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SOME GENERALIZATIONS OF STRONGLY REGULAR RINGS I

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SHORT ELEMENTARY PROOF OF A RINGTHEORETICAL RESULT

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Ring means in this note an associative ring (cf. [1]). The subring, generated by a subset S of the ring A, will be denoted by $\{S\}$. Furthermore, $|S|$ means the cardinality of a set S, and p denotes a prime number.

Following L. Rédei [2], a ring A is called one-step nonregular, if A has divisors of zero, but every proper subring of A is without divisors of zero. The paper [2] determines all one-step nonregular rings, which here will be shortly called P-rings, in a little complicated way. Later, O. Steinfeld [3] has communicated a shorter proof for the theorem of [2]. Various characterizations of the P-rings are given by R. Wiegandt [6]. The P-rings are also P_1-rings (i.e. they satisfy $aA=aAa$ for any $a\in A$) in the sense of author's paper [5].

The aim of this note is to give a short elementary proof for L. Rédei's result:

Theoren (*cf.* [2]). *A ring A is a P-ring if and only if either $A^2=0$ such that $|A|=p$, or A is a direct sum $B\oplus C$ of some fields B and C such that $|B|=|C|=p$.*

Proof. Assume that A is a P-ring. Then there are nonzero elements a_1 and a_2 of A such that $a_1\cdot a_2=0$ and $A=\{a_1, a_2\}$. If $a_2\cdot a_1\neq 0$, then $(a_2\cdot a_1)^2=0$ yields $A=\{a_2\cdot a_1\}$, $A^2=0$ and $|A|=p$. In the case $a_2\cdot a_1=0$ we treat $\mathcal{I}=\{a_1\}\cap\{a_2\}$. If $\mathcal{I}\neq 0$, then by $\mathcal{I}^2=0$ one has $\mathcal{I}=A$, and again $A^2=0$ with $|A|=p$. If $\mathcal{I}=0$ (and $a_1\cdot a_2=a_2\cdot a_1=0$) then the direct sum $D=\{a_1\}\oplus\{a_2\}$ can be constructed. We have $D=A$. Furthermore, $S_1=\{a_1\}\oplus T_2$ and $S_2=T_1\oplus\{a_2\}$ coincide by $a_1\cdot T_2=T_1\cdot a_2=0$ with A, for any nonzero subring T_i of $\{a_i\}\,(i=1,2)$. Hence $\{a_i\}\,(i=1,2)$ cannot contain nontrivial subrings. By $a_i^2\neq 0$ and by (the shortly and elementarily proved) Lemma 1 of [4] any $\{a_i\}$ is a field such that $|\{a_i\}|=p$. Conversely, all rings mentioned in the theorem are P-rings, indeed.

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References

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