65, 66, 67

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# *Notes on Modules. I, II, III*

By

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# 80. Notes on Modules. I

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Sharpening a result of Kertész [3], who showed the semisimplicity (in the sense of Jacobson) of the total endomorphism ring $E(M)$ of a completely reducible module $M$ over an arbitrary associative ring $A$, we prove in our paper the Neumann regularity of this ring $E(M)$. The result also generalizes a theorem of Johnson-Kiokemeister [2], and is an English version of our earlier result [4], written in Hungarian.

**Theorem.** *Assume that $M$ is a completely reducible module over an arbitrary ring $A$, and $E(M)$ is the total ring of endomorphisms of $M$. Then $E(M)$ is regular in the sense of Neumann.*

**Proof.** Let $M$ be homogeneous. Supposing that the elements of $A$ are right operators, and the elements of $E(M)$ left operators for the module $M$, for any fixed element $\gamma \in E(M)$ there exists an $A$-submodule $K$ of $M$ satisfying:

$$M=\gamma M \oplus K \tag{1}$$

being $M$ completely reducible. Denote $L_\gamma$ the kernel of the endomorphism $\gamma$ in $M$, that is

$$L_\gamma=\{m\,;\, m \in M,\ \gamma m=0\}$$

Then $L_\gamma$ is an $A$-submodule of $M$, and there exists another $A$-submodule $N$ of $M$ with

$$M=L_\gamma \oplus N \tag{2}$$

Being also $N$ completely reducible, we have

$$N=\sum \oplus \{n_\alpha\} \qquad \{\alpha \in \Gamma\}$$

with simple $A$-modules $\{n_\alpha\}$. By (2) our module can be generated by the set of all elements $\gamma n_\alpha$ $(\alpha \in \Gamma)$.

Assume that we have a linear connection

$$\gamma n_{\alpha_1} a_1 + \cdots + \gamma n_{\alpha_k} a_k = 0 \qquad (a_i \in A) \tag{3}$$

then for the element

$$n^* = n_{\alpha_1} a_1 + \cdots + n_{\alpha_k} a_k$$

obviously $\gamma n^*=0$ and $n^* \in L_\gamma$ holds, which yields by (2) also $n^*=0$. The direct sum $\sum \oplus \{n_\alpha\}$ can be built, therefore $n^*=0$ implies $n_{\alpha_1}a_1 = \cdots = n_{\alpha_k}a_k=0$ and thus also $\gamma n_{\alpha_1}a_1 = \cdots = \gamma n_{\alpha_k}a_k=0$. Consequently, the set of all $\gamma n_\alpha$ is a basis of $\gamma M$. Furthermore, let the set of all $k_\beta (\beta \in \Gamma')$ be a basis for $k$, then by (1) one has

$$M=\sum_{\alpha \in \Gamma} \oplus \{\gamma n_\alpha\} \oplus \sum_{\beta \in \Gamma'} \oplus \{k_\beta\}$$

Evidently, every element of $E(M)$ can be determined by his effect on the basis elements $\gamma n_\alpha$ and $K_\beta$ of $M$ ($\alpha \in \Gamma$, $\beta \in \Gamma'$). Because $M$ is now by assumption homogeneous, there exists an element $\delta \in E(M)$ with

(4) $$\delta(\gamma n_\alpha) = n_\alpha, \qquad \delta k_\beta = 0 (\alpha \in \Gamma, \beta \in \Gamma')$$

Define $\vartheta = \gamma\delta\gamma - \gamma$. Then by (4) one has $\vartheta n_\alpha = 0$ and $\vartheta N = 0$, furthermore by $\gamma L_\gamma = 0$ and (2) also $\vartheta M = 0$. Hence $\vartheta = 0$ and $\gamma = \gamma\delta\gamma$ which means the regularity Neumann for $E(M)$ in the homogeneous case.

If $M$ is not homogeneous, then $M$ is a discrete direct sum of its homogeneous components $H_\iota$, and $E(M)$ is the complete direct sum of the rings $E(H_\iota)$. But any $E(H_\iota)$ is regular by the above, and thus also their complete direct sum is regular in the sense of Neumann.

This completes the proof of Theorem.

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# 81. *Notes on Modules. II*

By Ferenc A. SZÁSZ

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Generalizing a well known important result (cf. Jacobson [1], Chapter IV, p. 93) for vector spaces, in our paper all twosided ideals of the total endomorphism ring $E(M)$ of a homogeneous completely reducible module $M$ over an arbitrary ring $A$ are determined. Our result is an English version of the earlier paper of the author [2].

**Theorem.** *Let $E(M)$ be the total endomorphism ring of a homogeneous completely reducible right $A$-module $M$ over an arbitrary ring $A$. Then for every nonzero twosided ideal $\mathcal{J}$ of $E(M)$ there exists an infinite cardinality $\mathfrak{M}$ such that $\mathcal{J}$ coincides with the set of all endomorphisms $\gamma$ of $M$ with* rang $\gamma M < \mathfrak{M}$

**Proof.** We assume that rang $M \geqq \aleph_0$ over $A$, being $E(M)$ a simple total matrix ring over a division ring for the particular case

$$\text{rang } M < \aleph_0.$$

**1.** Firstly we assert that if $\mathcal{J}$ is a twosided ideal of $E(M)$ with $\gamma_2 \in \mathcal{J}$ and

$$\text{rang } \gamma_1 M \leqq \text{rang } \gamma_2 M \tag{1}$$

for an arbitrary $\gamma_1 \in E(M)$, then $\gamma_1 \in \mathcal{J}$

Namely, for $i=1$ and $i=2$ let $N_i$ be the kernel of the endomorphism $\gamma_i$ in $M$. Then there exists a completely reducible submodule $K_i$ of $M$ with $M = N_i \oplus K_i$. Then (1) implies

$$\text{rang } K_1 \leqq \text{rang } K_2 \tag{2}$$

If $K_i = \sum \oplus \{\underset{(i)}{k_{\alpha j}}\}$, then by (2) and by the fact that $M$ is homogeneous, there exists an endomorphism $\delta_1 \in E(M)$ such that holds

$$\delta_1 \underset{(1)}{k_{\alpha_1}} = \underset{(2)}{k_{\alpha_1'}} \quad \text{and} \quad \delta_1 N_1 = 0 \tag{3}$$

Here $\alpha_1'$ denotes an uniquely determined index $\alpha_2$ from $\Gamma_2$, and for $\alpha_1 \neq \beta_1$ one has obviously $\alpha_1' \neq \beta_1'$ ($\alpha_1, \beta_1 \in \Gamma_1$; $\alpha_2, \beta_2 \in \Gamma_2$, being $\Gamma_i$ the set of indices of fixed basis elements of $K_i$). Consequently, the restriction of $\delta_1$ on $\delta_1 K_1$ has an inverse element $\delta_1^{-1}$.

From an assumed linear connection

$$\sum_{j=1}^{n} \gamma_2 \delta_1 \underset{(1)}{k_{\alpha_j}} a_j = 0 \quad (a_j \in A) \tag{4}$$

follows $\gamma_2 k^* = 0$ for the element

$$k^* = \sum_{j=1}^{n} \delta_1 \underset{(1)}{k_{\alpha_j}} a_j \in K_2$$

Therefore $k^* \in N_2 \cap K_2$, and $k^* = 0$. There exists an inverse element

$\delta_1^{-1}$ of the restriction of $\delta_1$ on $\delta_1 K_1$, so one has

(5) $$\delta_1^{-1} k^* = \sum_{j=1(1)}^{n} k_{\alpha_j} a_j = 0$$

which yields $\underset{(1)}{k_{\alpha_j}} a_j = 0$ for every $j = 1, 2, \cdots, n$, forming $\sum\{k_\alpha\}$ a direct sum. Therefore, the elements $\gamma_2 \delta_1 \underset{(1)}{k_\alpha}$ are linearly independent over $A$ $(\alpha \in \Gamma_1)$. By the fact that $M$ is homogeneous, there exists an element $\delta_2 \in E(M)$ satisfying

(6) $$\delta_2(\gamma \delta_1 \underset{(1)}{k_\alpha}) = \gamma_1 \underset{(1)}{k_\alpha}.$$

Analysing the difference $\gamma_0 = \delta_2 \gamma_2 \delta_1 - \gamma_1$, we conclude, $\gamma_0 = 0$, that is

(7) $$\gamma_1 = \delta_2 \gamma_2 \delta_1 \in \mathcal{J},$$

which completes the proof of Assertion 1.

**2.** Secondly, it can be shown that for rang $M \geqq \aleph_0$ and for every nonzero twosided ideal $\mathcal{J}$ of $E(M)$, the endomorphisms $\gamma$ with condition rang $\gamma M < \aleph_0$ are contained in $\mathcal{J}$, and all these endomorphisms $\gamma$ form a twosided ideal $F$ of $E(M)$.

Namely, for the direct composition $M = \Sigma \oplus \{m_\alpha\} (\alpha \in \Gamma)$ we define the endomorphisms $\varepsilon_\beta \in E(M)$ by

$$\varepsilon_\beta m_\alpha = \delta_{\alpha\beta} m_\beta.$$

(8) $$\varepsilon_\beta m_\alpha a = \delta_{\alpha\beta} m_\alpha a (\alpha, \beta \in \Gamma, a \in A)$$

where $\delta_{\alpha\beta}$ denotes Kronecker's delta symbol. Clearly rang $\varepsilon_\alpha M = 1$ and thus by Assertion 1, holds $\varepsilon_\beta \in \mathcal{J}$ for every $\beta$. Consequently

(9) $$\delta_{\beta_1} + \varepsilon_{\beta_2} + \cdots + \varepsilon_{\beta_3} \in \mathcal{J}$$

which verifies the existence of endomorphisms $\gamma \in \mathcal{J}$ with rang $\gamma M = n$ for every $n$.

From this follows already every statement of Assertion 2.

**3.** Thirdly, we prove that there exists for every nonzero ideal $\mathcal{J}$ of $E(M)$ an infinite cardinality $\mathfrak{M}$, such that $\mathcal{J}$ consists of every endomorphism $\gamma \in E(M)$ satisfying rang $\gamma M < \mathfrak{M}$

Let $\mathfrak{M}$ be namely the least (infinite) cardinality satisfying rang $\gamma M < \mathfrak{M}$ for every $\gamma \in \mathcal{J}$. Clearly there exists such a cardinality. By Assertion 2, one has $F \subseteq \mathcal{J}$ and thus $\mathfrak{M} \geqq \aleph_0$.

If rang $M < \mathfrak{M}$, then by definition of $\mathfrak{M}$ there exists an element $\gamma \in \mathcal{J}$ with the condition rang $\gamma M =$ rang $M$ and by Assertion 1 also $\mathcal{J} = E(M)$.

Assuming that $\mathcal{J} \neq E(M)$ and $\mathcal{J} \neq 0$, in case $\mathfrak{M} = \aleph_0$ one has $\mathcal{J} = F$ by Assertion 2.

Furthermore, in case $\mathfrak{M} > \aleph_0$ and $\mathfrak{M} \leqq$ rang $M$ the condition rang $\gamma_i M < \mathfrak{M}$ and definition of $\mathfrak{M}$ imply the existence of an endomorphism $\vartheta \in \mathcal{J}$, with

(10) $$\text{rang } \vartheta M \geqq \text{rang } \gamma_i M$$

whence by Assertion 1 follows $\gamma_i \in \mathcal{J}$.

These Assertions 1, 2 and 3 complete the proof of the Theorem.

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## 82. *Notes on Modules. III*

By Ferenc A. SZÁSZ

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In this paper we discuss the Kertész' radical for modules, and among other we show that this radical fails to be a ring radical in the sense of Amitsur and Kurosh. We refer yet concerning this topic to our earlier papers [6], [7].

Following Kertész [3], for an arbitrary ring $A$ and for any right $A$-module $M$, we consider the set

(1) $$K(M)=\{X; X\in M, \quad XA\subseteq \Phi(M)\}$$

where $\Phi(M)$ denotes the Frattini $A$-submodule of $M$. (That is, $\Phi(M)$ is the intersection of all maximal submodules of $M$, and $\Phi(M)=M$ for modules $M$ having no maximal $A$-submodules.) Obviously, $K(M)$ is an $A$-submodule of $M$. Calling an $A$-submodule $N$ of $M$ homoperfect, if

(2) $$MA+N=M$$

holds, then (1) implies by Kertész [3], that $K(M)$ coincides with the intersection of all homoperfect maximal $A$-submodules of $M$

**Example.** For a prime number $p$ let $A$ be the ring generated by the $3\times 3$ matrices over the field of $p$ elements:

(3) $$x=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \qquad y=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}$$

Then $A$ is a noncommutative ring with $p^2$ elements and with the multiplication:

(4)

| | $x$ | $y$ |
|---|---|---|
| $x$ | 0 | $x$ |
| $y$ | 0 | $y$ |

By a routine calculation it can be verified that the principal right ideal $(y)_r$ of $A$ is a homoperfect maximal right ideal, but $(y)_r$ is neither modular, nor quasimodular in $A$.

Furthermore, for the Kertész radical $K_r(A)$ of the $A$-right module $A$, one has by

(5) $$(x)_r\cap(y)_r=0$$

obviously $K_r(A)=0$, being also $(x)_r$ homoperfect and maximal in $A$. The Jacobson radical $F(A)$ of $A$ now coincides with $(x)_l=K_l(A)$, denoting $K_l(A)$ the left-right dual of $K_r(A)$

Therefore, this ring $A$ has the property, that

(6) $$0=K_r(A)\neq K_l(A)=F(A)$$

**Remark 1.** For an antiisomorphic image $A'$ of the ring $A$ of the above example evidently holds

$$0=K_l(A')\neq K_r(A')=F(A') \tag{7}$$

**Theorem 1.** *For an arbitrary cardinality $\mathfrak{M}$ there exists a ring $A$ with $\mathfrak{M}$ different elements and with conditions $0=K_r(A)\neq K_l(A)=F(A)$ if and only if $\mathfrak{M}$ is not a quadratfree finite number.*

**Proof.** If $\mathfrak{M}$ is a quadratfree finite number, and $A$ has exactly $\mathfrak{M}$ different elements, then $A$ is a ringdirect sum of rings of prime order. These components are commutative rings, therefore also $A$ is commutative, consequently $K_r(A)=F(A)$.

But in the case, when $\mathfrak{M}$ is finite and not quadratfree, then $\mathfrak{M}=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_m^{\alpha_m}$ with $\alpha_i\geqq 2$ at least for an $i$, with different prime numbers $p_j$. Assume that $i=1$ and $p_1=p$. Let our ring $B$ be the ringdirect sum of the ring $A$ from the above example, of $(\alpha_1-2)$ copies of fields of order $p$ and of $\alpha_j$ copies of fields of order $p_j$ for every $p_j\neq p$. Then one has obviously $|B|=\mathfrak{M}$ and $0=K_r(B)\neq K_l(B)=F(B)$.

Thirdly, if $\mathfrak{M}$ is an infinite cardinality, then let $C$ be the ringdirect sum of the ring $A$ from the example and of a field with $\mathfrak{M}$ elements. This field can be taken, as a field extension of the rational number field with the transcendence grad $\mathfrak{M}$. Then evidently $|C|=\mathfrak{M}$ and

$$0=K_r(C)\neq K_l(C)=F(C), \tag{8}$$

which completes the proof of Theorem 1.

**Remark 2.** The above ring $C$, constructed for an infinite $\mathfrak{M}$ as a right $C$-module $C$, is completely reducible, without nonzero left annihilators, but with the nonzero right annihilator $(x)_r=F(C)$. A right completely reducible ring $A$ has no nonzero right annihilators if and only if $C$ is semisimple in the sense of Jacobson, and $C$ satisfies the minimum condition for principal right ideals. (Cf. F. Szász [7].)

**Remark 3.** By the present author [8] was proved the existence of a right having a quasimodular maximal, but not modular right ideal. Calling an ideal $Q$ of a ring $A$ quasiprimitive, if there exists a quasimodular maximal right ideal $R$ of $A$ satisfying $Q=\{x\,;\,x\in A,\ Ax\subseteq R\}$, the equivalence of primitive and quasiprimitive ideals can be verified (cf. Steinfeld [5], and in a sharper form F. Szász [9]). But, for a maximal right ideal of a ring "homoperfect", "quasimodular" and "modular" are three different concepts.

**Theorem 2.** *The twosided ideals $K_r$ and $K_l$ (Kertész radicals) satisfy $AK_r\subseteq\Phi_r\subseteq K_r\subseteq F$ and $K_lA\subseteq\Phi_l\subseteq K_l\subseteq F$ for any ring $A$, furthermore $K_r$ and $K_l$ are not radicals in the sense of Amitsur and Kurosh.*

**Proof.** By the definition (1) it is sufficient to verify only the last statements (cf. yet F. Szász [8]).

Assume that $K_r$ is a radical in the sense of Amitsur and Kurosh.

Then by Theorem 47 of Divinsky's book [1], any twosided ideal of a semisimple ring is also semisimple. But the ring $A$ of the earlier example of the present paper satisfies $K_r(A)=0$ with $K_r(F(A))=F(A) \neq 0$ for the Jacobson radical of $A$.

This completes the proof of Theorem 2.

**Theorem 3.** *For any ring $A$ the following conditions are equivalent:*

a) *$A$ is a semisimple Artin ring,*

b) *$A$ is a ring with twosided unity satisfying the minimum condition on principal right ideals and yet with the condition that $K(M)\cdot A=0$ for the Kertész $K(M)$ radical of every right $A$-module $M$ holds.*

**Proof.** a) implies b). By assumption a) follows, that is also a ring with twosided unity and with minimum condition on principal right ideals. Furthermore, any $A$-right module $M$ can be decomposed into a form

$$M=M_0 \oplus M_1 \tag{9}$$

where $\oplus$ is a module direct sum, $M_0A=0$ and $M_1$ is an unitary $A$-module. This can be proved by Peirce decompositions. Moreover $M_1$ is a completely reducible $A$-right module, which implies $K(M_1)=0$ and $K(M)=M_0$ whence

$$K(M)\cdot A=0$$

Conversely, also b) implies a). Let $A$ be a ring having twosided unity, satisfying the minimum condition on principal right ideals and with $K(M)\cdot A=0$ for every right $A$-module $M$. Then $K_r(A)$ coincides with the Jacobson radical $F$ of $A$, and $FA=0$ implies by $1 \in A$ evidently $F(A)=0$. Therefore, the right $A$-module $A$ is completely reducible by the author's paper [7]. Consequently $A$ is by $1 \in A$ a semisimple Artin ring.

This completes the proof of Theorem 3.

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