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Let A be an associative ring (with or without twosided unity element). Furthermore, let M be an A-right module. The *Frattini submodule* $\Phi(M)$ of M is the intersection of all maximal A-submodules of M, and $\Phi(M)=M$, if M does not have maximal submodules. An A-right module M is said to be *trivial*, if $MA=0$ holds. Moreover, M is called a *perfect* A-right module, if we have $MA=M$. Instances of perfect modules are the unitary modules M over rings A with two-sided unity element; then $m\cdot 1=m$ holds for $1\in A$ and for every $m\in M$.

Following A. Kertész [4], an A-submodule S of M is said to be *homoperfect*, if M/S is a perfect A-right module. In [4] the subset

$$K(M)=[m;\ m\in M,\ mA\subseteq\Phi(M)]$$

of the A-right module is introduced, and in [4] is proved, that $K(M)$ coincides with the intersection of all homoperfect maximal submodules of M. Then $K(M)/\Phi(M)$ is a trivial A-module. Author [5] has called $K(M)$ the *Kertész radical* of the module M.

Let us consider the ring A, as an A-right module A. Then the A-submodules of A are the right ideals of A. The Jacobson radical $F(A)$ of the ring A is, by N. Jacobson [3], the intersection of all modular maximal right ideals, which are obviously homoperfect submodules of the A-module A, whence one has $K_r(A)\subseteq F(A)$ for the Kertész radical $K_r(A)$ of the A-*right* module A. Theorem 22. 15. 3 of E. Hille [2, page 486] asserts

$$A\cdot F(A)\subseteq\Phi_r(A)\subseteq F(A),$$

which implies, that both of $\Phi_r(A)$ and

$$K_r(A)=[a;\ a\in A,\ aA\subseteq\Phi_r(A)]$$

are twosided ideals of A.

Author [5], [6] has proved:

(1) For a cardinality $\boldsymbol{m}$ there exist at least one ring consisting of $\boldsymbol{m}$ elements such that $K_r(A)\neq F(A)$ if and only if $\boldsymbol{m}$ is not a square-free finite

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number;

(2) $K_r(A)$ is not a radical in the sense of Amitsur and Kurosh (see N. Divinsky [1]). We have namely rings A having a nonzero ideal B such that $K_r(A)=0$, but $K_r(B)=B\neq 0$;

(3) $K(M)$ is suitable for characterizing the Jacobson semisimple right Artin rings in the class of the rings with twosided unity element and with minimum condition on principal right ideals. cf. [6].

Theorem.* *$K_r(A)=A$ holds if and only if A is a Jacobson radical ring, i.e., $F(A)=A$.*

Proof. Assume $K_r(A)=A$. Then $K_r(A)\subseteq F(A)$ yields $F(A)=A$. Therefore $K_r(A)=A$ implies $F(A)=A$.

Conversely, assume $F(A)=A$. If $\Phi_r(A)=A$, then $\Phi_r(A)\subseteq K_r(A)\subseteq F(A)$ gives $K_r(A)=A$. If $\Phi_r(A)\neq A$, then A has a proper maximal right ideals R. We shall show, that the existence of an element $x\in A$ and of a maximal right ideal R with $xA\nsubseteq R$ will yield $x\in R$, however $xA\nsubseteq R$ trivially implies $x\notin R$, and from this contradiction we will verify $xA\subseteq\phi_r(A)$ for every $x\in A$; that is $K_r(A)=A$ will be proved. Namely, by the condition $xA\nsubseteq R$ and the maximality of R we have $A=xA+R$, whence there exist elements $a\in A$ and $r\in R$ with $x=xa+r$. By $F(A)=A$ there exists an element $b\in A$ with $a+b-ab=0$, whence by $r=x-xa$ we have

$$\begin{aligned} x &= x-x\cdot(a+b-ab) \\ &= (x-xa)-(x-xa)b = r-rb\in R. \end{aligned}$$

Consequently, also $F(A)=A$ implies $K_r(A)=A$.

This completes the proof.

Remark. Prof. A. Kertész and Dr. A. Widiger have proved $K_r(N)=N$ for nil rings N.

References

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* This result solves a problem raised by Prof. A. Kertész and by Dr. R. Wiegandt.