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On right residuals in lattice ordered groupoids

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To Professor LÁSZLÓ KALMÁR on his 65th birthday

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Following G. BIRKHOFF [2] and L. FUCHS [3, p. 191] a lattice ordered groupoid, or shortly a l. o. groupoid, is defined as a groupoid G, which is at the same time also a lattice, satisfying the distributivity requirements:

(1) $$(a \cup b)\, c = a\, c \cup b\, c \quad \text{and} \quad a\,(b \cup c) = a\, b \cup a\, c$$

for every $a,\ b,\ c \in G$. The monotonity laws, i. e. $a \leqq b$ always implies $a\, c \leqq b\, c$ and $c\, a \leqq c\, b$, follow obviously from (1). Here the duals of (1) are neither assumed, nor can be derived from (1), therefore for l. o. groupoids the duality principle fails generally to be valid. If we assume that for all $a,\ b \in G$ an element $c = a : b$ there exists such that

(2) $$x \leqq c \quad \text{is equivalent to} \quad b\, x \leqq a,$$

then $c = a : b$ is called the right residual of a and b, furthermore G is said to be a right residuated l. o. groupoid. More generally, $a : b$ can be defined also in some partially ordered groupoids. The right residual generalizes the concept of the right-sided ideal quotient in the ring theory. Some properties of l. o. groupoids and of right residuals are collected in G. BIRKHOFF [2] and L. FUCHS [3].

If the l. o. groupoid G is a complete lattice and it fulfils the infinite distributive laws:

(3) $$a\,(\bigvee_\alpha b_\alpha) = \bigvee_\alpha a\, b_\alpha \quad \text{and} \quad (\bigvee_\alpha b_\alpha)\, a = \bigvee_\alpha b_\alpha a,$$

then G is called a complete l. o. groupoid.

For results on some l. o. groupoids we refer the reader e. g. to B. A. ANDRUNAKIEVICH-JU. M. RJABUKHIN [1], G. BIRKHOFF [2], L. FUCHS [3], L. LESIEUR [7], O. STEINFELD [8, 9, 10], I. V. STELLECKIY [11], E. G. ŠUL'-GEYFER [12], F. SZÁSZ [13], M. WARD [17] and M. WARD-R. P. DILWORTH [18]. For notions of lattice theory see yet G. SZÁSZ [16]; furthermore for notions of ring theory and group theory see N. JACOBSON [4] and A. G. KUROSH [5], respectively.

Obviously, an infinite union of right residuals of a complete residuated l. o. groupoid G fails generally to be again a right residual of some elements a and b of G.

On the other hand, if, in particular, G is the l. o. semigroup of all ideals (with respect to the usual lattice and semigroup operations) of an associative commutative principal ideal ring A with unity element and without divisors of zero, then any finite union of arbitrary right residuals $a_i : b_i$ ($a_i, b_i \in G$, $i = 1, 2, \ldots, n$) equals to a right residual $a : b$ for some a and b of G.

The purpose of this paper is to give a sufficient condition, which implies that any finite union of right residuals of G is again a right residual. Our result has some applications to the theory of rings, of semigroups, of groups and of (rational or algebraic) numbers.

In what follows we assume that our l. o. groupoid G satisfies yet further axioms.

Axiom 1. G contains a maximal element e and a minimal element 0, satisfying $x\,e = e\,x = x$ and $x\,0 = 0\,x = 0$ for every $x \in G$.

Remark 2. Axiom 1 seems to be essentially different that the lattice G is complete.

Proposition 3. *The l. o. groupoid G, satisfying Axiom 1, is negatively ordered, i. e.* $x\,y \leqq x \frown y$ *holds for every* $x, y \in G$.

Proof. We obtain by $x \leqq e$, $y \leqq e$ and by the monotonity laws

$$x\,y \leqq e\,y = y \quad \text{and} \quad x\,y \leqq x\,e = x,$$

consequently $x\,y \leqq x \frown y$.

Axiom 4. G is a right residuated l. o. groupoid, $x : y$ denoting the right residual of x and y.

Proposition 5. *For* $x \leqq y$ *in a l. o. groupoid satisfying Axiom 1 holds* $y : x = e$.

Proof is by Axioms 1 and 4 trivial.

Axiom 6. For any nonzero element x and nonzero element y, and arbitrary element z ($x, y, z \in G$) of the right residuated l. o. groupoid holds

$$(x : y)\,z : x = z : y.$$

Proposition 7. *For any nonzero element x, and for arbitrary element y of the l. o. groupoid, satisfying Axioms 1, 4 and 6, holds* $x\,y : x = y$.

Proof. If $x : y = u$ and $y = e$, then we have $y\,u \leqq x$, that is

$$u = e\,u = y\,u \leqq x,$$

whence by $y\,x \leq x \frown y$ it follows also $x \leq u \leq x$, consequently

$$x : y = u = x.$$

Now axiom 6 implies $z : e = x\,z : x$, but $v = z : e$ equals by $z \leq v$ and $v = e\,v \leq z$ to z, consequently one has $x\,z : x = z$, indeed.

Proposition 8. *Any l. o. groupoid satisfying the axioms 1, 4 and 6 fulfils the left cancelling rule, that is $x\,y = x\,z$ and $x \neq 0$ imply $y = z$.*

Proof. $x\,y = x\,z$, $x \neq 0$ and Proposition 7 imply $y = z$, being

$$x\,y : x = y.$$

Proposition 9. *Any l. o. groupoid G satisfying the Axioms 1, 4 and 6 has no nonzero divisors of zero, furthermore G is an Ore lattice, that is $x \neq 0$ and $y \neq 0$ always imply $x \frown y \neq 0$.*

Proof is, by Propositions 8 and 3, trivial.

Axiom 10. For every element x and y of the right residuated l. o. groupoid G holds:

$$((x \frown y) : x) \smile ((x \frown y) : y) = e.$$

Proposition 11. *For a l. o. groupoid G satisfying the Axioms 1, 4, 6 and 10 condition $x\,y = y\,x = x \frown y$ implies $x \smile y = e$.*

Proof. If $x\,y = y\,x = x \frown y$, then Proposition 7 yields

$$(x \frown y) : x = y \quad \text{and} \quad (x \frown y) : y = x,$$

whence by Axiom 10 we have $x \smile y = e$, indeed.

Proposition 12. *For the l. o. groupoid G satisfying Axiom 1 the conditions $x \smile y = e$ and $x \smile z = e$ imply*

$$x \smile y\,z = x \smile z\,y = e.$$

Proof is, by the requirements (1), by $e^2 = e$ and by the negative ordering, trivial.

Axiom 13. The lattice $\boldsymbol{L}$ of all left ideals, contained in $G^2 = G.\,G$, of the l. o. groupoid G, is an Ore lattice, that is for the nonzero left ideals L_1 and L_2, contained in G^2, of G always $L_1 \frown L_2 \neq 0$ holds.

Remark 14. Obviously, Axiom 13 is equivalent to the condition that for arbitrary nonzero elements g_1 and g_2 of G some elements f_1 and f_2 of G there exist such that $f_1\,g_1 = f_2\,g_2 \neq 0$ holds, the left ideals $G\,g_1$ and $G\,g_2$ of G being contained in $G^2 = G.\,G$.

We can now formulate our

Theorem 15. *Let G be a l. o. groupoid satisfying the Axioms 1, 4, 6, 10 and 13, containing some elements x_1, x_2, y_1 and y_2 such that*

$$x_1 \smile y_1 = e \quad \textit{and} \quad x_2 \smile y_2 = e \tag{4}$$

hold. Then there exist elements f_1 *and* f_2 *of* G *such that* $f_1 x_1 = f_2 x_2 = g$, *and that the union* $(x_1 : y_1) \cup (x_2 : y_2)$ *of the arbitrary two right residuals* $(x_i : y_i)$ $(i = 1, 2)$ *equals to the right residual* $(g : f_1 y_1 \cap f_2 y_2)$, *i. e.*

$$(5) \qquad (x_1 : y_1) \cup (x_2 : y_2) = (g : f_1 y_1 \cap f_2 y_2).$$

Proof. The negative ordering $y_1 y_2 \cup y_2 y_1 \leqq y_1 \cap y_2$ for arbitrary $y_1, y_2 \in G$ implies by Axiom 4 evidently $\overline{y_1} = (y_1 \cap y_2) : y_2 \geqq y_1$ and $\overline{y_2} = (y_1 \cap y_2) : y_1 \geqq y_2$. Furthermore, by Axiom 10 we have $\overline{y_1} \cup \overline{y_2} = e$. Now, Axiom 1 yields $u = e\,u = \overline{y_1}\, u \cup \overline{y_2}\, u$ for an arbitrary element $u \in G$, whence Axiom 6 gives at once

$$(6) \qquad u : (y_1 \cap y_2) = (u : y_2) \cup (u : y_1),$$

substituting into Axiom 6 the elements $x = y_1 \cap y_2$, $y = y_i$ $(i = 1, 2)$, $z = u$. On the other hand, by Axiom 13 there exist elements f_1 and f_2 of G for the given $x_1 \in G$ and $x_2 \in G$, such that

$$(7) \qquad f_1 x_1 = f_2 x_2 = g \neq 0$$

holds. Therefore, for $r_i = (x_i : y_i)$ $(i = 1, 2)$, we have

$$(8) \qquad r_1 \cup r_2 = (f_1 x_1 : f_1 y_1) \cup (f_2 x_2 : f_2 y_2),$$

whence the equation (6) yields immediately the desired relation (5), that is:

$$r_1 \cup r_2 = r \quad \text{with} \quad r = (g : f_1 y_1 \cap f_2 y_2) \text{ (see (7))}.$$

This completes the proof of Theorem 15, indeed.

Remarks 16. The requirements (1), Axiom 1, therefore also the negative ordering, furthermore Axiom 4 are satisfied for the l. o. groupoid G of all twosided ideals of a (not necessarily associative) ring with unity element. The nilpotent rings A, for which $A^n = 0$ for some n holds, show that a ring without unity element has such a l. o. groupoid G in which $e^2 \neq e$ for its maximum element e holds, therefore Axiom 1 is not satisfied, being now $e = A$. Furthermore, for the l. o. groupoid of all twosided ideals of nilpotent rings A, also Axiom 6 fails generally to be valid. On the other hand, the ringtheoretical direct sum $e = x \cup y \cup z$ of three division rings x, y and z shows always $uv = vu = u \cap v$ for $u = x, y$ or z and $v = x, y$ or z, being the ring e strongly regular (cf. S. Lajos-F. Szász [6]), but generally $u \cup v \neq e$ holds, consequently also Axiom 10 is generally not satisfied for l. o. groupoids of twosided ideals of rings. Finally, it can be observed that Axioms 6 and 13 are satisfied by the l. o. semigroup of all twosided ideals of an associative, commutative principal ideal integrity domain.

In what follows, we discuss some examples, which can be considered also as some corollaries for Theorem 15.

E 1. Let A be a not necessarily associative ring with Axioms 6, 10 and 13 in the l. o. groupoid of all twosided ideals of A, such that $JA = AJ = J$ for any twosided ideal J of A holds. If the ideals B_1, B_2, C_1 and C_2 of A satisfy $B_1 + C_1 = B_2 + C_2 = A$, then one has also

$$(9) \qquad (B_1 : C_1) + (B_2 : C_2) = (F_1 B_1 : F_1 C_1 \frown F_2 C_2),$$

where $F_1 B_1 = F_2 B_2 \neq 0$ for some ideals F_1 and F_2 of A. Evidently this (9) is a particular case of (5).

E 2. Let A be an associative and commutative principal ideal ring with unity element and without divisors of zero. Then $JA = AJ = J$ holds for all ideals. Furthermore $B_1 + C_1 = B_2 + C_2 = A$ for the twosided ideals B_1, B_2, C_1 and C_2 of A imply by Theorem 15 that

$$(10) \qquad ((b_1) : (c_1)) \smile ((b_2) : (c_2)) = ((f_1 b_1) : (f_1 c_1) \frown (f_2 c_2))$$

for some elements f_1 and f_2 of A, being any ideal $D = (d)$ principal in A. Let (b, c) denote the greatest common divisor of b and c, furthermore $[b, c]$ the least common multiple of b and c. On the other hand, let F_0 denote the quotient field of the ring A and $\{\ldots, b_\varkappa, \ldots\}$ the A-submodule generated by the elements $\ldots, b_\varkappa, \ldots$ of the A-module F_0.

Then, for $b_1, b_2, c_1, c_2 \in A$, the equations $(b_1, c_1) = (b_2, c_2) = A$ imply in A

$$(11) \qquad (b_1 c_2, b_2 c_1) = (b_1, b_2) \cdot (c_1, c_2),$$

furthermore in F_0 also

$$(12) \qquad \left\{\frac{b_1}{c_1}, \frac{b_2}{c_2}\right\} = \left\{\frac{(b_1, b_2)}{[c_1, c_2]}\right\}.$$

Remark 16. A consequence of (12) is that any finitely generated torsion-free commutative group of rank one is cyclic. This follows, on the other hand, also from the fundamental theorem for finitely generated commutative groups (cf. A. G. Kurosh [5]).

E 3. Let Γ be a groupoid having zero such that the l. o. groupoid G of all twosided ideals of Γ satisfies the Axioms 1, 4, 6, 10 and 13. Let us assume that for twosided ideals B_1, B_2, C_1 and C_2 of Γ the conditions

$$B_1 \smile C_1 = B_2 \smile C_2 = \Gamma$$

hold, where the lattice union $\smile$ coincides naturally with the set theoretical union. Then our Theorem 15 yields:

$$(B_1 : C_1) \smile (B_2 : C_2) = (F_1 B_1 : (F_1 C_1 \frown F_2 C_2))$$

for some twosided ideals F_1 and F_2 of the groupoid Γ.

E 4. Let now Γ be, in particular, an arbitrary group, furthermore let G be the lattice of all its normal subgroups, such that G by the building of mutual commutator subgroups $[N_1, N_2]$ generated in Γ by all commutators $n_1^{-1} n_2^{-1} n_1 n_2$ with $n_1 \in N_1$ and $n_2 \in N_2$ becomes also a l. o. groupoid. Let us assume that Axioms 1, 4, 6, 10 and 13 for G are satisfied. Then

$$B_1 \cup C_1 = B_2 \cup C_2 = \Gamma$$

imply by our Theorem 15 that

$$(B_1 : C_1) \cup (B_2 : C_2) = ([F_1, B_1] : ([F_1, C_1] \cap [F_2, C_2]))$$

with some normal subgroups $F_1, F_2 \in G$, and for the normal subgroups B_1, B_2, C_1 and C_2 of Γ holds. Here one has, by definition, for normal subgroups B and C of Γ

$$B : C = \{x;\ x \in \Gamma,\ x^{-1} c^{-1} x\ c \in B \quad \text{for any} \quad c \in C\},$$

furthermore $B \cup C$ is the normal subgroup of Γ, generated by all group products $b\,c$ with $b \in B$, $c \in C$. On the other hand $B \cap C$ is the set theoretical intersection.

E 5. Let Γ be a group, such that with the notions discussed at **E 4**, $[\Gamma, N] = N$ for any normal subgroup N of Γ holds. Furthermore, let us assume that the group Γ is a so-called „principal normal subgroup integrity domain" in the sense that for any normal subgroup N of Γ there exists an element $\gamma \in \Gamma$ such that $N = [\gamma, \Gamma]$ holds, furthermore

$$[N_1, N_2] = \{1\} \quad \text{implies} \quad N_1 = \{1\} \quad \text{or} \quad N_2 = \{1\},$$

where $\{1\}$ is the unity subgroup of Γ. Then, for the l. o. groupoid G, with operations of **E 4**, of all normal subgroups of Γ, all Axioms 1, 4, 6, 10 and 13 are satisfied. Therefore **E 5** is a particular case of **E 4**, consequently our Theorem 15 can be applied to **E 5**.

Obviously, any simple noncommutative group belongs to the class C_0, discussed at **E 5**.

Furthermore, if a group of this class C_0 is meta-abelian, that is the condition

$$(13) \qquad [\Gamma, [\Gamma, \Gamma]] = \{1\}$$

holds, then the l. o. groupoid G for Γ is, by A. G. Kurosh [5], a semigroup, because then one has $[x, [y, z]] = [[x, y], z] = 1$ for every element $x, y, z \in \Gamma$.

Remark 17. A very particular form of Theorem 15, when we have discussed some (principal) ideals of the ring J of all rational integers, has occurred ten years ago on page 130 of author's paper [14].

Finally, we point out some open problems:

P 1: Can be given a generalization of Theorem 15 for categories C, satisfying some axioms (e. g. the normal subobjects of any object of C form a set, which is a compactly generated complete lattice etc., and yet similar

axioms to Axioms 1,
[15]))?

P 2: What is a ne
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P 3: Let us inves
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axioms to Axioms 1, 4, 6, 10 and 13 hold (cf. e. g. F. Szász-R. Wiegandt [15]))?

P 2: What is a necessary and sufficient condition for a l. o. residuated groupoid G that any finite (or any infinite) union of right residuals let be again a right residual of G?

P 3: Let us investigate more in detail the „principal normal subgroup integrity domain groups" Γ, discussed at our example **E** 5, for which at the same time also $[\Gamma, N] = N$ with any normal subgroup N of Γ holds! (E. g. is the class C_0 of these groups closed, with respect to subdirect products, or to building of „trans-free images"? (cf. F. Szász-R. Wiegandt [15]). Or, what is the lower radical class determined by this class of groups?)

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