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ON SOME WEAKLY SUPERNILPOTENT RADICALS OF RINGS

BY

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All rings considered here are associative. For the fundamental notions used in this paper we refer to Jacobson [9]. In particular, notions concerning radicals can be found in the books by Divinsky [6], by Gray [8], and by the author [15].

The author's paper [21] explicitly determines the class C_1 of all rings, every additive subgroup of which is multiplicatively idempotent. Satz 1 of the author's paper [18] solves a problem orally raised by R. Wiegandt and shows the existence of a non-trivial homomorphically closed semisimple class C_2 of rings. Then the inclusion $C_1 \subsetneq C_2$ holds. On the other side, independently of [18], Stewart [11] has determined all non-trivial semisimple radical classes C_3, and it is interesting that the equality $C_2 = C_3$ holds. The author [20] characterizes this class C_2 $(= C_3)$ by five further equivalent conditions.

The upper radical UC_2, determined by C_2, is supernilpotent, which also is "very strong" radical in the sense that in any ring A the radical $UC_2(A)$ contains every UC_2-radical subring of A (cf. Wiegandt [24]). Let us mention that generalizations called "strong" radicals of "very strong" radicals were discussed before in the fundamental joint paper of Divinsky, Krempa and Suliński [7].

It is the purpose of this note to discuss some weakly supernilpotent radicals and, in particular, to find lower bounds and upper bounds for them.

Definition 1. A ring A is called a *generalized Jacobson radical ring*, shortly a *GF-ring*, if $A(1-a)+(1-a)A = A$ holds for every element $a \in A$.

PROPOSITION 2. *Every Jacobson radical ring is a GF-ring and every commutative GF-ring is a Jacobson ring. The class of all GF-rings is homomorphically closed.*

Proof is trivial.

PROPOSITION 3. *Let A be a GF-ring and $I_{a,b}$ the two-sided ideal of A generated by the differences $ab-b$ and $ba-b$, where a and b are arbitrary fixed elements of A, i.e.*

$$I_{a,b} = (ab-b,\ ba-b).$$

Then $I_{a,b}$ contains the bi-ideal bAb of A.

We remark that bi-ideals are discussed in papers [10] and [22]. Generalized bi-ideals of rings are investigated in paper [23].

Proof. In the factor ring $\bar{A} = A/I_{a,b}$ we obviously have $\bar{b} = \bar{a}\bar{b} = \bar{b}\bar{a}$, where $\bar{x} = x + I_{a,b}$ for $x \in A$. By Proposition 2, also $\bar{A}$ is a GF-ring. Hence we have

$$(*) \qquad (\bar{1}-\bar{a})\bar{A} + \bar{A}(\bar{1}-\bar{a}) = \bar{A}.$$

Now multiplying (*) from left and right by $\bar{b}$, we get $\bar{b}\bar{A}\bar{b} = \bar{0}$, because the equations $\bar{b}(\bar{1}-\bar{a}) = (\bar{1}-\bar{a})\bar{b} = \bar{0}$ hold true. This implies $bAb \subseteq I_{a,b}$, which completes the proof.

COROLLARY 4. *Under the notations of Proposition 3, the inclusions $(bA)^2 \subseteq I_{a,b}$, $(Ab)^2 \subseteq I_{a,b}$ and $b^3 \in I_{a,b}$ hold.*

The proof follows from the inclusion $(bA)^2 + (Ab)^2 \subseteq I_{a,b}$ and from the fact that $b^3 \in bAb$.

COROLLARY 5. *A GF-ring does not contain non-zero idempotent elements.*

Proof. Let us assume that $e^2 = e$ in a GF-ring. Then, by definition of $I_{a,b}$, we obviously have $I_{e,e} = 0$ and Corollary 4 implies $e^3 = 0$, thus $e = 0$.

Definition 6. A ring is called *Zorn ring* or an *I-ring* if every its non-nil ideal contains a non-zero idempotent element.

COROLLARY 7. *Every Zorn GF-ring is nil.*

COROLLARY 8. *Every GF-ring with minimum condition on principal right ideals is a nil ring.*

Proof follows from Corollary 7 and from paper [16].

THEOREM 9. *Let LGF be the lower radical class determined by the class of all GF-rings. Let F and Be denote the class of all Jacobson radical rings and that of all Behrens radical rings (cf. [4] and [6]), respectively. Then*

$$F \leqslant LGF \leqslant Be.$$

Proof. $F \leqslant LGF$ immediately follows from Proposition 2 and from Chapter I of paper [6]. Furthermore, Be is by definition the upper radical determined by the special class (see Andrunakevič's [2] results in Chapter VII of Divinsky [6]) of all subdirectly irreducible rings having a non-zero idempotent element in their hearts, where *heart* is the non-zero intersection of all non-zero two-sided ideals. Consequently, *Be* coincides with the *F*-regular radical of Brown and McCoy [4], where we choose $F(a) = (a^2 - a)$, and *F*-regularity of *a* is defined by $a \in F(a)$. Therefore, an application of Corollary 4 yields $LGF \leqslant Be$ since a *GF*-ring cannot be mapped homomorphically onto an *e*-primitive ring of Gray [8].

PROBLEM 10. Is *LGF* supernilpotent? (**P 859**)

Remark 11. The definition of the *e*-primitivity of a ring, on p. 106 of [8], is incorrectly formulated: "A ring *R* is *e*-primitive if every non-zero ideal in *R* contains a non-zero idempotent element". Correct definition reads e.g. as follows: A ring is *e*-primitive if there exists a non-zero idempotent element, which is contained in every non-zero ideal. Under Gray's original definition, the second assertion in her Theorem 26, p. 106 of [8], would be false, since the semisimple but non-primitive rings with minimum condition on principal right ideals (e.g. the direct sum of two division rings) obey her original definition, but they do not satisfy the second assertion of her Theorem 26. On the other hand, a ring is *e*-primitive if and only if it is a Behrens semisimple [4] subdirectly irreducible ring.

COROLLARY 12. *Every GF-ring, as well as every LGF-ring, is a Brown-McCoy radical ring* [5].

Remark 13. An important and deep theory has been developed for the Brown-McCoy radical, for the Brown-McCoy semisimple rings, and for the strongly Brown-McCoy semisimple rings of Suliński (see [12]-[14]). In [17] we investigated some surprising properties by Brown-McCoy radical rings *A* which satisfy the following condition: a submodule *N* of every *A*-right module *M* which is maximal in *M* with respect to the property $NA = 0$, always is a direct summand of *M*, i.e. $M = N \oplus K$ holds with a submodule *K* of *M*.

Definition 14. The subring $I(S)$ of a ring which is maximal for the property that it contains the subring *S* as a two-sided ideal is called the *idealizer* of *S*.

COROLLARY 15. *Let S be a subring which is maximal among all LGF-radical subrings of an arbitrary ring A. Then the idealizer* $I\big(I(S)\big)$ *of the idealizer* $I(S)$ *coincides with* $I(S)$, *i.e.* $I\big(I(S)\big) = I(S)$ *holds.*

The proof follows from the elementary fact that every nilpotent ring is an *LGF*-ring and from a theorem of [19].

Our [19] another theorem uses the notion of right *T*-nilpotent and left *T*-nilpotent rings. These rings were also discussed in [16].

Definition 16. A ring *A* is said to be *right* (*left*) *T-nilpotent* if for every countably infinite subsets of elements $a_1, a_2, \ldots$ of *A* there exists 0 in the sequence of the products $a_1 a_2 \ldots a_i$ (or $a_j a_{j-1} \ldots a_2 a_1$). *A* is *two-sided T-nilpotent* if it is right *T*-nilpotent as well as left *T*-nilpotent.

By our paper [16], the Jacobson radical of every ring with minimum condition for principal right ideals is right T-nilpotent.

PROPOSITION 17. (1) *A ring A is T-nilpotent (or right T-nilpotent or left T-nilpotent, respectively) if there exists an ideal I such that both I and A/I are T-nilpotent (or right T-nilpotent or left T-nilpotent, respectively).*

(2) *Every nilpotent ring is T-nilpotent.*

(3) *Every homomorphic image of a right T-nilpotent ring is a right T-nilpotent.*

Proof. (1) The necessity of the condition being trivial, we show only its sufficiency. Let I and A/I be e.g. right T-nilpotents. Then, for every ω-sequence $x_1, x_2, \ldots, x_m, \ldots$ of A, $y_1 = x_1 x_2 \ldots x_{m_1}$ belongs to I for a suitable index m_1 since A/I is right T-nilpotent. Furthermore, the product $y_2 = x_{m_1+1} x_{m_1+2} \ldots x_{m_2}$ also belongs to I for an index m_2 etc. But I is also right T-nilpotent, so that

$$y_1 y_2 \ldots y_k = x_1 x_2 \ldots x_{m_k} = 0$$

holds for a suitable k. Thus A itself is a right T-nilpotent.

(2) Every nilpotent ring is trivially T-nilpotent.

(3) The last assertion is also evident.

PROPOSITION 18. (1) *The sum of two right (left or two-sided) T-nilpotent ideals is of the same type.*

(2) *The sum of all right T-nilpotent ideals (of all T-nilpotent ideals) is a nil ideal of the ring.*

Proof. (1) Assume that I_1 and I_2 are right T-nilpotent ideals. By the first isomorphism theorem,

$$I_1 + I_2/I_1 \cong I_2/I_1 \cap I_2.$$

Now, $I_2/I_1 \cap I_2$ is, by assertion (3) of Proposition 17, right T-nilpotent. Thus $I_1 + I_2/I_1$ and I_1 are, at the same time, right T-nilpotent, hence assertion (1) of Proposition 18 follows from assertion (1) of Proposition 17.

(2) Let I be an arbitrary right T-nilpotent ideal of A and $x \in I$ be an arbitrary element. Then $x, x^2, x^3, \ldots, x^m, \ldots$ is an ω-sequence. Consequently, for the sequence of products $p_i = x x^2 x^3 \ldots x^i$ we obviously have $p_j = 0$ for a suitable j, I being right T-nilpotent. This yields

$$x^k = 0 \quad \text{with } k = 1 + 2 + \ldots + j = \frac{j(j+1)}{2}.$$

Therefore I is a nil ideal and thus also the sum of all right T-nilpotent ideals is a nil ideal, which completes the proof of assertion (2) as well.

Remark 19. The sum of all right T-nilpotent ideals of a ring is, generally, not right T-nilpotent, as example 3 of Divinsky [6], p. 19, shows. Namely, if x_a is an arbitrary basis element of the algebra A, then the ideal (x_a) is nilpotent, consequently also T-nilpotent, but their sum A is not T-nilpotent because

$$x_{1/2}x_{1/4}x_{1/8}\dots x_{1/2n}\neq 0 \quad \text{for every } n.$$

THEOREM 20. *Let LT (or LT_r or LT_l) be the lower radical class, determined by the class of all T-nilpotent (or right T-nilpotent or all left T-nilpotent, respectively) rings. Then $B \leqslant LT \leqslant LT_r \leqslant U$ and $B \leqslant LT \leqslant LT_l \leqslant U$ hold, where B is Baer's lower nil radical and U is Baer's upper nil radical.*

The proof follows immediately from assertion (2) of Proposition 17 and from assertion (2) of Proposition 18.

COROLLARY 21. *A ring with minimum condition on principal right ideals is LT_r-radical if it is a Jacobson radical ring, or if it is a nil ring.*

PROPOSITION 22. *The weakly supernilpotent radicals LT, LT_r and LT_l are supernilpotent.*

The proof follows from [3] and from the elementary facts that the classes T, T_r and T_l are hereditary.

PROBLEM 23. Are LT, LT_r and LT_l also special radicals? (**P 860**)

COROLLARY 24. (1) *The idealizer $I(S)$ of every T-nilpotent proper subring S of A is properly larger than S.*

(2) *If the subring S is maximal among all LT- (or all LT_r- or all LT_l-) subrings of A, then $I(I(S)) = I(S)$ holds.*

The proof follows from [19].

PROBLEM 25. Is $LT_r = LT_l$ true? (**P 861**)

PROBLEM 26. Is every T-nilpotent (or right T-nilpotent) right ideal of a ring contained in a T-nilpotent (or right T-nilpotent) two-sided ideal? (**P 862**)

The following assertion can be considered as well known (cf. [9]) but because of its importance we insert it with one of its proofs:

PROPOSITION 27. *In an arbitrary ring A the sum N_r of all nil right ideals coincides with the sum N_l of all nil left ideals of A.*

Proof. Let R be a nil right ideal of A and $r \in R$. Then for every element $x \in A$ there exists an exponent $n = n(x)$ such that $(rx)^n = 0$ holds. Thus we have also $(xr)^{n+1} = 0$ and the inclusion $(r)_l^2 \subseteq A_r$, for the principal left ideal $(r)_l$ generated by r in A, implies that $(r)_l^2$ and $(r)_l$ are nil left ideals of A. But $r \in R$ is chosen in an arbitrary way and, consequently, $N_r \subseteq N_l$. Similarly, also $N_l \subseteq N_r$ can be shown, which implies $N_r = N_l$.

Definition 28. We denote in the sequel the two-sided ideal $N_r = N_l$ by N. Furthermore, let LN be the lower radical class determined by the class of all rings A for which $N = A$.

Remark 29. The elements of N need not be nilpotent. G. Koethe's famous problem on the possibility of embedding nil right ideals into nil ideals being yet unsolved, but $N = N_r = N_l$ obviously contains the upper nil radical U.

THEOREM 30. *We have $U \leqslant LN \leqslant F$ for the Jacobson radical F.*

Proof. $U \leqslant N$ implies $U \leqslant LN$ and $N \leqslant F$ implies $LN \leqslant F$.

PROPOSITION 31. *LN is supernilpotent.*

The proof follows from the hereditariness and the fact that $U \leqslant LN$.

PROBLEM 32. Is LN a special radical? (**P 863**)

PROPOSITION 33. *The idealizer $I(I(S))$ of the idealizer $I(S)$ of every maximal LN-radical proper subring coincides with $I(S)$.*

The proof follows from [19].

PROPOSITION 34. *On the class of all rings with minimum condition on principal right ideals LN coincides with the Jacobson radical F.*

The proof follows from [16].

Definition 35. An ideal I of a ring is called *transfinitely nilpotent* if there exists an (infinite) ordinal number γ such that $I^\gamma = 0$. Here $I^\alpha I = I^{\alpha+1}$, and for a limit ordinal number β let I^β be $\sum I^\alpha$ for all $\alpha < \beta$.

Remark 36. The Jacobson radical of a ring with minimum condition on principal right ideals yields an instance of a transfinitely nilpotent ideal (see [16]).

Definition 37. Let LTR be the lower radical class determined by the class TR of all transfinitely nilpotent rings.

THEOREM 38. *$B \leqslant LTR \leqslant AS$ holds, where AS is the antisimple radical of Andrunakevič [1], and LTR is supernilpotent.*

The proof is almost trivial and we omit it.

PROBLEM 39. Is LTR a special radical ? (**P 864**)

PROPOSITION 40. *The idealizer $I(I(S))$ of the idealizer $I(S)$ of every maximal LTR-radical proper subring S coincides with $I(S)$.*

The proof follows from [19].

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