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The radical property of rings such that every homomorphic image has no nonzero left annihilators

To Professor OTT-HEINRICH KELLER on his 65th birthday

By FERENC A. SZÁSZ of Budapest

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The fundamental notions, used in this paper, can be found in N. DIVINSKY [12], N. JACOBSON [17] and N. H. MCCOY [18]. All rings, considered here, will be associative. For a radical property R , a ring A is said to be strongly R -semisimple, if any homomorphic image of A is R -semisimple (cf. V. A. ANDRUSIAKIEVICH [3]). Furthermore, a ring A is called strongly idempotent (or antisimple) if any ideal of A is idempotent (or if A cannot be homomorphically mapped onto a subdirectly irreducible ring having idempotent heart, respectively) (cf. V. A. ANDRUSIAKIEVICH [2]). Instance for a strongly idempotent ring is any biregular ring (cf. V. A. ANDRUSIAKIEVICH [1]), in which any principal twosided ideal has twosided unity element, and also any VON NEUMANN regular ring A , in which $a \in aAa$ for any element $a \in A$ holds. The principal left ideal (or twosided ideal), generated by a , of A , will be denoted by $(a)_l$ (or by (a) , respectively). The ideal $aA + Aa$ for any element a of the ring A will be denoted by $R(a)$. Let A^∞ denote the intersection of all powers A^n of the ring A , for $n = 1, 2, 3, \dots$ etc.

In studying some criteria (see Sätze 3.1 (1), (2), (3), (4); 3.2 (1), (2); 3.3 (1), (2) and 3.4 (1), (2) from [22]) for the existence of the twosided unity element of a ring, the author [22] has investigated six classes of rings, namely the class of E_i -rings for $i = 0, 1, 2, 3, 4$ and 5. According to this terminology, a ring A is called an E_i -ring (or an E_j -ring) if any homomorphic image has no nonzero left annihilators (or $a \in aA$ for any $a \in A$ holds, respectively). The E_i -rings, in other terminology, have been discussed before by R. BAER [4], [5] and [6], furthermore N. DIVINSKY [10] and [11]. By author [22] a ring A is called an E_j -ring, if the maximal trivial submodule M_0 of any A -right module M is a direct summand of M , where the submodule $N \subseteq M$ is trivial, if $NA = 0$ holds. Various properties of nonzero BROWN-MCCOY radical E_j -rings were asserted in author's paper [24].

Instances for E_i -rings are all E_i -rings for $0 \leq i \leq 4$, furthermore all von NEUMANN-regular rings [8], all biregular rings [1], and all strongly idempotent rings [2], respectively. E. SASIADA's example [19] for a simple, nonnilpotent JACOBSON radical ring, shows the existence of an E_5 -ring, which is not an E_3 -ring. It can be proved that any E_2 -ring is an E_3 -ring, which more, any ring with twosided unity element is an E_2 -ring, and any ring with a right unity element is an E_3 -ring.

The aim of this paper is to point out that the class of all E_i -rings forms a radical class \mathbf{R} (see Theorem 3).

In what follows \mathbf{R} always means this radical, and \mathbf{R}' the radical, defined left-right dually to \mathbf{R} .

Any strongly idempotent being an \mathbf{R} -radical ring, one can expect, that some \mathbf{R} -semisimple rings will be \mathbf{S} -radical rings for a suitable supernilpotent radical \mathbf{S} . Corollaries 7, 8, 9, 10, 11 and 12 show that this is the case, indeed.

Among others three characterizations of the nilpotent right artinian rings will be given (see Corollaries 10, 11 and 12).

As it is well-known, among the supernilpotent radicals of rings the JACOBSON radical [17] and the BROWN-McCOY radical [7] are most useful. H. J. HOEHKE's important papers [14], [15] and [16] have developed the theories of two concrete radicals for semigroups with zero element, which correspond to the JACOBSON radical and the BROWN-McCOY radical of rings. In addition, on HOEHKE's radical of first type, for semigroups with zero element, see yet H. SEIDER, [20], [21], author's paper [23] and its references. Some results on general radicals of semigroups with zero can be found in author's paper [25].

Proposition 1. *A ring A is an E_i -ring if and only if $a \in \mathbf{R}(a) = aA + Aa$ for any $a \in A$ holds.*

Proof. Assume that A is an E_i -ring. Then A/LA has no nonzero left annihilators for any left ideal L of A , consequently $L \subseteq LA$ holds (see Satz 2.1 (3) from [22]). In particular, put $L = (a)_l$. Then $L \subseteq LA$ gives $a \in aA + Aa = \mathbf{R}(a)$ (see Satz 2.2 (2) from [22]).

Conversely, assume that A is not an E_i -ring. Then there exists a homomorphic image A/B of A having nonzero left annihilators of the form $a + B$ ($a \in A$). Hence one has $aA \subseteq B$ and $a \notin B$, which obviously imply $\mathbf{R}(a) = aA + Aa \subseteq B$ and $a \notin \mathbf{R}(a)$.

This completes the proof of Proposition 1.

Remarks 2. (1) Since the relationship "to be a homomorphic image" is transitive, any homomorphic image of an E_i -ring is again an E_i -ring (see Satz 2.1 (1) from [22]).

(2) One has $I = \phi_i$ for the JACOBSON radical I and for the intersection ϕ_i of all maximal left ideals of any E_i -ring (see Satz 2.2 (3) from [22]). Namely, Theorem 22.15.3 of E. HILLE [13], page 486, yields $IA \subseteq \phi_i \subseteq I$, which implies, by the validity of $L \subseteq LA$ for any left ideal L , at once $I \subseteq IA \subseteq \phi_i \subseteq I$, consequently $I = \phi_i$.

(3) The ideal ϕ_i can be considered, as a FRATTINI left ideal of A . Namely $x \in \phi_i$ holds if and only if $A = (x, T)_l$ implies $A = (T)_l$ for any subset T of A . For a survey paper on FRATTINI one-sided ideals and subgroups we refer the reader to author's paper [26].

Theorem 3. *The class \mathbf{R} of all E_i -rings is a radical class in the sense of AMITSUR and KUROSH. Any \mathbf{R} -semisimple ring A is a subdirect sum of subdirectly irreducible \mathbf{R} -semisimple rings S_α such that $H_\alpha S_\alpha = 0$ holds for the heart H_α of S_α .*

Proof. Assume $B \in \mathbf{R}$. Then, according to Proposition 1, one has $b \in \mathbf{R}(b) = bB + Bb$ for any $b \in B$. The relationship $b \in \mathbf{R}(b)$ defines a so-called F -regularity in the sense of BROWN-McCOY [7] (see yet [9]); $\mathbf{R}(b)$ being an ideal of B with $(\mathbf{R}(b))\varphi = \mathbf{R}(b\varphi)$ for any ring homomorphism φ of B . Let A be now an arbitrary ring, and let $\mathbf{R}(A)$ be the set of all elements x of A such that $y \in \mathbf{R}(y) = yA + Ay$ for any element y of the principal ideal (x) of A holds. Then $\mathbf{R}(A)$, by BROWN-McCOY [7], is a twosided ideal of A such that $\mathbf{R}(A/\mathbf{R}(A)) = 0$ holds, and $\mathbf{R}(A)$ contains, by definition, all \mathbf{R} -regular ideals of A . Therefore the class \mathbf{R} is a radical class, indeed, because \mathbf{R} is also homomorphically closed by Remark 2 (1). $\mathbf{R}(A)$ coincides, again by BROWN-McCOY [7], in any ring A with the intersection of all ideals M_α such that $S_\alpha = A/M_\alpha$ is subdirect irreducible and S_α is \mathbf{R} -semisimple. Let H_α be the heart of S_α . Then, S_α being \mathbf{R} -semisimple, an element \bar{h}_α , differing from zero, of H_α there exists such that $\mathbf{R}(\bar{h}_\alpha) = 0$ consequently $\bar{h}_\alpha S_\alpha + S_\alpha \bar{h}_\alpha S_\alpha = 0$ holds. This implies $H_\alpha S_\alpha = 0$, which completes the proof of Theorem 3.

Corollary 4. *$\mathbf{R}(B) \subseteq B \cap \mathbf{R}(A)$ holds for any ideal B of any ring A . (Cf. Theorem 4.7 of N. DIVINSKY [12].)*

Remark 5. If A is the ring of the rational integers, then for the ideal B of the even integers one has $\mathbf{R}(B) \neq B \cap \mathbf{R}(A)$ by $B \cap \mathbf{R}(A) = B$ and $\mathbf{R}(B) \neq B$.

Corollary 6. *DIVINSKY's D -regular radical [11] and the maximal von NEUMANN regular ideal, as a radical of the ring [8], are contained in the \mathbf{R} -radical.*

Corollary 7. *Any nilpotent ring is strongly \mathbf{R} -semisimple.*

Corollary 8. *Any strongly \mathbf{R} -semisimple ring is antisimple.*

Corollary 9. Any strongly R -semisimple ring with minimum condition on principal twosided ideals is a nil ring.

(Cf. V. A. ANDRUSAKIEVITCH [2], Theorem 7).

Corollary 10. Any strongly R -semisimple ring with minimum condition on all twosided ideals is nilpotent.

(Cf. V. A. ANDRUSAKIEVITCH [2], Theorem 8.)

Corollary 11. For any ring A with minimum condition on right ideals the following two requirements are equivalent:

(I) A is nilpotent.

(II) A is R -semisimple, and it has no nonzero left annihilators, contained in the intersection A^w .

(Cf. the similar Theorem 5 from N. DIVINSKY [11].)

Corollary 12. For any ring A with minimum condition on right ideals the following two requirements are equivalent:

(I) A is nilpotent.

(II) A is both R -semisimple and R' -semisimple, and it has no nonzero twosided annihilators, contained in the intersection.

(Cf. the similar Theorem 6 from N. DIVINSKY [11].)

Example 13. Let A be the algebra, generated by the elements a and b , over the field of two elements, with the table of multiplication:

	a	b
a	a	0
b	b	0

Then one has $R(A) = A \neq 0 = R'(A)$ for the radicals R and R' , defined right left dual after the explication of the aim of this paper, and occurring in Corollary 12.

Remarks 14.

(1) It would be interesting to give a ring A such that the radical $R(A)$ of A does not contain a right ideal I , which is an R -radical ring, of A .

(2) It is yet an open question the validity of $R(A_n) = (R(A))_n$ for any full matrix ring A_n .

(3) Since a discrete direct sum A of rings A_i , with right (or twosided) unity elements e_i , where i runs over an infinite set I of indices, does not have a right (or twosided, respectively) unity element, the class of all rings with right (or twosided, respectively) unity elements is not a radical class of rings in the sense of AMITSUR and KUROSHI.

(4) An interesting task would be to investigate the class of all semigroups S such that $s \in s \cup S s S$ for any $s \in S$ holds. This class of semigroups corresponds to the class of E_S -rings.

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