121

COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
6. RINGS, MODULES AND RADICALS. KESZTHELY (HUNGARY), 1971

An almost subidempotent radical property of rings

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All rings, considered in this paper, are associative (with or without unity element). Unity element and ideal here always mean two sided ones. For arbitrary subsets B and C of a ring A, the product $B \cdot C$ will denote the additive subgroup, generated by all products bc with $b \in B$ and $c \in C$ of A. We denote the intersection of the powers A^n, taken for all natural n, by A^{ω}. For every element $a \in \in A$, the product $(a)_l \cdot A$ of the principal left ideal $(a)_l$ and of A, will be denoted by $R(a)$.

Then one has $R(a) = aA + AaA$, which is obviously an ideal of A. Generally, the condition $a \in R(a)$ does not hold for an element a of an arbitrary ring A. It can be easily shown, that $a \notin R(a)$ holds if and only if the homomorphic image $H = A/R(a)$ of A has a nonzero left annihilator $a + R(a)$.

The class $\boldsymbol{C}_5$ of all rings (formerly called by the author in [25] E_5-rings), such that every homomorphic image has no nonzero left annihilators, is suitable to give a sequence of criteria for the existence of the unity element of a ring. For this sequence see Sätze 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.2.1, 3.2.2, 3.3.1, 3.3.2, 3.4.1 and 3.4.2

Following de la Rosa [22, page 13], an ideal Q of the ring A is said to be *quasi-semi-prime*, if $A \cdot I \cdot A \subseteq Q$ implies $I \subseteq Q$ for every ideal I of A. The semi-prime ideals, which are arbitrary intersections of prime ideals of A, are instances for quasi-semi-prime ideals; calling an ideal P prime in A, if $BC \subseteq P$ implies $B \subseteq P$ or $C \subseteq P$ for arbitrary ideals B and C of A. Furthermore, a ring A is called in [22] a λ-ring, if all its ideals are quasi-semi-prime in A. Let $\mathbf{\Lambda}$ denote the class of all λ-rings. Theorem 5.1 of de la Rosa [22] asserts, that a ring A is a λ-ring if and only if $a \in AaA$ holds for every $a \in A$, whence one obviously has:

$$\mathbf{\Lambda} \subseteq \boldsymbol{C}_5 .$$

Consequently, every λ-ring A is idempotent, but in general it is not strongly idempotent.

A common generalization of the E_5-rings and of the λ-rings are the ***F**-regular rings* in the sense of B. Brown and N.H. McCoy [10]. These authors further generalized the $\boldsymbol{F}$-regularity for some (noncommutative) groups with operators. For this see B. Brown – N.H. McCoy [11]. To define the $\boldsymbol{F}$-regularity of a ring A, assume, that A satisfies the following conditions:

(1) There exists a mapping $a \to \boldsymbol{F}(a)$ of the set of all elements a of A into the set of all ideals of A (i.e. $\boldsymbol{F}(a)$ is here an ideal of A).

(2) $\boldsymbol{F}(a\varphi) = (\boldsymbol{F}(a))\varphi$ holds for every $a \in A$ and for every (ring-theoretical) homomorphism φ of A.

Now, a ring A is said to be $\boldsymbol{F}$-regular, if $a \in \boldsymbol{F}(a)$ holds for every $a \in A$. The E_5-rings (and the λ-rings, too) are evidently F-regular for $\boldsymbol{F}(a) = \boldsymbol{R}(a)$ (or $\boldsymbol{F}(a) = AaA$, respectively). But the Brown – McCoy $\boldsymbol{G}$-radical rings [10] are also $\boldsymbol{F}$-regular for

$$\boldsymbol{F}(a) = (1-a)A + A(1-a)A ,$$

where we use the following notation, even for $1 \notin A$:

$$(1-a)A = [x - ax \,;\; x \in A] .$$

A *radical property of rings* in this paper is always understood in sense of S.A. Amitsur [1] and A.G. Kuroš [20]. For this notion also see the good elaboration of the theory in the book of N. Divinsky [14].

Moreover, A is idempotent, however it is $\hat{\boldsymbol{R}}$-semisimple.

Following V.A. Andrunakievič [3], a ring A is called *antisimple*, if it cannot be homomorphically mapped onto a subdirectly irreducible ring $\boldsymbol{S}$ having an idempotent heart, H, which is the nonzero intersection of all nonzero ideals of $\boldsymbol{S}$. Every nilpotent ring is obviously antisimple. The class of all antisimple rings forms a radical class, which is supernilpotent. Let us mention, a famous unsolved problem: Does the antisimple radical in every ring contain the upper nil radical of G. Koethe?

Following V.A. Andrunakievič [4], a ring A is said to be strongly $\boldsymbol{T}$-semisimple for a radical $\boldsymbol{T}$, if every homomorphic image of A is $\boldsymbol{T}$-semisimple. An example for a strongly $\boldsymbol{T}$-semisimple ring is every simple ring, which is also $\boldsymbol{T}$-semisimple. A. Suliński [23] in his fundamental paper has characterized the [strongly Brown – McCoy semisimple rings, with the aid of an] interesting system of invariants, using also topological methods.

The author [29] has explicity given supernilpotent radicals $\boldsymbol{S}$ such that the class of all $\boldsymbol{S}$-semisimple rings is homomorphically closed. If $\boldsymbol{C}$ is a radical class of rings for a radical $\boldsymbol{S}$ such that $\boldsymbol{C}$ is also a semisimple class for another radical $\boldsymbol{T}$, then $\boldsymbol{C}$ is called a semisimple radical class, which must obviously be also homomorphically closed. Trivial instances for semisimple radical classes are: (1) the class of all rings and (2) the class containing only the ring $A = \{0\}$. All nontrivial semisimple radical classes of rings were explicitly determined by P.M. Stewart [24]. It is surprising, that P.M. Stewart's classes essentially coincide with the examples of the author [29]. A characterization of the union of these classes, with the aid of five equivalent conditions, has been given recently by the author [28].

Let us mention, that if the class of all $\boldsymbol{T}$-semisimple rings is homomorphically closed for a radical $\boldsymbol{T}$, then the mapping:

$$I \to \boldsymbol{T}(I) ,$$

where I is an arbitrary ideal of an alternative or associative ring A, and $\boldsymbol{T}(I)$ denotes the $\boldsymbol{T}$-radical of the ring I, is a join-endomorphism ([26]) of the lattice of all ideals of A, that is, we always have:

$$\boldsymbol{T}(I_1 + I_2) = \boldsymbol{T}(I_1) + \boldsymbol{T}(I_2) ,$$

which fails to be correct for every radical without the condition on homomorphically closedness.

Proof. Let us assume $A \in \boldsymbol{C}_5$. Then, by [25], we have $a \in \boldsymbol{R}(a) = aA + AaA$ for every $a \in A$. Furthermore, $\boldsymbol{R}(a\varphi) = (\boldsymbol{R}(a))\varphi$ holds for every $a \in A$, and for every homomorphism φ of the ring A onto another ring. Therefore A is F-regular for $F(a) = R(a)$ in the sense of B. Brown – N.H. McCoy [10].

Let C be an arbitrary ring (for which $C \in \boldsymbol{C}_5$ or $C \notin \boldsymbol{C}_5$ holds). Let us consider, following B. Brown – N.H. McCoy [10], the set $\boldsymbol{R}(C)$ of all elements $c \in C$ such that every element d of the principal ideal (c) of C is F-regular in C, i.e. one has

$$d \in \boldsymbol{R}(d) = dC + CdC, \quad \text{for every} \quad d \in (c)\,.$$

Then $\boldsymbol{R}(C)$ is an ideal, which contains every $\boldsymbol{F}$-regular ideal of C for $\boldsymbol{F}(x) = \boldsymbol{R}(x)$. Also $\boldsymbol{R}(C/\boldsymbol{R}(C)) = 0$ can be proved. Therefore $\boldsymbol{C}_5 = \boldsymbol{R}$ is a radical class, and $\boldsymbol{R}(A)$ is an $\boldsymbol{F}$-regular ideal.

Obviously, $H \in \boldsymbol{L}_3$ holds for a homomorphic image H of an arbitrary ring A if and only if there exists an element $a \in A$ satisfying $a \notin \boldsymbol{R}(a)$. Now, if B is a nonzero $\boldsymbol{R}$-semisimple ring then B is, by the $\boldsymbol{F}$-regularity of $\boldsymbol{R}$, a subdirect sum of nonzero subdirectly irreducible $\boldsymbol{R}$-semisimple rings S_a. But this condition for S_a is, by [10], equivalent to the existence of a nonzero element $h \in H_a$, satisfying $\boldsymbol{R}(h) = 0$, where H_a is the heart of S_a. Obviously $hS_a + S_a h S_a = 0$ implies $H_a \cdot S_a = 0$, which completes the proof.

Proposition 3. *Denote by $\Phi_l(A)$ for a ring A the left Frattini submodule of the A-left module A, i.e. the intersection of all maximal left ideals of A, or $\Phi_l(A) = A$, if a does not have maximal left ideals. Then $\Phi_l(A) = J(A)$ holds for the Jacobson radical $J(A)$ of every $\boldsymbol{R}$-radical ring A.*

Proof. By E. Hille [16, Theorem 22.15.3, page 486], we have $J(A) \cdot A \subseteq \Phi_l(A) \subseteq J(A)$, whence $\Phi_l(A)$ is a two-sided ideal of A.

Furthermore, $A/\Phi_l(A)$ does not have nonzero left annihilators, whence $J(A) \subseteq \Phi_l(A)$ which implies $J(A) = \Phi_l(A)$.

Remark 4. $\boldsymbol{R}(A) = 0$ and $\Phi_l(A) = 0 \neq A = J(A)$ hold for any ring A consisting of p elements, with $A^2 = 0$ where p is a prime number.

Proposition 5. *The radical properly $\boldsymbol{R}$ is almost subidempotent, but it is not subidempotent.*

Remark 10. In corollary 9 we have considered rings satisfying minimum conditions for principal or for all ideals. In what follows, we will discuss some $\boldsymbol{R}$-semisimple rings with minimum condition on *right* ideals, i.e. some $\boldsymbol{R}$-semisimple *right* Artinian rings.

Proposition 11. *For an arbitrary right Artinian ring A the following two conditions are equivalent:*

(I) *A is nilpotent;*

(II) *$\boldsymbol{R}(A) = 0$ holds (i.e. A is $\boldsymbol{R}$-semisimple) and A has no nonzero left annihilators, contained in the intersection A^ω.*

Proof. (I) implies (II). If A is nilpotent, then, by Proposition 7 $\boldsymbol{R}(A) = 0$ holds, and $A^n = 0$ for an exponent n evidently implies $A^\omega = 0$, consequently we have condition (II).

Conversely, condition (II) implies (I). Let us assume condition (II), for the right Artinian ring A. By E. Artin – C. Nesbitt – R.M. Thrall [6 Theorem 9.3 C, page 100], we have for A the additively direct decomposition:

$$A = e_1A + e_2A + \ldots + e_mA + N_1 ,$$

where the right ideals e_iA (with $e_i^2 = e_i$ for $i = 1, 2, \ldots, m$), are directly indecomposable and the right ideal N_1 of A is nilpotent. Therefore N_1 is contained in the nilpotent Jacobson radical $N = J(A)$ of A, i.e. $N_1 \subseteq J(A)$ holds. We shall prove $e_i = 0$ for every i $(1 \leqslant i \leqslant m)$, as follows, which will imply $A = N_1 = N = J(A)$, i.e. condition (I) will be derived from (II).

By condition (II) we have $\boldsymbol{R}(A) = 0$, and the assumption $e_1 \neq 0$ yields $e_1 \notin \boldsymbol{R}(A)$.

Now, we shall use four well-known assertions, which can be easily verified, (see e.g. R. Baer [7], N. Divinsky [13]) to finish the proof of Proposition 11:

(1) If we have $a \in aA$, $b \in bA$ and $x \in xA$ for every $x \in bA$ in an arbitrary ring A, and then $a + b \in (a + b)A$ holds.

Namely, starting from $a = a \cdot a_1$ and $b = b \cdot b_1$, we define $c = b(b_1 - a_1)$. Then $c \in bA$ holds, whence our assumption in (1) implies $c = c \cdot c_1$ with an element $c_1 \in A$. If $d = c_1 + a_1 - a_1 \cdot c_1$, then $ad = a$ and $bd = b$ yield $a + b = (a + b)d \in (a + b)A$.

no left annihilator of A. By $e_1 bA \subseteq N = J(A)$ we have $e_1 bA \subseteq e_1 N \neq 0$. Now, A being a right Artinian ring, there exists an exponent n such that $e_1 N^n \neq 0$, but $e_1 N^{n+1} = 0$ holds for the Jacobson radical N of A.

Let c be an arbitrary element of N^n such that $e_1 c \neq 0$. Let $e_1 c$ be denoted by d. Then $dN \subseteq e_1 N^n \cdot N = 0$ holds, but $d \neq 0$. Since we have the inclusions $d = e_1 c = e_1^{k-1} c \in A^k$, for every k, condition (II) evidently implies $dA \neq 0$. Now $dN = 0$ yields $dN_1 = 0$, whence, by $dA \neq 0$, one has $de_i A \neq 0$ for at least one i.

Now, we shall verify, that $de_i A$ is a minimal right ideal of A. Let us assume the existence of a right ideal R of A such that $0 \neq R \subseteq de_i A$ holds. Then we define the set

$$S = [x \colon x \in e_i A, \quad dx \in R] .$$

Obviously, S is a right ideal of A such that $S \subseteq e_i A$ holds. Assuming $S \neq e_i A$, the directly indecomposable property of $e_i A$ yields by E. Artin – C. Nesbitt – R.M. Thrall [6] at once $S \subseteq N$, whence we have $dS \subseteq dN = 0$. If r is an arbitrary element of R, then $R \subseteq de_i A$ implies $r = de_i a$ with an element $a \in A$, and the definition of S yields $e_i a \in S$, which implies $r = de_i a \in dS = 0$ and $R = 0$, contradicting $R \neq 0$. Consequently we have $S = e_i A$ and therefore $R = de_i A$ is a minimal right ideal of A.

Let f be an element of A such that $de_i f \neq 0$. If $g = e_i f$, then $dg \in de_i A$ holds, so $dgN \subseteq dN = 0$. But $dg = e_i^{k-2} dg \in A^k$, for any k yields, by condition (II), obviously $dgA \neq 0$. We have the inclusion $dgA = de_i fA \subseteq de_i A$, which implies, by $dgA \neq 0$ and by the minimality of $de_i A$ the equation $dgA = de_i A$, consequently $dg = de_i f \in de_i A = dgA$.

For an arbitrary element $h \in A$, assuming $dgh \neq 0$ the inclusion $dgh = e_i^{k-3} dgh \in A^k$, for every k yields, $dgh \in A^\omega$, by condition (II) $dghA \neq 0$ and the minimality of the ring ideal $de_i A$ in A implies $dghA = de_i A$, whence one has $dgh \in dghA$ for every $h \in A$.

Assertion (4), pointed out and proved before yields by condition (II) $dg \in \boldsymbol{T}(A) \subseteq \boldsymbol{R}(A) = 0$ and $dg = 0$, contradicting to $dgA \neq 0$. Consequently, $e_i A = 0$ holds for every i and one has $A = N_1 = J(A)$, which shows the implication (II) $\Rightarrow$ (I).

Let j denote the maximum of all k_i and l_i. Furthermore, if we take $e = e_1 + e_2 + \ldots + e_n$, then we have

$$eN^{j+1} = N^{j+1}e = 0$$

but either $eN^j \neq 0$ or $N^j e \neq 0$. If e.g. we assume $eN^j \neq 0$, then there exists an e_i with $e_i N^j \neq 0$.

Let us consider an element $b \in N^j$ such that $e_i b \neq 0$. Let $e_i b$ be denoted by c. Then one evidently has $c = e_i b = e_i c = ec$ and $cN = 0$. Let $\hat{T}(A)$ be defined left-right dually to the ideal $T(A)$, defined in the proof of Proposition 11, in assertion (3). Then $\hat{T} \subseteq \hat{R}$ holds, and condition (II′) implies also $\hat{T}(A) = 0$. The left-right dualization of assertion (4) yields, by $\hat{T}(A) = 0$ and $c = ec \in eA$, that there exists an element $d \in A$ such that one has $dc \notin Adc$. We may take $d = de_i = de$.

We shall verify $d \in N$. By $dc \notin Adc$ one has $d \notin Ad = Ade_i$. Therefore Ade_i is properly contained in Ae_i, which is directly indecomposable, whence by E. Artin – C. Nesbitt – R.M. Thrall [6] $Ade_i \subseteq N$ follows. This implies, that $(de_i)^2$ is nilpotent, whence also de_i must be nilpotent. But $Ade_i \subseteq N$ and $d = de_i$ imply $d \in N$.

Then $dc \in Ne_i N^j \subseteq N^{j+1}$ yields $dce = 0$. But $dcN = 0$ implies $dcA = 0$ and by $dc \in N^{j+1}$ we have also $edc = 0$. On the other hand the inclusions $dc = de_i c = de_i^{k-2} c \in A^k$, for any k, $dc \in A^\omega$, yield, by condition (II′) evidently $Adc \neq 0$.

Since $edc = 0$, one has $Ndc \neq 0$. Consequently, there exists an element $g \in N$ such that $gdc \neq 0$ holds. As above, we have $gdcA = egdc = 0$. If gdc is not a two-sided annihilator of A, then $Ngdc \neq 0$ holds.

We can continue this process until $g^* = g_{s-1} \ldots g_1 dc \neq 0$, where $g^* A = 0$ and $eg^* = 0$. Then $g^* \in N^j e N^j$ and $Ng^* \subseteq N^{j+1} e_i N^{j+1} = 0$. This implies $Ag^* = g^* A = 0$ and $g^* \in A^\omega$, which is by $g^* \neq 0$ a contradiction to condition (II). Therefore, one has $e_i = 0$ for every i, and $A = N_2 = N = J(A)$.

Consequently, condition (II′) implies (I′). This completes the proof.

Remarks 13. (1) An interesting task would be to investigate the $\boldsymbol{R}$-radical of a full matrix ring A_n $(n \geqslant 2)$ for an arbitrary ring A.

(2) We mention in connection with Propositions 11 and 12, that T. Szele

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