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On the Idealizer of a Subring

By

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In this note, by a ring we always mean an associative ring. For a nonzero subring S of a ring A the left idealizer $L(S)$ of S is the greatest subring of A which contains S as a left ideal, i. e. such that $L(S).S \subseteq S$ holds. If S is a left ideal of A, then, by definition, one has $L(S) = A$. The right idealizer $R(S)$ of a subring S of A can be defined right-left dually. Furthermore the intersection $J(S) = L(S) \cap R(S)$ is called the idealizer of the subring S of the ring A. Obviously $J(S) = A$ holds if and only if S is a twosided ideal of A.

For the fundamental notions of the theory of rings we refer to N. H. McCoy [7]. Furthermore, we recall that a general radical property in the sense of Amitsur and Kurosh (see N. Divinsky [2]) is hereditary, if any twosided ideal of a radical ring is again a radical ring; moreover it is supernilpotent (see Andrunakievitch [1]) if it is hereditary and every nilpotent ring is a radical ring. The ring A is called twosided T-nilpotent, if every sequence of right products $p_n = a_1 . a_2 . . . a_n$ and left products $p_n' = a_n . a_{n-1} \ldots a_2 . a_1$ of ring elements $a_i \in A$ for a large index n coincides with zero. Furthermore A is a twosided subcommutative ring if $aA = Aa$ holds for every $a \in A$.

It should be remarked that the idealizer of a subring has been discussed before, for instance by P. A. Freidman [3] and L. Fuchs [4]. Moreover, some special cases, where any subring S is a twosided ideal of the ring A, i. e. $J(S) = A$ holds for all S, were discussed by L. Rédei [8], [9] and the author [12]. Rings satisfying the milder condition, that every subring S of A is a right ideal of A, i. e. $R(S) = A$ holds for all S, were discussed by the author [11], [13].

The purpose of this note is to communicate some further statements on the idealizer of a subring of a ring.

Theorem 1. *The idealizer $J(S)$ of every nonzero subring S of a ring A coincides with S itself if and only if either A is a zero-ring of prime number order, or it is a commutative, absolute algebraic field of prime characteristic.*

Proof. Since the conditions mentioned in the theorem are almost trivially sufficient, we shall prove only that these conditions are also necessary.

In what follows, let A be any ring, in which for every nonzero subring S the equation $J(S) = S$ holds. Then, obviously, every nonzero subring S of A must be a simple ring. For every prime number p the subring $pS = [ps; s \in S]$ is a twosided ideal of S; consequently $pS = 0$ or $pS = S$. But if $pS = S$ for a nonzero subring S and for all p, then the additive subgroup S^+ is divisible. Being a simple ring S cannot contain a quasicyclic zero subring $C(p^\infty)$, therefore in the case $pS = S \neq 0$ for every p, the additive group S^+ must be torsionfree, but this is again a contradiction, because the additive group of the ring $\{x\}$ generated by a single element x of infinite additive order is not divisible.

Consequently one has $pA = 0$ for some prime number p.

Assume first that A contains a nonzero nilpotent subring N. Then $N^2 = 0$, because for $N^2 \neq 0$ it follows at once that $N \subseteq J(N^2) = N^2$, and consequently the contradiction $N^2 = N$. Now $J(S) = S$ for every nonzero subring S of A implies that A can contain at most one nonzero subring $\{a\}$ generated by a with $a^2 = 0$. For every element $x \in A$, $(axa)^2 = ax.a^2.xa = 0$. Consequently $axa \in \{a\}, ax \in J(\{a\}) = \{a\}$. Similarly we get $xa \in \{a\}$, whence at once one has $x \in J(\{a\}) = \{a\}$, that is $A = \{a\}$. Therefore, if A contains a nonzero nilpotent subring, then A itself is a zero-ring of prime number order.

Assume secondly, that A does not contain any nonzero nilpotent subring. Since $\{a\}$ is a commutative simple ring for every $a \in A$, $a \neq 0$, it is now a field. Furthermore $a \in \{a^2\} + a^2\{a\}$ and $pA = 0$ imply that $\{a\}$ is a finite field. Consequently, by N. JACOBSON [5, Theorem 10. 1. 1], A is a commutative, absolute algebraic field.

This completes the proof.

Corollary 2. *If every nonzero subring S of a ring A coincides with its idealizer $J(S)$, then every nonzero subring S also coincides with its left idealizer $L(S)$ and with its right idealizer $R(S)$.*

Theorem 3. *Let $\mathbf{R}$ be a general radical property, for which all nilpotent rings are radical rings. Furthermore let S be a subring of the ring A, which is maximal among all $\mathbf{R}$-radical subrings of A. Then $J(J(S)) = J(S)$.*

On

Proof. The $\mathbf{R}$-radical
$J(S).J(J(S)) \subseteq J(S)$, it
being contained in $J(S)$
right annihilator of $J(S)/S$,
immediately implies $J(J($
$S.J(J(S)) \subseteq S$, consequent
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Remark 4. Obviously
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Remark 5. The radical
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Theorem 6. *If a prope*
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Remark 7. Two import
1. S is a nilsubring
right ideals (see F. SZÁSZ
2. S is nilpotent.

Theorem 8. *A ring A*
following three conditions:
1. *A has no divisors of*
2. *A contains a chain*
$0 =$
where S_1 is a twosided sub
$i = 0, 1, 2, \ldots, n-1$;

Proof. The $\mathbf{R}$-radical of the ring $J(S)$ coincides with S. Since $J(S).J(J(S)) \subseteq J(S)$, it follows that $J(S).J(J(S))\,S \subseteq S$. $J(J(S))\,S$ being contained in $J(S)$ by definition, $J(J(S)) + S/S$ obviously is a right annihilator of $J(S)/S$, and therefore our assumption on the radical $\mathbf{R}$ immediately implies $J(J(S)).S \subseteq S$. Similarly it can be shown that $S.J(J(S)) \subseteq S$, consequently S is a twosided ideal in the subring $J(J(S))$ of A. By definition of $J(S)$ we have $J(J(S)) \subseteq J(S) \subseteq J(J(S))$ and this completes the proof.

Remark 4. Obviously Theorem 3 corresponds to a statement in the theory of groups, by which the normalizer $N(N(S))$ of the normalizer $N(S)$ of a Π-Sylow subgroup S of a group G coincides with $N(S)$. (See A. G. KUROSH [6, page 344].)

Remark 5. The radical property $\mathbf{R}$ in Theorem 3 need not be hereditary, and thus it need not be supernilpotent. But by Zorn's lemma there exists a maximal $\mathbf{R}$-radical subring S of the ring A, if $\mathbf{R}$ is taken nil or locally nilpotent. The nil and locally nilpotent radicals obviously are supernilpotent.

Theorem 6. *If a proper subring S of a ring A is twosided T-nilpotent, then $J(S)$ is properly larger than S.*

Proof. Assume that $J(S) = S$, and we shall deduce a contradiction. Since $S \neq A$, there exists an element $x \in A$ with $x \notin S = J(S)$. Therefore there exist elements $s_1, s_1' \in S$ such that $xs_1 \notin S$ or $s_1'x \notin S$ holds. Assume for instance that we have $xs_1 \notin S = J(S)$. Then there exist elements $s_2, s_2' \in S$ such that $xs_1s_2 \notin S$ or $s_2'xs_1 \notin S$ holds. Continuing this procedure we find sequences $\bar{s}_1, \bar{s}_2, \bar{s}_3, \ldots$ and $\bar{s}_1', \bar{s}_2', \bar{s}_3', \ldots$ such that for instance $\bar{s}_m' \ldots \bar{s}_1' x \bar{s}_1 \ldots \bar{s}_n \notin S$. But this contradicts the assumed T-nilpotence of the subring S of A. Therefore $J(S) \neq S$, indeed.

Remark 7. Two important particular cases of T-nilpotent subrings S:

1. S is a nilsubring with minimum condition on the principal right ideals (see F. SZÁSZ [14]), and
2. S is nilpotent.

Theorem 8. *A ring A is a division ring if and only if A satisfies the following three conditions:*

1. *A has no divisors of zero;*
2. *A contains a chain of subrings*

$$0 = S_0 \subset S_1 \subset \ldots \subset S_n = A$$

where S_1 is a twosided subcommutative subring, and $S_{i+1} \subseteq J(S_i)$ holds for $i = 0, 1, 2, \ldots, n-1$;

3. *This chain cannot be refined.*

Proof. S_1 contains no proper ideals, and by the twosided subcommutativity and validity of the cancelling rule, S_1 is obviously a division ring with the unit element e. Now for any $x \in S_2 \subseteq J(S_1)$ one has $xe = s_1 \in S_1$ and $ex = s_1' \in S_1$. But $xe = s_1 e$, $ex = es_1'$, and the cancelling rules imply $x = s_1 = s_1' \in S$ and $S_2 = S_1$. By induction on n we can also show $A = S_n = S_1$, which together with the trivial converse statement concludes the proof.

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