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A Class of Regular Rings

By

Ferenc Szász, Budapest

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The fundamental notions used in this paper can be found in N. JACOBSON [4], I. KAPLANSKY [6] and N. H. MCCOY [9]. All rings considered here will be associative. Their commutativity is not assumed, but it will be proved. As is well known, a ring A is called regular in the sense of Neumann (or strongly regular, respectively) if $a \in aAa$ (or $a \in a^2A$, respectively) for every element $a \in A$. Every strongly regular ring is also regular. A full $n \times n$ matrix ring ($n \geq 2$) over a division ring is regular, but it is not strongly regular. For some results on strongly regular rings we refer the reader to A. FORSYTHE - N.H.MCCOY [3], T. KANDO [5], L. G. KOVÁCS [7], S. LAJOS and the author [8] and the author [12]. The product BC of the subsets B and C of the ring A means the additive subgroup, generated by all $b.c$ with $b \in B$ and $c \in C$, of the additive group A^+ of A.

In studying the two-sided idealizer of a subring of a ring, the author [13] has determined, among others, all rings such that every nonzero subring coincides with its two-sided idealizer. If a ring A with this property differs from the ring B of prime order with trivial multiplication (i. e. $B^2 = 0$), then every subring S of A is idempotent, i. e. $S^2 = S$ holds.

The aim of this paper is to characterize all rings such that every subring is idempotent. Namely we have:

Theorem 1. For an arbitrary ring A the following five conditions are equivalent:

(I) *Every subring S of A is idempotent.*

(II) *No homomorphic image of any subring S of A has nonzero left annihilators.*

(III) *Every subring S of A is regular in the sense of Neumann.*

(IV) *Every subring S of A is strongly regular.*

(V) *The additive group A^+ of A is a torsion group, and $pA_p = 0$ holds for every p-component A_p of A. Every subring $\{a\}$, generated by a single element a of A, is finite, and is a direct sum of finite fields. Furthermore A is commutative*.*

Proof. Assume condition (V), and we prove that condition (IV) holds. Let a be an arbitrary element of the ring A. Then there exist some finite fields F_i $(i = 1, 2, \ldots, n)$ such that

$$\{a\} = F_1 \oplus F_2 \oplus \ldots \oplus F_n.$$

If $a = f_1 + f_2 + \ldots + f_n$, with $0 \neq f_i \in F_i$, then let g_i be the inverse element of f_i in F_i .Obviously we have

$$a = (f_1 + \ldots + f_n)(f_1 g_1 + \ldots + f_n g_n) = \\ = a^2 (g_1 + \ldots + g_n) \in a^2 \{a\}.$$

Therefore the subring $\{a\}$ is strongly regular, which implies condition (IV).

Condition (IV) implies (III) trivially.

Since any homomorphic image of a Neumann regular ring is again Neumann regular, and since a Neumann regular ring has no nonzero left annihilators, condition (III) also implies (II).

Assume that condition (II) holds for the ring A. If J is an arbitrary ideal of the ring $\{a\}$, then $\{a\}/J.\{a\}$ has no nonzero left annihilators, consequently $J \subset J\{a\}$ holds. In particular, $J = \{a\}$ yields $\{a\} \subset \{a\}^2$, therefore $\{a\}^2 = \{a\}$ for every element $a \in A$, which implies condition (I).

We shall now prove that condition (I) implies (V). Assume that A is a ring satisfying condition (I). Let a be an arbitrary element of A. Then $\{a\} = \{a\}^2$ implies $a = a.a_1$ with an element $a_1 \in \{a\}$, consequently also $a = a.a_1^2$. Let a_2 be an element of $\{a\}$ such that $a_1^2 = aa_2$. Then $a = a^2.a_2$ holds. Therefore A is strongly regular, and it has no nonzero nilpotent elements. Furthermore $a = aa_1$ and $a_1 \in \{a\}$ imply $a_1^2 = a_1 \neq 0$. Consequently any nonzero subring contains a nonzero idempotent element.

Let T be the maximal torsion ideal of the ring A. Then $B = A/T$ is torsion free (cf. I. KAPLANSKY [6]), and B also satisfies condition(I). Let b_1 be a nonzero idempotent which belongs to the nonzero subring

* Added in proof (February 25, 1971): In the meantime there appeared P. N. STEWART's paper "Semisimple radical classes", Pacific J. Math. **32** (1970), 249–254, in which it is asserted that there exists a natural number n for every $a \in A$ such that $a^n = a$ holds. (But cf. our Remark 6(4) at the end of this paper.)

$\{b\}$ of $B = A/T$. Then we have $\{2b_1\}^2 = 4\{b_1\} + 2\{b_1\}$, B^+ being torsion free; consequently the subring $\{2b_1\}$ is not idempotent. Therefore $b = 0$, $B = 0$ and $T = A$. Thus A^+ is a torsion group, and we have $A = \sum_p A_p$, where A_p is a nonzero p-component of A. Then $A_p^2 = A_p$ and $(pA_p)^2 = pA_p$ obviously imply $p(pA_p) = pA_p$, i. e. pA_p is a divisible subgroup, which is a direct sum of Prüfer quasicyclic groups $C(p_\infty)$. But any $C(p_\infty)$ admits only trivial multiplication (i. e. $x.y = 0$ holds for all $x, y \in C(p_\infty)$), whence condition (I) implies $pA_p = 0$ for any prime p. Therefore A^+ is an elementary torsion group (cf. I. Kaplansky [6]).

Consequently any p-component A_p is an algebra over the finite prime field K_p, and A_p is algebraic since $a \in \{a\}^2$ for every $a \in A$. A, having no nonzero nilpotent elements, is commutative by Corollary X.1 of N. Jacobson [4]. Since $\{a\}$ is finite and semisimple in the sense of N. Jacobson [4], it is a direct sum of finite fields. Therefore condition (V) is indeed satisfied.

This completes the proof of Theorem 1.

Corollary 2. Every ring satisfying condition (I) *is commutative and regular, and consequently, strongly regular.*

Corollary 3. Every ring satisfying condition (I) *is a subdirect sum of fields.*

Corollary 4. Any p-component of a ring with condition (I) *is an algebraic PI-algebra* (*in the sense of Chapter X of N. Jacobson* [4]) *over the finite prime field* K_p, *and consequently it is locally finite* (*i. e. any finitely generated subalgebra has finite dimension*).

Examples 5. (1) Let K be a field which is the algebraic closure of the finite prime field K_p. Then K is infinite, locally finite over K_p, and it satisfies condition (I), any subring S of K being a field. We mention that every subring S of K coincides with its two-sided idealizer $J(S)$ in K (see the author [13]).

(2) Let A be the direct sum of the finite prime fields $\{a\}$ and $\{b\}$ with $a^2 - a = b^2 - b = ab = ba = pa = pb = 0$, where p is a prime number. Then A satisfies condition (I). Furthermore, the additive subgroup S, generated by $a + b$, is a subring, but S is not a two-sided ideal of A since $a = a(a+b) \notin S$ and $b = b(a+b) \notin S$.

(3) Let A be an arbitrary ring with trivial multiplication, i. e. $A^2 = 0$. Then any subgroup of A is a two-sided ideal of A, but A does not satisfy condition (I).

Remarks 6. (1) Let C_1 denote the class of all rings satisfying condition (I). L. RÉDEI [10] has determined the class C_2 of all full ideal rings, i. e. rings such that every additive subgroup is a twosided ideal. Now $C_1 \not\supset C_2$ holds, by Example 5.(2). Furthermore we have also $C_2 \not\supset C_1$, by Example 5.(3), consequently C_1 and C_2 are two different classes of rings.

(2) A third class, which is a proper subclass of C_1, of strongly regular rings has been determined in the author's paper [14] such that every ring of this subclass is a certain subdirect sum of finite prime fields.

(3) Every ring satisfying condition (I) is F-regular in the sense of B. BROWN-N. H. MCCOY [1], page 308, by $a \in \{a\}^2$, according to their example 4. Namely we have $\{a\}^2 \subset F(a)$, where $F(a)$ is the set of all polynomial expressions of the form $n_2a^2 + n_3a^3 + \ldots + n_ka^k$.

(4) Let R be the upper radical (cf. N. DIVINSKY [2]), determined by the class C_1 of all rings satisfying condition (I). Then any homomorphic image of any R-semisimple ring is again R-semisimple, consequently the mapping $J \to R(J)$ is a join endomorphism of the lattice of the ideals J of any ring A (cf. the author's paper [11]). Namely $R(J_1 + J_2) = R(J_1) + R(J_2)$ holds for every ideal J_2 and J_2 of an arbitrary ring.

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Author's address:
Dr. F. Szász
Magyar Tudományos Akadémia
Matematikai Kutató Intézete
Reáltanoda u. 13–15
Budapest V
Ungarn

Printed in Austria