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Ferenc A. SZÁSZ

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Budapest

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The term "semigroup" means in this note always a semigroup with zero element (see [3]). Several concrete types of radicals for semigroups were proposed (see for instance [2], [3], [5]–[9] and [11]). By a ring theoretical analogy (see [4]) also a general theory of radicals for semigroups can be developed.

For any class C of semigroups a C -semigroup S means a semigroup belonging to C . If a semigroup S has a C -ideal $C(S)$ such that $C(S)$ contains any further C -ideal of S , then $C(S)$ is called the C -radical of S . Semigroups S with $C(S)=0$ are called C -semisimple. A class R of semigroups is called *radical*, if the following conditions are satisfied:

- 1) R is homomorphically closed /not only with respect to forming of Rees factor semigroups/
- 2) in any semigroup S there exists the R -radical $R(S)$
- 3) the Rees factor semigroup $S/R(S)$ is R -semisimple.

The aim of this note is to generalize for semigroups some ring-theoretical results of [1] and [10].

Theorem 1. For any radical class R of semigroups, and for any ideal J of a semigroup S , the R -radical $R(J)$ of J is an ideal of S .

Proof. Assuming that $R(J)$ is not an ideal of S , there exists an element $s \in S$ satisfying either $sR(J) \not\subseteq R(J)$ or $R(J)s \not\subseteq R(J)$. If $sR(J) \not\subseteq R(J)$, then the union $U = sR(J) \cup R(J)$ properly contains $R(J)$ and $U \subseteq J$ holds. By $JU = JsR(J) \cup JR(J) \subseteq R(J)$ and $UJ \subseteq U$ this union U is an ideal of J . Being $J/R(J)$ R -semisimple, $U/R(J)$ is not an R -semigroup.

By $\varphi_1(r) = sr \cup R(J)$ ($r \in R(J)$) is given a mapping of $R(J)$ onto $U/R(J)$, which by the associativity and

$$\begin{aligned} \varphi_1(r_1, r_2) &= sr_1r_2 \cup R(J) = R(J) \\ &= sr_1s, r_2 \cup R(J) = \varphi_1(r_1), \varphi_1(r_2) \end{aligned}$$

is a homomorphism. Being $R(J)$ radical and $U/R(J)$ nonradical non-zero semigroups, respectively, this contradiction shows $sR(J) \subseteq R(J)$. Similarly can be verified also $R(J)s \subseteq R(J)$.

Corollary 2. With the above notations $R(J) \subseteq J \cap R(S)$ holds.

Proof. $R(J)$ is an R -ideal of S , contained in $R(S)$.

Corollary 3. Any ideal of an R -semisimple semigroup is again

R-semisimple.

Let now $C=C_1$ an arbitrary, homomorphically closed class of semigroups. Assume that for the ordinal numbers $\alpha < \beta$, the classes C_α are already defined. Let C_β be the class of all semigroups every non-zero Rees factor semigroup of which contains a non-zero ideal belonging to a class C_α for some $\alpha < \beta$.

Furthermore let LC_1 be the union of all classes C_α .

A subsemigroup A of S is called accessible, if there exists a finite chain of subsemigroups A_1, A_2, \dots, A_{n-1} of S such that

$$A = A_0 \subset A_1 \subset A_2 \subset \dots \subset A_{n-1} \subset A_n = S_1$$

and A_i is an ideal of A_{i+1} for $i=0, 1, 2, \dots, n-1$. Such a minimal n is the index of A in S .

Proposition 4. *For any homomorphically closed class $C=C_1$ of semigroups, and for the above class LC_1 any non-zero LC_1 -semigroup S has a nonzero accessible subsemigroup A contained in C_1 .*

Proof. Being S a set, by $S \in UC_\alpha$ there is an ordinal number α with $S \in C_\alpha$ for some α . Assume α to be minimal. We prove the existence of a desired A by induction on α . If $\alpha=1$, then $S \in C_1$, and let A be S itself. Assume that our assertion is verified for all semigroups T from C_β with $\beta < \alpha$. If $S \in C_\alpha$, and α is minimal, then any non-zero factor semigroup of S , and thus also S itself, contains a non-zero ideal J contained in C_β for some $\beta < \alpha$. By induction J has a non-zero accessible subsemigroup A in C_1 . But A is a non-zero accessible subsemigroup of S , contained in C_1 .

Proposition 5. *For any non-zero accessible subsemigroup A of a semigroup S with $A \in C_1=C$, the ideal (A) of S generated by A_1 is contained in C_n for some finite n .*

Proof. Let n be the index of A in S . We proceed by induction on n . If $n=1$, then $(A)=A \in C_1$. Assume that our proposition is proved for all accessible subsemigroups of index $< n$. Let now A^* be the ideal of A_{n-1} generated by A . By induction A^* lies in C_{m-1} for some finite m , and $A^* \subseteq (A) \subseteq A_{n-1}$. Thus A^* is an ideal of (A) , and by $(A)=A \cup AS \cup SA \cup SAS$ we have also $(A)=A^* \cup A^*S \cup SA^* \cup SA^*S$.

We shall prove $(A) \in C_m$. Let $(A)/B$ be any non-zero Rees factor semigroup of (A) . If $A^* \not\subseteq B$, then $(A)/B$ with respect to forming of Rees factor semigroups has the nonzero ideal $(A^* \cup B)/B \cong A^*/A^* \cap B$. The righthand side of this isomorphism-relation is a homomorphic image of A^* from the class C_{m-1} , and thus $A^*/A^* \cap B \in C_{m-1}$, being every C_α closed. Then $(A)/B$ has a non-zero ideal in C_{m-1} .

Consequently we assume $A^* \subseteq B$. If $A^*S \not\subseteq B$, then there is an element $s \in S$ such that $A^*s \not\subseteq B$. It is easy to prove that $(A^*s \cup B)/B$ is a non-zero twosided ideal of $(A)/B$. Furthermore $\varphi_a a = as \cup B (a \in A^*)$

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maps A^* onto the factor-semigroup $(A^*s \cup B)/B$, and this mapping φ_2 is by

$$\varphi_2 a_1 a_2 = a_1 a_2 s \cup B \subseteq A^* A^* S \cup B \subseteq A^*(A) \cup B \subseteq B$$

and by

$$\varphi_2 a_1 \cdot \varphi_2 a_2 \subseteq A^* S A^* S \cup B \subseteq A^*(A) \cup B \subseteq A^* \cup B = B$$

a homomorphism. Consequently $(A^*s \cup B)/B$ is a non-zero ideal of $(A)/B$ in C_{m-1} . Similarly it can be shown that $(sA^* \cup B)/B$ is a non-zero ideal of $(A)/B$, contained in C_{m-1} , for some $s \in S$, if $SA^* \not\subseteq B$.

In what follows, we assume $A^* \cup SA^* \cup A^*S \subseteq B$. By $(A) \neq B$ holds $SA^*S \not\subseteq B$, and thus there exist elements $s \in S, t \in S$ such that $sA^*t \not\subseteq B$. It can be verified, that $(sA^*t \cup B)/B$ is an ideal of $(A)/B$. Furthermore by $\varphi_3 a = sat \cup B$ ($a \in A^*$) the semigroup A^* is homomorphically mapped onto $(sA^*t \cup B)/B$, being $\varphi_3 a_1 a_2 = B = \varphi_3 a_1 \cdot \varphi_3 a_2$. Consequently (A) lies in a class C_m for some finite m .

Theorem 6. *The class LC_1 , determined by any homomorphically closed class $C_1 = C$ of semigroups, coincides with the class C_{ω_0} , where ω_0 is the first infinite ordinal number.*

Proof. Obviously $C_{\omega_0} \subseteq LC_1$ holds. Furthermore, if S is any nonzero semigroup from LC_1 , then any non-zero Rees factorsemigroup S/T of S lies in LC_1 , and S/T contains by Proposition 4 a nonzero accessible subsemigroup A , lying in C_1 . By Proposition 5 S/T contains a nonzero ideal $(A)/T$ contained in C_m for a finite m . Consequently $LC_1 = C_{\omega_0}$.

Theorem 7. *If C_1 is a homomorphically closed and hereditary class of semigroups (i.e. any ideal of a semigroup from C_1 lies again in C_1), containing all nilpotent semigroups, then $LC_1 = C_2$ holds.*

Proof. If S is any semigroup from C_3 , then any non-zero factor semigroup $S' = S/T$ of S contains a nonzero ideal J belonging to C_2 . Thus also J contains a nonzero ideal K lying in C_1 . If (K) is the ideal of S' generated by K , then $K \subseteq (K) \subseteq J$. Being

$$(K)^3 \subseteq J(K \cup KS' \cup S'K \cup S'KS')J \subseteq K$$

and C_1 a hereditary class, one has $(K)^3 \in C_1$ and $(K) \neq 0$. If $(K)^3 \neq 0$, then it is a nonzero C_1 -ideal of S' . In the case $(K)^3 = 0$ and $(K)^2 \neq 0$ obviously $(K)^2 \in C_1$ being $(K)^2$ nilpotent. If $(K)^2 = 0$, then also $(K) \in C_1$, and thus S' has a nonzero C_1 -ideal. Consequently $LC_1 = C_2$.

It would be interesting to investigate that in what case LC_1 is a radical class for a homomorphically closed class C_1 of semigroups. (A homomorphism generally cannot be given by a Rees factor semigroup.)

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