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MATHEMATICS
(ASSOCIATIVE RINGS AND ALGEBRAS)

Further Characterization of Strongly Regular Rings

by

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Summary. Eleven conditions are used to characterize strongly regular rings.

The fundamental notions used here can be found in [1] or [4]. "Ring" will always mean an associative ring. As it is well known, a ring A is said to be strongly regular, if $a \in a^2 A$ holds for every $a \in A$. These rings always are Neumann regular, i.e. also $a \in aAa$ holds for every $a \in A$. $(a)_r$ and $(a)_l$ denote the principal right and the principal left ideals generated by a, respectively.

In [8] I have explicitly determined the class of all rings, every finitely generated proper subring of which is a principal right ideal. It is interesting that this class forms a set, of cardinality of the continuum, and every ring from this ring set is of cardinality at most $\aleph_0$. Furthermore, paper [9] discusses some properties of rings in which every left ideal is a principal left ideal and every right ideal is a principal right ideal. In an arbitrary strongly regular ring every finitely generated two-sided ideal is a principal right ideal, and at the same time, it is also a principal left ideal. Theorem 2 in [10] characterizes the strongly regular rings using twenty-one equivalent conditions.

It is the purpose of this note to give ten *further* equivalent conditions, which partially summarize some of my results for the strong regularity of rings.

THEOREM. *For an arbitrary ring A the following conditions are mutually equivalent:*

(1) *A is strongly regular;*

(2) *A has no nonzero nilpotent element and A satisfies one of the following conditions:*

  (I) *For every element $a \in A$ there exists an exponent $n = n(a) \geqq 2$ such that $aAa = a^n Aa^n$ holds;*

  (II) *$aA = aAa$ holds for every $a \in A$;*

  (III) *$Aa = aAa$ holds for every $a \in A$;*

*) Author's proofs received on February 1, 1974.

(IV) $aA=a^2A$ *holds for every* $a\in A$;

(V) $Aa=Aa^2$ *holds for every* $a\in A$;

(3) $(ab)_r=(a)_r\wedge(b)_r$ *holds for every* $a,b\in A$;

(4) $(ab)=(a)_l\wedge(b)_l$ *holds for every* $a,b\in A$;

(5) $(a)_r=(a^2)_r$ *holds for every* $a\in A$;

(6) $(a)_l=a^2$ *holds for every* $a\in A$;

(7) *There is an exponent* $m\geqq 2$ *for every right ideal* R *of* A *such that* $(R+AR)^m=R$ *holds, and an exponent* $n\geqq 2$ *for every left ideal* L *of* A *such that* $(L+LA)^n=L$ *holds.*

Proof. $(x)\Leftrightarrow(y)$ will, as usual, denote the equivalence of the conditions (x) and (y).

$(1)\Leftrightarrow(2)$ (I) was proved in Theorem 9 of [11].

$(1)\Leftrightarrow(2)$ (II) was proved in Theorem 6 of [12].

$(1)\Leftrightarrow(2)$ (III) follows from the left-right selfduality, implied by the subcommutativity, of the strongly regular rings (cf. Theorem 2 of [10]).

$(1)\Leftrightarrow(2)$ (IV) was proved in Theorem 9 of [13].

$(2)\Leftrightarrow(2)$ (V) again follows from Theorem 9 of [13].

$(1)\Leftrightarrow(3)\Leftrightarrow(4)\Leftrightarrow(5)\Leftrightarrow(6)$ has been shown in Theorem 2 of [14].

(1) implies (7), since, by (1) and Theorem 2 of [10] every one-sided ideal of A is a two-sided ideal, and it is also idempotent.

(7) implies (1), since, by (7) every right ideal R and left ideal L of A is a two-sided ideal of A. Moreover, by the trivial inclusions

$$(R+AR)^m\subseteq(R+AR)^2\subseteq R$$
$$(L+LA)^n\subseteq(L+LA)^2\subseteq L$$

and by (7), every right ideal and every left ideal is idempotent. Thus A is a subcommutative regular ring, which is, by Theorem 2 of [10], strongly regular.

This completes the proof.

Remark 1. The "Satz" in [8] implies that if in a ring the Brown-McCoy radical G is properly larger than Baer's lower nil radical B, that is $G\supsetneqq B$ holds, then A contains a finitely generated proper ($\neq A$) subring S, which is not a principal right ideal of Suliński [5] has given an important characterization for the Brown—McCoy radical.

Remark 2. Márki [3] has investigated some categories, satisfying also Grothendieck's axiom AB5, in which the sums of subobjects coincide with some unions of the same subobjects. Now, in rings the sets $B+C$ and $B\cup C$, for subrings B and C, with $B\nsubseteq C$ and $C\nsubseteq B$, generally are different subsets of the ring. Let us remark that for semigroups Lajos [2] has shown the equivalence of similar conditions to (1) and (7), but only for $m=n=2$, and naturally using set-theoretic unions in the analogue of (7) instead of sums, used in (7) for $m\geqq 2$ and $n\geqq 2$.

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Remark 3. Every homomorphic image of every strongly regular ring is again strongly regular. On the other hand, every strongly regular ring is Jacobson semisimple, and also Brown—McCoy semisimple. Thus every strongly regular ring A is strongly Brown—McCoy semisimple, and consequently every two-sided ideal I of A is "representable" in the sense of Suliński [6]; in this particular case $I=\bigwedge_{\alpha} I_{\alpha}$ holds, where A/I_{α} is a division ring.

Problem. Classify the strongly regular rings using Suliński's methods [6, 7].

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Ференц А. Сас, **Дальнейшие характеристики сильно регулярных колец**

Содержание. В настоящей работе представлены характеристики сильно регулярных колец при помощи определения эквивалентности одиннадцати условий.