On minimal biideals of rings

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In this paper by a ring we always mean an associative ring (cf. N. JACOBSON [4]). For arbitrary subsets C and D of a ring A the product CD will mean the subgroup generated by all products $c \cdot d$ with $c \in C$ and $d \in D$. By a biideal B of a ring A we understand a subring B of A satisfying the condition $BAB \subseteqq B$.

Obviously, every one-sided ideal is a biideal. The biideals for semigroups are special cases of the (m, n)-ideals, introduced by S. LAJOS [5]. The concept of biideal for semigroups was introduced by R. A. GOOD and D. R. HUGHES [3] (in addition A. H. CLIFFORD—G. B. PRESTON [2]). For biideals of rings we refer the reader to [7], whose Proposition 3 asserts that for any biideal B and any subset T of a ring A the products BT and TB are again biideals of A. Biideals of rings occurred earlier also in the author's papers [9] and [10]. Obviously, any biideal B of a two-sided regular ring A is, by $B \subseteqq BA = AB = BA \cap AB = BA$, $AB = BAB \subseteqq B$, a two-sided ideal of A (cf. S. LAJOS and the author [6]). Important particular cases of biideals are the quasiideals which were studied for rings by O. STEINFELD [8]. The quasiideal Q of a ring A is in fact a submodule satisfying $QA \cap AQ \subseteqq Q$.

J. CALAIS [1] gave an example of a biideal, which is a product of two quasiideals, but which itself is not a quasiideal. But it is still an open problem whether there exists a ring A having a minimal biideal B such that $B^2 = 0$ and B is not a quasiideal of A.

In this paper we are interested in minimal biideals of rings.

Theorem 1. *If the biideal B of a ring A is a division ring, then B is a minimal biideal of A.*

Proof. Assume that C is an arbitrary biideal of A satisfying $C \subseteqq B$. Then $CAC \subseteqq C$ implies $CBC \subseteqq C$ and thus C is also a biideal of B. Consequently, by Theorem 1 of [7] C is a left ideal of a right ideal of B. But the division ring B has only trivial left ideals and right ideals, and therefore either $C = 0$ or $C = B$. Consequently B is a minimal biideal of A.

Conversely, we have also the following

Theorem 2. *For any minimal biideal B of a ring A the following holds: either $B^2=0$, or B is a division ring.*

Proof. By Proposition 3 of [7], B^2 is a biideal of A, and by the assumed minimality of B, we have either $B^2=0$ or $B^2=B$. We assume $B^2=B$ which implies $B^3=B$.

First we prove the existence of the two-sided unity element of the subring B and then we show that B is a division ring.

We note that the definition of the biideal B of a ring A implies that the set S of all elements xby (b runs over B, x and y are fixed elements of B) coincides with the subring generated by the set S.

Since $B^3=B$, there exist elements $b_1, b_2 \in B$ with $b_1Bb_2 \neq 0$. B being a subring, by Proposition 3 of [7] we have $b_1Bb_2=B$ and, by $0 \neq B=B^2=b_1Bb_2b_1Bb_2$, obviously $b_2b_1 \neq 0$, too. Since $b_1Bb_2=B$, there exist two elements $b_3, b_4 \in B$ satisfying $b_1=b_1b_3b_2$ and $b_2=b_1b_4b_2$, whence

$$0 \neq b_2b_1=b_1b_4b_2b_1b_3b_2=b_1b_4(b_2b_1)=(b_2b_1)b_3b_2 \in Bb_2b_1 \cap b_2b_1B$$

follows. $Bb_2b_1 \subseteqq B$, $b_2b_1B \subseteqq B$ being true, Propositions 1 and 3 of [7] imply that $Bb_2b_1 \cap b_2b_1B$ is also a biideal of A which is contained in B. Thus the fact $b_2b_1 \neq 0$ and the minimality of B give $B = Bb_2b_1 \cap b_2b_1B$. Consequently, there exist four further elements b_5, b_6, b_7 and b_8 of B satisfying

$$b_1=b_5b_2b_1=b_2b_1b_6 \neq 0 \quad \text{and} \quad b_2=b_7b_2b_1=b_2b_1b_8 \neq 0.$$

As for the element $e=b_5b_2b_1b_8$, since $b_1 \neq 0$ and $b_2 \neq 0$, we first observe that

$$0 \neq e=b_5b_2b_1b_8=b_1b_8=b_5b_2$$

and

$$e^2=(b_5b_2)(b_1b_8)=e \in B$$

hold. Furthermore $e=e^3 \in eBe \subseteqq B$, thus Proposition 3 of [7] and the minimality of B imply $B=eBe$.

Therefore e is the two-sided unity element of the subring B.

Let ebe be any nonzero element of eBe. Then $B'=eBe$, ebe is contained in eBe. Furthermore, since $e^3 \cdot ebe \neq 0$, by virtue of Proposition 3 of [7], and the minimality of B, B' is a nonzero biideal of A, consequently $B'=B$. Thus there exists an element $eb'e \in B$ satisfying $eb'e \cdot ebe = e$.

Therefore B is a division ring indeed, which completes the proof.

Theorem 3. *If a minimal biideal B of a ring A contains an element b such that b is neither a left divisor of zero, nor a right divisor of zero in A, then A must have a two-sided unity element.*

Proof. Evidently $b^3 \neq 0$. Then, since $b^3 \in bAb \subseteqq B$, in virtue of Proposition 3 of [7] and the minimality of B we have $bAb = B$. Hence there exists an element $a \in A$ such that $b = bab$ holds. Then for any $x \in A$ and $y \in A$, by making use of the two-sided cancelling rule concerning b, we obtain from $xb = xbab$ and $by = baby$ that $x = xba$ and $y = aby$. Consequently $e = ba$ is a right unity element and $f = ab$ a left unity element of A, therefore $e = fe = f$ is the two-sided unity element of the ring.

Theorem 4. *If R is a minimal right ideal and L a minimal left ideal of a ring A, then either $RL = 0$ or RL is a minimal biideal of A.*

Proof. Assume $RL \neq 0$. If B' is a biideal of A satisfying $0 \neq B' \subset B = RL$, then from $B' \subset RL \subseteqq R$ we conclude that $B'A \subseteqq R$. The minimality of R also implies $B'A = R$, because in the case $B'A = 0$ the biideal B' is also a nontrivial right ideal of A which is contained in R. Similarly one also has $L = AB'$ and thus the contradiction

$$B = RL = B'A \cdot AB' \subseteqq B'AB' \subseteqq B' \subset B$$

completes the proof of Theorem 4.

In some special cases the converse statement to Theorem 4 also holds. In fact we have

Theorem 5. *Any minimal biideal B of a ring A without nonzero nilpotent ideals can be represented in the form $B = RL$, where R is a minimal right ideal and L is a minimal left ideal of A.*

Proof. By virtue of $BAB \subseteqq B$ and Proposition 3 of [7] we have $BAB = B$. In fact, in case $BAB = 0$, the right ideal BA is nilpotent, consequently $BA = 0$, $B^2 = 0$, $B = 0$, which is impossible. Therefore $B = BABAB$, which, by virtue of $BABAB \subseteqq \subseteqq BA^2B \subseteqq BAB$, implies $B = BA \cdot AB$.

We shall prove that $R = BA$ is a minimal right ideal, and $L = AB$ is a minimal left ideal of A.

If R' is a right ideal of A satisfying $0 \subset R' \subset R$, then by Proposition 3 of [7] $B' = R'AB$ is a biideal of A such that $B' \subseteqq BAAB \subseteqq B$ holds. By the minimality of B we have either $B' = 0$ or $B' = B$. But $B' = 0$ implies $R'AR' \subseteqq R'ABA = B'A = 0$, $(R'A)^2 = 0$, $R'A = 0$, $(R')^2 = 0$, $R' = 0$, which is impossible. Therefore $B' = B$, consequently $B = R'AB \subseteqq R'$, $BA \subseteqq R'A \subseteqq R' \subset BA$. This is a contradiction, and thus the verification of the minimality of the right ideal $R = BA$ is complete. For $L = AB$ the proof is similar.

Theorem 6. *Any ring A without nonzero nilpotent ideals and with minimum condition on principal right ideals is a sum of minimal biideals of A.*

Proof. By [9] we have $A=\sum_\alpha R_\alpha=\sum_\beta L_\beta$, where R_α are minimal right ideals and L_β minimal left ideals of A. Then $A=A^2=\sum_{\alpha,\beta} R_\alpha L_\beta$, and Theorem 4 implies Theorem 6.

References

[1] J. CALAIS, Demi-groupes quasi-inversifs, *C. R. Acad. Sci. Paris*, **252** (1961), 2357—2359.

[2] A. H. CLIFFORD—G. B. PRESTON, *The algebraic theory of semigroups.* I, II (Providence, 1961, 1967).

[3] R. A. GOOD—D. R. HUGHES, Associated groups for a semigroup, *Bull. Amer. Math. Soc.*, **58** (1952), 624—625.

[4] N. JACOBSON, *Structure of rings* (Providence, 1964).

[5] S. LAJOS, Generalized ideals in semigroups, *Acta Sci. Math.*, **22** (1961), 217—222.

[6] S. LAJOS—F. SZÁSZ, Some characterizations of two-sided regular rings, *Acta Sci. Math.*, **31** (1970), 223—228.

[7] S. LAJOS—F. SZÁSZ, Biideals in associative rings, *Acta Sci. Math.*, **32** (1971), 185—193.

[8] O. STEINFELD, Über die Quasiideale von Ringen, *Acta Sci. Math.*, **17** (1956), 170—180.

[9] F. SZÁSZ, Über Ringe mit Minimalbedingung für Hauptrechtsideale. II, *Acta Math. Acad. Sci. Hung.*, **12** (1961) 417—439.

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