

104  
Monatshefte für Mathematik 77, 354—356 (1973)  
© by Springer-Verlag 1973

## **Rings with Radical Maximal Submodules**

By

**Ferenc Szász**, Budapest

*(Received July 13, 1972)*

In what follows, we consider only the case  $\mathbf{R}=\mathbf{J}$ , where  $\mathbf{J}$  denotes the Jacobson radical (see N. JACOBSON [3] and yet [1] and [9], respectively). We point out, that ION D. ION [2] before has discussed, for some other purposes, the particular case  $\mathbf{R}=\mathbf{B}$ .

where  $\mathbf{B}$  is the Baer McCoy lower nil radical. Finally, let us remark that some ringtheoretical properties of another radical for modules were studied in the mentioned part III of author's paper [5].

The aim of this note is to characterize the class of rings with radical maximal submodules in every module, and it will be here verified, that this class coincides with that of Jacobson radical rings. (It may be remarked, that by using some Green equivalence relations, a characterization of  $\mathbf{J}$  can be found in author's paper [7]. For another module-theoretical characterization of Jacobson's radical rings see author's note [8].) Thus we have:

**Theorem.** *For an arbitrary associative ring  $A$  the following three conditions are equivalent:*

- (1)  *$A$  is a Jacobson radical ring, that is  $A = \mathbf{J}(A)$  holds;*
- (2) *For every right  $A$ -module  $M$  every maximal submodule  $N$  of  $M$  is  $\mathbf{J}$ -radical, that is  $\mathbf{J}(N) = A$ ;*
- (3) *For every right  $A$ -module  $M$  every homoperfect maximal submodule  $N$  of  $M$  is  $\mathbf{J}$ -radical.*

*Proof* is cyclic. (1) implies (2), as follows. Since from  $A = \mathbf{J}(A)$  it follows also  $\mathbf{J}(A/(N:M)) = A/(N:M)$ ,  $\mathbf{J}$  trivially being homomorphically closed, we have (2), indeed.

(2) trivially implies (3).

But also (3) implies (1), as follows. Let us assume (3), and let  $M$  be, in particular,  $A$  itself, as a right  $A$ -module. Now, the assumption  $A \neq \mathbf{J}(A)$  will yield a contradiction, which will prove  $A = \mathbf{J}(A)$ . Namely, if  $A \neq \mathbf{J}(A)$  then  $A$  has a nonzero primitive homomorphic image  $B = A/P$ , where  $P$  is a primitive ideal of  $A$ . Consequently there exists a modular maximal right ideal  $R$  of  $A$  such that

$$P = R:A = [x; x \in A, Ax \subseteq R]$$

holds. But the modularity of right ideals implies also their homoperfectness, hence, by condition (3),  $R$  is a  $\mathbf{J}$ -radical submodule of  $A$ . Therefore

$$\mathbf{J}(A/(R:A)) = \mathbf{J}(A/P) = A/P.$$

On the other side, the primitive  $B = A/P$  is also  $\mathbf{J}$ -semisimple, which implies  $\mathbf{J}(A/P) = P/P$ , contradicting to  $P \neq A$ . Hence (3)  $\Rightarrow$  (1) is proved.

This completes the proof.



*Remark.* However the Jacobson radical is supernilpotent in the sense of V. A. ANDRUNAKIEVICH (see e. g. N. DIVINSKY [1]), the lower radicals, generated by the classes of Neumann regular rings or of strongly regular rings, respectively, and the lower radicals, generated by the primitive classes  $J_n$  of the Jacobson-Herstein-Andrunakievich-Stewart rings (that are rings for which exists a fixed  $n$  such that  $x^n = x$  holds for every element of the ring) almost trivially are subidempotent in the sense of V. A. ANDRUNAKIEVICH. (For these latter classes of rings see yet author's paper [10] and its footnote on page 169.) The Neumann regular rings, the strongly regular rings and the  $J_n$ -rings obviously are Jacobson semisimple.

#### References

- [1] DIVINSKY, N.: Rings and Radicals. London. 1965.
- [2] ION, ION D.: O prezentare simpla a teoriei descompunerilor primare in neocomutativ. Analele Univer. Bucuresti, Ser. Matemat. **16**, 109—112 (1967).
- [3] JACOBSON, N.: Structure of Rings, 2. ed., Providence. 1964.
- [4] KERTÉSZ, A.: Vizsgálatok az operatormodulusok elméletében, III. Magyar Tudományos Akadémia III. Osztályának Közleményei **9**, 105—120 (1959), (in Hungarian).
- [5] SZÁSZ, F.: Notes on modules, I, II, III. Proc. Japan Acad. **46**, 349—350, 351—353, 354—357 (1970).
- [6] SZÁSZ, F.: Das im Operatorring enthaltene allgemeine Radikal eines Untermoduls. Acta Sci. Math. Szeged (to appear).
- [7] SZÁSZ, F.: Äquivalenzrelation für eine Charakterisierung des Jacobson'schen Radikals. Acta Math. Acad. Sci. Hungar. **22**, 85—86 (1971).
- [8] SZÁSZ, F.: Rings, which are radical modules. Math. Japonicae (to appear).
- [9] SZÁSZ, F.: Radikale der Ringe. Budapest (Akadémiai Kiado, to appear).
- [10] SZÁSZ, F.: A class of regular rings. Monatsh. Math. **75**, 168—172 (1971).

Author's address:

F. A. Szász

Mathematisches Institut der Ungarischen Akademie der Wissenschaften  
Realtanoda u. 13—15

Budapest 5, Ungarn