

On Generalized Subcommutative Regular Rings

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All rings discussed in this note are associative. For the notions used here, we refer to N. DIVINSKY [3], N. JACOBSON [5] and N. H. MCCOY [8]. For arbitrary subsets C and D of a ring A we mean by the product $C \cdot D$ the additive subgroup of A^+ generated by the set of all products $c \cdot d$ with $c \in C$ and $d \in D$. By a generalized biideal B of a ring A we understand an additive subgroup B such that $BA B \subseteq B$ holds. If a generalized biideal is a subring, then it is called a biideal. For properties of these notions see S. LAJOS—F. SZÁSZ [7] and the author's papers [15]. Following D. BARBILIAN [2] a ring A is said to be subcommutative if $aA = Aa$ holds for every $a \in A$. A ring A is called regular in the sense of J. NEUMANN [9] if we have $a \in aAa$ for every $a \in A$. Following R. F. ARENS—I. KAPLANSKY [1], a ring A is said to be strongly regular if $a \in a^2A$ holds for every $a \in A$. Regular rings and strongly regular rings play an important role also in the theory of Banach algebras (cf. C. E. RICKART [10]).

Any subcommutative regular ring is, by S. LAJOS—F. SZÁSZ [6], [7], strongly regular. Furthermore, any strongly regular ring is regular, but there exist regular rings which fail to be strongly regular. Regular and strongly regular rings are characterized, by strings of equivalent conditions, in the second part of the author's paper [15] and in S. LAJOS—F. SZÁSZ [6], [7]. A particular class of strongly regular rings, which first was investigated by N. JACOBSON [4], is characterized by five equivalent conditions in the author's paper [14]. For further properties of these strongly regular rings, in which for every $a \in A$ there exists an exponent $n = n(a) \geq 2$ such that $a^{n(a)} = a$ holds, see also I. SUSSMANN—A. L. FOSTER [12]. These particular strongly regular rings are unions of homomorphically closed semisimple classes for some radicals, which are also radical

classes for other radicals. (See P. N. STEWART [11] and „Satz 1“ from the author's paper [13].)

We say that a ring A is a generalized subcommutative regular ring, or shortly a *gsr-ring*, if for every element $a \in A$ there exists an exponent $n = n(a) \geq 2$ such that $aAa = a^n A a^n$ holds. Obviously aAa is a biideal of A .

We shall study in this note some properties of the *gsr-rings*. Among other things it will be shown here that the class of *gsr-rings* without nonzero nilpotent elements coincides with the class of all subcommutative regular rings. On the other side the class of all *gsr-rings* is properly larger than the class of all subcommutative regular rings.

Proposition 1. *Every homomorphic image of a gsr-ring is again a gsr-ring.*

The proof follows immediately from the definition.

Proposition 2. *For every non-nilpotent element a of a gsr-ring A there exists an element $b \in A$ such that $e = a^3 b$ and $f = b a^3$ are nonzero idempotent elements with $a^3 = e a^3 = a^3 f$.*

Proof. If $aAa = a^n A a^n$, then since $n \geq 2$ there exists a natural number k such that $kn - k \geq 2$. Therefore, by $aAa = a^{kn-k+1} A a^{kn-k+1}$ we may assume $n \geq 3$. Then there exists an element $c \in A$ with $a^3 = a^n c a^n$, hence $b = a^{n-3} \cdot c \cdot a^{n-3}$ satisfies $a^3 = a^3 b a^3$. Obviously, $e = a^3 b$ and $f = b a^3$ are idempotent.

Proposition 3. *If $a^3 = a^3 b a^3$, then also $g = a b a^3 b a^2$ and $h = a^2 b a^3 b a$ are idempotent.*

The proof is trivial.

Proposition 4. *If a is a nilpotent element of a gsr-ring A , then $a^3 = 0$ holds, and a generates a nilpotent ideal of A .*

Proof. Assume $a^m = 0$ with an exponent $m \geq 2$. If $aAa = a^n A a^n$ holds with $n \geq 2$, then we have $kn - k + 1 \geq m$ for a suitable k , hence $aAa = a^{kn-k+1} A a^{kn-k+1}$ implies $aAa = 0$, $a^3 = 0$ and $(aA)^2 = 0$. Since the square $(a)_r^2$ of the principal right ideal $(a)_r$ is contained in aA , one has $(a)_r^4 = 0$. Consequently also the principal ideal (a) is nilpotent.

Theorem 5. *Any prime gsr-ring A is a division ring.*

Proof. Since A is prime, it does not have, by Proposition 4, nonzero nilpotent elements a , the ideal (a) being nilpotent. Therefore A has no nonzero divisors of zero. If A is a nonzero element of A ,

then $e = a^3 b$ is, by Proposition 2, for a suitable $b \in A$, an idempotent of A . Namely, $e^2 = e$ and $ye = y$ for every x, y element of A , hence $e = 1$.

Theorem 6. *The class of gsr-rings without nonzero nilpotent elements forms an ideal, which coincides with Köthe's and Baer's class of rings with the Brown-McCoy radical.*

Proof. Let N be the Brown-McCoy radical of A . Then N contains the upper nilradical of A , with the sum $B \subseteq U \subseteq N$ also with the Brown-McCoy radical of A , then N contains all elements or nonzero elements implies $J = B$. On the other side N is a subdirect sum of prime rings, which are division rings. Therefore N is the Brown-McCoy radical T .

This completes the proof.

Theorem 7. *If A is a gsr-ring whose characteristic is not 2, then the Baer-McCoy radical holds for the Baer-McCoy radical.*

Proof. Theorem 6 implies that the Baer-McCoy radical T is contained in N , hence the Baer-McCoy radical T is contained in N . Appendix C, page 2.

Proposition 8. *If $a^3 = a^4 b a^2$, then $a^3 = a^4 b a^2$.*

Proof. By $a^3 = a^4 b a^2$ whence $(a - a^4 b)^4 = 0$ consequently $a^3 = a^4 b a^2$.

Theorem 9. *For a gsr-ring A the following conditions are equivalent:*

- (i) A is a gsr-ring
- (ii) A is a subdirect sum of division rings
- (iii) A is a strongly gsr-ring

[11] and „Satz 1“

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Theorem 6. *The set of all nilpotent elements of a gsr -ring A forms an ideal, which coincides with Baer's lower radical, with Köthe's and Baer's upper nil radical, with the Jacobson radical, with the Brown—McCoy radical and also with Thierrin's corpoidal radical.*

Proof. Let N be the set of all nilpotent elements of A . Then N contains the upper nil radical U , and N coincides, by Proposition 4, with the sum of all nilpotent ideals of A , and hence by $B \subseteq U \subseteq N$ also with Baer's lower nil radical B . If J is the Jacobson radical of A , then J/B cannot contain either nonzero idempotent elements or nonzero nilpotent elements, whence Proposition 3 implies $J = B$. On the other hand, A/B is, by Proposition 1, a subdirect sum of prime gsr -rings, each of which is, by Theorem 5, a division ring. Therefore $N = B = U = J$ coincides with the Brown—McCoy radical G and also with Thierrin's [16] corpoidal radical T .

This completes the proof.

Theorem 7. *If A is a gsr -algebra over a (commutative) field whose characteristic is zero or a prime number $p \geq 5$, then $T^7 = 0$ holds for the Baer—Jacobson—Thierrin radical T of A .*

Proof. Theorem 6 and Proposition 4 yields $t^3 = 0$ for every $t \in T$, hence the Nagata—Higman theorem (see N. JACOBSON [5], Appendix C, page 274) implies Theorem 7.

Proposition 8. *In a gsr -ring $a^3 = a^3ba^3$ implies $a^3 = a^2ba^4$ and $a^3 = a^4ba^2$.*

Proof. By $a^3 = a^3ba^3$ we obviously have $(a - a^4b)^2 = a^2 - a^4ba$ whence $(a - a^4b)^4 = 0$. Now, Proposition 4 yields $(a - a^4b)^3 = 0$, consequently $a^3 = a^4ba^2$. Similarly one proves also $a^3 = a^2ba^4$.

Theorem 9. *For an arbitrary ring A the following three conditions are equivalent:*

- (i) A is a gsr -ring without non-zero nilpotent elements;
- (ii) A is a subcommutative regular ring;
- (iii) A is a strongly regular ring.

Proof. Since the equivalence of conditions (ii) and (iii) was proved by S. LAJOS—F. SZÁSZ [6], [7] (for this see also Theorem 2 of the second part of the author's paper [15]), it is sufficient to show here only the equivalence of (i) and (iii).

Assume that A is a ring with condition (i). Then by Proposition 2, for every non-zero element a there exists an element b such that $a^3 = a^3 b a^3$ holds, whence a direct calculation yields

$$(a - a^4 b)^4 = (a - b a^4)^4 = 0.$$

But condition (i) implies $a = a^4 b \in a^2 A$ and $a = b a^4 \in A a^2$, which means that (i) implies (iii), indeed.

Conversely, assume that A satisfies condition (iii). By Proposition 2 of the second part of the author's paper [15], every generalized biideal of a strongly regular ring is a two-sided ideal. Furthermore, by condition (iii) we have $a A a = a^2 A = a^4 A = a^2 A a^2$. Therefore, since A is without nonzero nilpotent elements, (iii) implies (i) as well.

This completes the proof.

Remarks 10. (1) Any ring A satisfying $a A a = 0$ for every $a \in A$ is a *gsr*-ring which is not strongly regular. $A^3 = 0$ is a particular case of this condition.

(2) Any discrete direct sum of division rings is a subcommutative regular ring.

(3) The ring of all linear transformations of a vector space of dimension ≥ 2 over a division ring is regular but not subcommutative.

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