Almost Right Quasiregular Adjoint Semigroups of Rings

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The fundamental notions, used in this paper, can be found in [3, 11, 14, 16 and 18].

As it is well-known, the important papers [5, 6, 7, 8, 9, 10, 11 and 12] of H. J. Hoehnke have developed a detailed theory of a radical for semigroups with zero, discussing representations, primitive congruences and modular right congruences etc. of semigroups. Interesting contributions to this radical theory were given by H. J. Hoehnke and H. Seidel [13]. The Hoehnke radical for semigroups plays the role of the Jacobson radical for rings. With the help of right quasiregular elements, which were introduced generally also for semigroups without zero, H. Seidel [18] has proved that the Hoehnke radical of a semigroup S with zero coincides with the nil radical of S. We remark that author's paper [23] has treated six further concrete radicals for a semigroup with zero, and the author has conjectured that the Hoehnke radical coincides with these six radicals. H. Seidel [19] could partially solve this author's conjecture, namely for author's four radicals. Various concrete radicals for semigroups were discussed for instance in J. Bosák [1], J. Luh [15], S. Schwarz [17], L. N. Shevrin [20] and R. Shulka [22]. Some results on general radicals of semigroups with zero were treated in author's paper [24].

We recall that on the basis of Seidel's paper [18] an element s of a semigroup S (generally without zero element) is right quasi-regular if and only if for arbitrary elements t and u of S there exist nonnegative rational integers m and n such that $s^m t = s^n u$ holds; where eventually $s^0 t$ means the element t of S.

A semigroup S with twosided unity element e will be called almost right quasiregular, if $S = e \cup Q$ with $e \notin Q$ holds, where Q is a subsemigroup of S consisting only of right quasiregular elements. Therefore the Hoehnke radical of an almost right quasiregular semigroup with zero can be very big.

A semigroup S having both twosided unity element e and zero, is called almost nil, if $S = e \cup N$ with $e \notin N$ holds, where N is a nil subsemigroup of S.

Throughout this paper ring always will mean an associative ring. As it is well-known, the elements of any ring A form with respect to the so-called circle operation $x \circ y = x + y - xy$ a semigroup S, which is said to be the adjoint semigroup of the ring A. On the basis of the equations $x \circ 0 = 0 \circ x = x$ for any $x \in A$, the zero element 0 of the ring A is the twosided unity element of S. Consequently an adjoint semigroup of any ring contains twosided unity element. Furthermore, an element $e \in A$ is a left, right or twosided zero element of the adjoint semigroup S if and only if e is a left, right or twosided unity element, respectively, of the ring A. For various results on the circle operation of a ring we refer the reader for instance to [2, 14, 22 and 25].

In what follows, we call an arbitrary ring A HOEHNKE ring, or shortly only H-ring, if the adjoint semigroup S of A is almost right quasiregular in the above sense. At this definition the existence of a one-sided unity element of A, that is of a one-sided zero element of the adjoint semigroup S, is not assumed, but its existence will be proved. If A is not assumed to contain twosided unity element, let 1 denote the twosided unity element of the canonic DORROH ring extension A_1 of A (see J. K. DORROH [4]), and even in this case the products and differences of the form

$$(1 + x)\, y \quad \text{and} \quad 1 - (1 + x)(1 + y) \quad (x, y \in A),$$

respectively, have sense, and they are contained in the ring A itself.

An important task of the semigroup theory is to determine all semigroups with twosided unity element, which occur as adjoint semigroups of rings.

The aim of this paper, in particular, is only to determine all almost right quasiregular adjoint semigroups, that is the adjoint semigroups of all H-rings.

Any nontrivial H-ring can be considered as a very strongly nonradical ring for the JACOBSON radical [14], since the adjoint semigroup of any JACOBSON radical ring is a group. It is evident that for adjoint semigroups S of nontrivial rings "right quasiregular" or "HOEHNKE radical" cannot be taken instead of "almost right quasiregular", because S always has twosided unity element, which is not right quasiregular in S, and S generally is not assumed to have zero, respectively.

First we discuss some preliminary results on H-rings.

Proposition 1. *For any nonzero element a and for arbitrary elements b and c of any H-ring A there exist nonnegative rational integers m and n such that $1 - (1 - a)^m (1 - b) = 1 - (1 - a)^n (1 - c)$ holds.*

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Proof follows immediately from $x \circ y = 1 - (1 - x)(1 - y)$, definition of an almost right quasiregular semigroup, and SEIDEL's criterium [18] for the right quasiregularity.

Proposition 2. *Any nonzero idempotent element e of any H-ring A is a left unity element of A.*

Proof. By $e^2 = e$ one has also $(1 - e)^2 = 1 - e$ (for instance in the DORROH ring extension A_1 [4] of A). Therefore Proposition 1 asserts for case $a = b = e$ and arbitrary element $c = x$ of A that

$$1 - (1 - e) = 1 - (1 - e)(1 - x)$$

holds. This implies obviously $ex = x$ for any $x \in A$, which completes the proof.

Proposition 3. *For any nonzero element x of any H-ring A there exist different natural numbers k and l such that $1 - (1 - x)^k = 1 - (1 - x)^l$ holds.*

Proof. Proposition 1 asserts for $a = b = x \circ x = 2x - x^2$ and $c = x$ obviously

$$1 - (1 - x \circ x)^m (1 - x \circ x) = 1 - (1 - x \circ x)^n (1 - x)$$

with certain m and n, where $x \neq 0$ implies by the definition of an almost right quasiregular semigroup also $x \circ x \neq 0$, and therefore Proposition 1 can be applied for $a = x \circ x \neq 0$, indeed. Consequently we have

$$1 - (1 - x)^{2m+2} = 1 - (1 - x)^{2n+1}$$

with

$$k = 2m + 2 \neq 2n + 1 = l.$$

Proposition 4. *For any nonzero element x of any H-ring A there exists a natural number m such that $(1 - x)^m \cdot y = 0$ for any $y \in A$ holds, and thus the subring $\{x\}$ of A contains a left unity element.*

Proof. By $x \circ y = 1 - (1 - x)(1 - y)$ and Proposition 3 the adjoint semigroup S of any H-ring is a torsion semigroup. Consequently the subsemigroup of S, generated by any element, which differs from $O \in A$, contains an idempotent element e of S. Then $e = 1 - (1 - x)^m$ holds with an m, and e is also an idempotent element of the ring A. Therefore e is by Proposition 2 a left unity element of A, whence we have at once $(1 - x)^m y = 0$ for any $y \in A$. Obviously $e = 1 - (1 - x)^m \in \{x\}$ holds, which completes the proof.

Proposition 5. *A nonzero element x of any H-ring A cannot have a quasi-inverse element y of the ring A.*

Proof. Assume $x + y - xy = x + y - yx = 0$. Then

$$1 - (1 - x)(1 - y) = 1 - (1 - y)(1 - x) = 0$$

implies also $1 - (1 - y)^m (1 - x)^m = 0$ for any natural number m. Furthermore, by Proposition 4 for this x there exists a number m such that $(1 - x)^m \cdot z = 0$ for any $z \in A$ holds. Consequently

$$z = 1 \cdot z = (1 - y)^m \cdot (1 - x)^m \cdot z = (1 - y)^m \cdot 0 = 0,$$

which is impossible. This contradiction concludes the proof.

Proposition 6. *For any element x of any H-ring A the equation $x + x = 0$ holds, that is A has characteristic two.*

Proof. Assume $x \neq 0$. Then the subring $\{x\}$ contains by Proposition 4 a left unity element e of the ring A. For this element

$$2e + 2e - 2e \cdot 2e = 0$$

holds, whence Proposition 5 gives $2e = 0$, consequently also

$$x + x = 2x = 2(ex) = (2e)x = 0 \cdot x = 0.$$

Proposition 7. *Any H-ring A contains twosided unity element.*

Proof. By Proposition 4 any nonzero subring $\{x\}$ contains a left unity element e of A. Then the equation $L^2 = 0$ holds for the left ideal

$$L = A(1 - e) = [y - ye; y \in A].$$

Therefore any element of L is nilpotent, consequently quasiregular in the ring A (see Jacobson [14]). Now Proposition 5 yields $L = 0$, hence e is a right unity element, and therefore the twosided unity element of the H-ring A.

Proposition 8. *Any H-ring A has an almost nil adjoint semigroup S.*

Proof. The zero element of A is the twosided unity element of S. Furthermore A has by Proposition 7 twosided unity element, which is the zero element of S. Now Sätze 3.3 and 3.4 of H. Seidel [18] yield that S is almost nil, indeed.

We have the following

Theorem. *For an adjoint semigroup S of a ring A the following seven conditions are mutually equivalent:*

(i) *S is almost right quasiregular.*

(ii) *S is almost right quasiregular with left zero.*

(iii) *S is almost right quasiregular with right zero.*

(iv) *S is almost right quasiregular with zero.*

(v) *S contains zero and S is almost nil.*

(vi) *S is the adjoint semigroup of the field of two elements.*

(vii) *S consists of two elements a and b with the multiplication.*

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Corollary 1. Any almost right quasiregular adjoint semigroup is commutative, consequently it is also almost left quasiregular.

Corollary 2. Any almost right quasiregular adjoint semigroup is finite.

Corollary 3. The class of all almost right quasiregular adjoint semigroups consists of one semigroup.

Corollary 4. Any H-ring is isomorphic to the field of two elements.

Proof of the Theorem. The implications (vii) $\Rightarrow$ (vi) $\Rightarrow$ (v) are trivial.

(v) implies (iv) by Theorem 3.6 of H. J. HOEHNKE [11].

(iv) implies (iii) obviously.

(iii) implies (ii) as follows. If S satisfies (iii), then A is an H-ring with a right unity element e. But e is by Proposition 2 also a left unity element of A, because it is a nonzero idempotent. Consequently e is a left zero element of S, and thus condition (ii) holds, indeed.

The implication (ii) $\Rightarrow$ (i) is trivial.

Finally, (i) implies (vii) as follows. Assume condition (i) for the adjoint semigroups S of a ring A. Then A is by definition an H-ring, which by Proposition 7 contains twosided unity element 1. Then S is by Proposition 8 almost nil, 1 being the zero of S. Consequently $x \circ y = 1 - (1 - x)(1 - y)$ yields that for any nonzero element x of A a natural number m there exists such that $1 - (1 - x)^m = 1$ holds. Therefore for any $x \neq 0$ we have $(x - 1)^m = 0$, consequently $x - 1$ is nilpotent. Hence $x - 1$ is also quasiregular in the ring A, which is possible by Proposition 5 only in case $x - 1 = 0$ and $x = 1$. Therefore any nonzero element x of A coincides with 1, consequently A is a field consisting only of two elements. Then S is isomorphic to the semigroup, mentioned in condition (vii) of the Theorem. Therefore the implication (i) $\Rightarrow$ (vii) is true, indeed.

This completes the proof of Theorem.

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