a "one-sided" analogon

$$\Phi_r(A \oplus B) = \Phi_r(A) \oplus \Phi_r(B),$$

denoting $\oplus$ ringtheoretical direct sum, A and B arbitrary rings!

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