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It is well known, that among the various concrete radicals of associative rings the Jacobson radical seems to be the most useful for investigation of the structure of rings (see N. Jacobson [2]). On the other side, for much structure theorems for rings, using subdirect sums and some concrete radicals, the so-called F-radicals by B. Brown and N. H. McCoy [1], the subdirectly irreducible rings play an important rôle. Among these rings the simple rings (i. e. the rings A with $A^2=A$ and without nontrivial ideals) are of extraordinary significance.

It is well known (cf. author [7]), that every simple Jacobson radical ring with minimum condition on principal right ideals is the ring (O), since it satisfies, by author [7] necessarily $A^2=(O)$, whence $A^2=A$ implies $A=(O)$, indeed. E. Sasiada [6] has constructed a nontrivial simple Jacobson radical ring. Jacobson's famous problem asks:

Does there exist nontrivial simple nil rings?

This problem is yet unsolved, but some interesting properties of these rings were asserted by W. A. McWorter [5]. For some characterizations of the Jacobson radical we refer yet e. g. to Chapter 5 of the book of A. Kertész [3], Chapter 4 of the book of author [9] and author [8]. For some properties of simple Jacobson radical rings we refer to author [10].

According to A. G. Kurosh's book [4], every group G, written multiplicatively, can be embedded into a group $\overline{G}$ such that every two elements $x, y \in \overline{G}$, which are different from $1 \in \overline{G}$, also are conjugate in $\overline{G}$, that is there exists an element $z \in \overline{G}$ satisfying $x = z^{-1}yz$

Considering this grouptheoretical property, we understand by an Ω-ring A a Jacobson radical ring, all nonzero elements of which are quasiconjugate, i. e. if $x \neq 0$, $y \neq 0$, and $x, y \in A$, then there exists an element $z \in A$ such that

$$(*) \qquad x = (1-z)y(1-z)^{-1}$$

Proposition 1. *Every Ω-ring satisfying $A^2=A$, is a simple Jacobson radical ring.*

Proof. It is sufficient, by the assumed Jacobson radical property, only shown the simplicity of A, which follows from $A^2=A$ and from $x \in (y)$, the principal

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ideal generated by y in A, since $(1-z)^{-1}$ has obviously a form $1-w$ with $w \in A$, and $z+w-zw=0$.

Proposition 2. *Every Ω-ring, which satisfies $A^2=A$, is an algebra over a prime field F_p ($p=0$ or a prime number).*

Proof is trivial, since either $pA=0$ for a suitable prime number p, or $pA=A$ holds for every p. In latter case A is of characteristic 0.

Proposition 3. *Every Ω-ring, which contains nonzero nilpotent elements, satisfies $A^2=0$ and it consists only of two elements: $A=[0, a]$ with $a^2=a+a=0$.*

Proof. If A has nonzero nilpotent elements, then there exists an $a \neq 0$ with $a \neq 0$, whence by (*) every element of A has a square zero, and thus $bc=-cb$ holds for every b, $c \in A$. Furthermore $(1-z)^{-1}=1+z$, whence (*) and $zy=-yz$, $z^2=0$ imply

$$x=(1-z)y(1+z)=y+2yc \in (y)_r,$$

the principal right ideal generated by y in A. Thus A coincides with the minimal right ideal $(y)_r$, which satisfies $A^2=0$. This implies $|A|=2$, indeed.

Proposition 4. *Every Ω-ring without nonzero nilpotent elements does not contain nonzero divisors of zero.*

Proof. Assume $xy=0$ and $x \neq 0$. Then, by our assumption, $(yAx)^2=0$ and $(yx)^2=0$ we have $yAx=yx=0$ and thus also $xAy=0$. This yields:

$$(x).(y)=(y).(x)=0,$$

which by the definition at (*) of Ω-rings implies in case $y \neq 0$ by $(x)=(y)$ the contradiction $x^2=0$, $x=0$.

Theorem 5. *If A is an Ω-ring without nonzero divisors of zero, then every its element is transcendental over the prime field F_p belonging to A (according to Proposition 2). Furthermore, for every polynomial $f(x) \in x, \mathfrak{J}[x]$ (here $\mathfrak{J}$ denotes the ring of rational integers) and for every nonzero element $a \in A$ there exists an element b, which satisfies $a=f(b)$ $(b \in A)$*

Proof. By Chapter I of N. Jacobson [2] the elements of the Jacobson radical of an algebra over a field are either nilpotent or transcendental over F_p, but our assumption excludes the nilpotent elements. Thus A is absolutely transcendental over F_p. Therefore, for every polynomial $f(x) \in x. \mathfrak{J}[x]$ and every nonzero element $a \in A$ we have $f(a) \neq 0$. According to the definition at (*) there exists an element $c \in A$ such that $f(a)=(1-c)a(1-c)^{-1}$. Let us define the desired element $b \in A$ by $b=(1-c)^{-1}a(1-c)$. Then

$$\varphi: x \longrightarrow (1-c)^{-1}x(1-c)$$

where $x \in A$, is an automorphism of the ring, whence

$$f(b)=f(\varphi a)=\varphi(f(a))=(1-c)^{-1}f(a)(1-c)=a$$

holds, indeed.

Corollary 6. ("Fermat's strong theorem", for Ω-rings). *Let A be an Ω-ring without nonzero nilpotent elements, furthermore let $n \geqq 3$ be a natural number and a and b be arbitrary elements of A such that $a^n+b^n \neq 0$. Then there exists an element $c \in A$ such that $a^n+b^n=c^n$ holds.*

Proof. By our assumption, Proposition 4 and Theorem 5, the algebra A over F_p is absolutely transcendental. Put $f(x)=x^n \in x.\mathfrak{J}[x]$, where $n \geqq 3$. If $a^n+b^n \neq 0$, then an application of Theorem 5 to this case completes the proof.

Remark 7. The assertion of Corollary 6 and its proof are true naturally also for $n=2$, and the case $n=1$ is trivial, but these cases naturally differ from Fermat's conjecture for the ring $\mathfrak{J}$.

Proposition 8. *In every Ω-ring A without nonzero nilpotent elements the centralizer C_x of an arbitrary nonzero element $x \in A$ is properly larger than the subring $\{x, x'\}$ generated by x and x', where $x' \in A$ and $x+x'-x.x'=0$.*

Proof. We have $(1-x)(1-x')=1$. Furthermore, there exists an element $z \in A$ with $x'=(1-z)x(1-z)^{-1}$. Therefore $(1-x')(1-(1-z)x'(1-z)^{-1})=1$, whence, by the unicity of quasiinverses, follows

$$x=(1-z)x'(1-z)^{-1},$$

consequently $x=(1-z)^2x(1-z)^{-2}$ and $x'=(1-z)^2x'(1-z)^{-2}$ In other words, $c=z^2-2z$ belongs to the centralizers of the subrings $\{x\}$ and $\{x'\}$, but, by the absolute transcendence of A over F_p holds $x \notin \{x'\}$ and $x' \notin \{x\}$, furthermore $c=z^2-2z \notin \{x, x'\}$. Therefore, the centralizer C_x of x is properly larger, than $\{x, x'\}$, indeed, since $C_x \supseteq \{x, x'\}$

Problem. Does there exist an absolutely transcendental Ω-ring?

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