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Remarks on my paper "The radical property of rings such that every homomorphic image has no nonzero left annihilators"

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1. We consider here only associative rings. For the arbitrary additive subgroups B and C of a ring A, let $B.C$ be the additive subgroup, generated by all products $b.c$ with $b \in B$ and $c \in C$, of A. Let $R(a)$ denote the product $(a)_l.A$, where $(a)_l$ is the principal left ideal, generated by the element a, of the ring A. Then $R(a) = aA + AaA$, which is a twosided ideal of A. The left-right dual of $R(a)$ is $R'(a) = Aa + AaA$. According to the author's paper [11], the rings such that every homomorphic image has no nonzero left annihilators are useful to give some criteria for the existence of the unity element of a ring, and these rings were called in [11] shortly E_5-rings. (Author [11] has discussed E_i-rings, for $i = 0, 1, 2, 3, 4$ and 5. See yet the author's paper [12]). Then the following six conditions on a ring A obviously are equivalent (see Satz 2.1 and 2.2 of [11]):

(i) A is an E_5-ring;

(ii) $L \subseteqq L.A$ for each left ideal L of A;

(iii) $a \in R(a)$ for each element a of A;

(iv) $(a) = R(a)$ for each principal twosided ideal (a) of A;

(v) $I = I.A$ for each twosided ideal I of A;

(vi) $(a) = (a).A$ for each principal twosided ideal (a) of A.

2. Let us mention, that G. Szász [14] before has discussed semigroups S such that $I = I.S$ holds for each twosided ideal I of S.

3. For the elements of a (not necessarily E_5-) ring A, the relationship $a \in R(a)$ defines a so-called F-regularity in the sense of B. Brown-N. H. McCoy [4], where F is a mapping of A into the set of ideals of A such that if φ is a homomorphism of A onto a ring A', then $F(a\varphi) = (F(a))\varphi$ in A'. Following [4], then $a \in A$ is F-regular, if $a \in F(a)$. Furthermore an ideal I of A is F-regular, if $a \in F(a)$ for all $a \in I$. The F-radical $\mathrm{rad}_F A$ of the ring A is the set of all elements b of A, which generate F-regular ideals (b) in A. Then $\mathrm{rad}_F A$ is an F-regular ideal of A, and by [4] also $\mathrm{rad}_F (A/\mathrm{rad}_F A) = 0$ holds. This $\mathrm{rad}_F A$ is in every ring A a radical in the sense of J. M. Maranda [8] and G. Michler [9]. Naturally $\mathrm{rad}_R A$ means $\mathrm{rad}_F A$ for the particular case $F(a) = R(a) = aA + AaA$. Let R denote the class of all E_5-rings. For an arbitrary class C of rings and for an arbitrary ring A, let $\bar{C}$ and $\bar{C}(A)$ denote the lower radical class, generated by C, in the sense of S. A. AMITSUR [1] and A. G. KUROSH [7] (see yet also N. DIVINSKY [5]), and the $\bar{C}$-radical of the ring A, respectively. Then $\mathrm{rad}_R\, A \supseteq \bar{R}(A)$ holds for every ring A.

4. My dear collega C. Roos from Delft (Holland) has kindly directed my attention to the situation, that however the first sentence of Theorem 3 of the author's paper [13] is true, its proof in the paper [13] fails to be correct. Namely, the first sentence of Theorem 3 of [13] asserts $\bar{R} = R$, which is true, but its incorrect proof shows only the existence of the F-radical $\mathrm{rad}_R\, A$ in every ring A, for $F(a) = R(a)$. On the other hand, C. Roos [10] has given an element free proof of $\bar{R} = R$, even for the not necessarily associative rings, and C. Roos [10] has shown $\mathrm{rad}_R\, A \supsetneqq R(A)$ for some concrete rings.

5. According to this observation, in the second sentence of Theorem 3 of the author's paper [13], "ring with F-radical $\mathrm{rad}_R\, A$ zero" must be read instead of "R-semisimple ring A."

6. The Corollaries 4, 6, 7 and 11 almost trivially remain true; and Corollary 7 holds even more generally: also with "locally nilpotent" instead of "nilpotent" (for this later see also C. Roos [10]).

7. The validity of the Corollaries 8, 9 and 10, for the originally formulated AMITSUR-KUROSH R-semisimplicity and strongly R-semisimplicity in the sense of V. A. ANDRUNAKIEVICH [3], follows from elementary facts. For, if C is the heart of an arbitrary subdirectly irreducible homomorphic image B of an arbitrary strongly R-semisimple ring A, then there exists an element $c \in C$ such that $c \notin c.\,C + C.\,c.\,C = C^*$ holds, the ring B being R-semisimple. If

$$C^{**} = C^* + B.\,C^* + C^*.\,B + B.\,C^*\,B,$$

then by V. A. ANDRUNAKIEVICH [2] one has $(C^{**})^3 \subseteq C^*$. Since C^{**} and $(C^{**})^3$ are ideals of B, the minimality of C in B, $c \notin C^*$ and $(C^{**})^3 \subseteq C^* \subsetneqq C$ imply $(C^{**})^3 = 0$. If $(C^{**})^m = 0$, but $(C^{**})^{m-1} \neq 0$ (where $m = 3$ or 2), then by the minimality of C in B we have $C \subseteq (C^{**})^{m-1}$, whence $C^2 = 0$ follows. Therefore A is antisimple, indeed.

8. Also Corollary 12 remains true, with the originally written "R-semisimple" and "R'-semisimple" (where R' is the left-right dualization of R; but naturally also with $\mathrm{rad}_R\, A = \mathrm{rad}_{R'}\, A = 0$), however I already before have observed, that for a proof of Corollary 12 of [13] "ring A with minimum condition on right ideals and on left ideals" must be read in Corollary 12, instead of "ring A with minimum condition on right ideals". On the other hand, it is yet an open problem, whether for an arbitrary right Artinian ring A, the nilpotence of A is implied by

$$R(A) = R'(A) = 0$$

and by the condition $A^\omega \cap (0:A)_r \cap (0:A)_l$, where $A^\omega = \cap A^n$, for all

$$n = 1, 2, 3, \ldots \text{ etc.}$$

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